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Mating of Polyhedrato Evince an Order ofthe All-Space-Filling Periodic Honeycombs

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Abstract

Earlier discovery and presentation of an inherent order of the regular and semi-regular polyhedra that displays three interrelated classes, together with consideration of the honeycombs, has led me to posit the existence of a coherent and integral metapattern that should relate the various all-space-filling periodical honeycombs, which I advance in an earlier paper that should be read in conjunction with thispaper, as they form part of a series.Here, I approach the periodic polyhedral honeycombs byexploring how pairs of polyhedra regularly combine or mate, whether proximally or distally, along the √1, √2 and √3 axes of their reference cubic and tetrahedral lattices. This is first performed for pairs of what I elsewhere term the Great Enablers (GEs), the positive and negative tetrahedra and truncated tetrahedra; secondly, for pairs of GEs and the Primary Polytopes (PPs); and thirdly, for pairs of PPs. This reveals that these three forms of mating, GE:GE, GE:PP and PP:PP, correlate with the three symmetry groups {2,3,3|2,3,3}, {2,3,3|2,3,4} and {2,3,4|2,3,4}, respectively, of the periodical honeycombs.These matings typically occur in naturallyoccurring pairs along each axis, so in general, a PP mates with just two PPs, though in certain cases one of these is the same as the original. These pairsof matings display a one-to-one correspondence with the possible periodic honeycombs. Differentiating the PPs into two groups of four according to their formal behavior suggests a pathway towards a proposed new order of the honeycombs.

Keywords*: all-space-filling;polyhedra; honeycomb; tessellation; spatial harmony; form; order*

1. Introduction

This research continues research into the regular and semiregular polyhedra that I have previously conducted [1-2]. In brief: contemplation of the fundamental spatial order that these polyhedra exhibit led me to intuit the existence of a comprehensive pattern that would properly accommodate these entities into a satisfactory order. This led to my evincing such an order, which consisted of three classes of polyhedra, according to the symmetry group *{2,3,3}*, *{2,3,4}*,or*{2,3,5}*each displayed.Each class was characterized by a pair of polar elements, a central neural element, and horizontal and vertical axes through the neutral element, which axes formed an inverted “T”. The horizontal axis showed a truncation sequence from one polar element, through its truncation, through a central neutral element, through the other truncated polar element, to the other polar element.The vertical transcendence axis showed a progression from neutral element, through a pair of snub enantiomorphs, through a small rhombic element, to culminate in a great rhombic element. The polar elements were regular polyhedra; the central neutral elements were the quasiregular polyhedra. Within any one class, an element correlated rigorously with its corresponding element in the two other classes. In addition, I extended the order to two

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further classes to properly accommodate the regular and semi-regular two-dimensional tilings of the plane, for *{2,3,6}* and *{2,4,4}* symmetry groups (excluding only $3^3.4^2$, which I consider degenerate). An interesting corollary of the order was the recognition that the 5 regular polyhedra, reconsidered as numbering 6 (3 pairs of 2, with positive and negative tetrahedra; octahedron and cube; and icosahedron and dodecahedron), rather than being considered as perfect forms (as evidenced throughout history), should instead be considered as extreme forms, about their central and therefore more perfect quasiregular polyhedron, *i.e*. tetratetrahedron (octahedron with colored faces), cuboctahedron, and icisidodecahedron.This was presented in another paper [3].

The apprehension of this elegant polyhedra order led me to suspect the existence of a comparable order that would embrace the periodic honeycombs – the very limited number of periodic arrays of regular and semi-regular polyhedra that fill space, of which the cubic lattice is the most obvious. By "comparable" here I mean an order that would exhibit a similar elegance, beauty, and integrity; and that would accommodate as particular cases each of the all-space-filling periodic honeycombs. As with the polyhedral order, I thought it likely that alternate "colorings" of polyhedra (paralleling alternate coloring of polygonal faces in two-dimensional tilings *e.g.* the checker-board pattern) would extend the number of such honeycombs (paralleling the extending of the number of regular and semi-regular polyhedra from 5 and 13 to 6 and 18, respectively,with counting each pair of enantiomorphs as one, in my earlier ordering of polyhedra [1]).

Here, my intimation was - and remains - that such an order ought to exist, given the intense regularity of each of its potential components. As far as I am aware,such a proper accounting of this order has yet to be presented, notwithstanding Critchlow's [4], Grünbaum's, and Shephard's valuable contributions [5-6]. This is potentially dangerous ground for a scholar, in that science and mathematics sometimes offer examples of such a would-be apprehension being later proven wrong; but the sophisticated integrity of my polyhedral order – which I do regard as proven - led me to believe such an order to be possible, necessary, and inevitable. The research consists of discerning such a meta-order.

In a paper that started to address the polyhedral honeycombs [7], which should be read in conjunction with this paper, I identify positive and negative tetrahedra and truncated tetrahedra asconstituting what I term the four Great Enablers $GE\colon\{T^{+/-}, D^{+/-}\} = \{T^+, T^-, D^+, D^-\}$.I identify eight Primary Polytopes,in which I include the 0-D VerTex VT, together with the Truncated Octahedron, Small Rhombic cuboctahedron, Great Rhombic cuboctahedron, CubOctahedron, CuBe and Truncated Cube - $PP\colon\{VT, TO, SR, GR, OH, CO, CB, TC\}$. I also identify three restricted Neutral Elements, where they develop as 3-D polyhedra, as $NE_{rst} = \{SP, RP, OP\}$ i.e. the Square Prism, Rotated (square) Prism (rotated by π/4), and Octagonal Prism, or alternatively seven complete secondary Neutral Elements of 0-D, 1-D, 2-D or 3-D polytopes $NE_{exp}\colon\{vt, ae, sq, og, SP, RP, OP\}$. I then present an overview of their honeycombs. Although the Square Prism and Rotated square Prism are simply cubes, and technically the Octagonal Prism is not a regular or semi-regular polyhedron, there are good reasons for including these as distinct entities.These reasons become evident when viewing my colored illustrations of the honeycombs, as they reveal a deeper theoretical consistency.

The present papersubstantially extends and revises an earlier paper [8], which developed the much briefer material in [9]. I am concerned with discerning a natural order among these various polytopes, according to how they mate with one another, by meeting in regular fashion along relevant $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$axes; and how this potential order relates to the honeycombs. So I am concerned with how theGEs relate one to another; how they relate to thePPs; and how thePPs relate one to another independently of the GEs.

I structure the paper as follows: Section 2 addresses axial mating of polytopes along $\sqrt{1}$, $\sqrt{2}$ and $\sqrt{3}$ axes of reference cubic and tetrahedral lattices. Section 3 investigates $GE\colon GE$ mating, and show how this characterizes the {2,3,3|2,3,3} honeycomb. Section 4 I investigatesGE: PP mating, and how this correlates with the four $\{2,3,3|2,3,4\}$honeycombs. Section 5 exploresPP: PP mating, and how this correlates with the ten $\{2,3,4|2,3,4\}$ honeycombs; andshows how the PPs can be formally differentiatedinto two groups of four. I conclude bysuggesting further research on an adequate formal model of the honeycombs.

2. The Possible Axial Relationships of Polytope Pairs

The periodic honeycombs exhibit obvious reference tetrahedral or cubic lattices (depending on the specific lattice).It therefore makes sense to situate individual polyhedra (or more generally, polytopes) within an orthogonal reference system, which can accommodate both (as in Figure 1 right). These polytopes exhibit relationships of one to another along the XYZ axes, diagonal axes, and long diagonal axes of the cube and cubic lattice, sofor convenience I refer to these axes as their $\sqrt{1}$, $\sqrt{2}$,and $\sqrt{3}$axes.I determine the relationship of polytope to polytope on the basis of whether they are compatible or not: *i.e.*, can they can mate together, at either a vertex, a transverse (or occasionally axial) edge, or an axial (or occasionally transverse) face?I then differentiate this mating as*proximal*, where they make actual contact (vertex, edge or face); or *distal*, *i.e.* through a secondary neutral intermediary element (which might be an axial edge, neutral face, or neutral polyhedron (a prism)). For example, CB and VTcannot mate along a $\sqrt{1}$ axis (square-to-vertex), nor can they mate along a $\sqrt{2}$ axis (edge-to-vertex), but they canmate along a $\sqrt{3}$ axis (vertex-to-vertex). Again, TO and OHcannot mate along a $\sqrt{1}$ axis (rotated square-to-vertex), nor can they mate along a $\sqrt{3}$ axis (hexagon-to-triangle), but they canmate along a $\sqrt{2}$ axis (diagonal transverse edge-to-edge).

3. How CanGE: GE Pairs Mate?

I first address the GEs. These differ from the PPs and NEs - they do not develop $\{2,3,4\}$ symmetry,and develop only $\{2,3,3\}$ symmetry on just the $\sqrt{1}$ and on alternating α and β tetrahedral $\sqrt{3}$ axes. Without loss of generality, define positive and negative Ts to be as shown in Figure 1 left; and the positive and negative Ds to be those that are developed from their respective solid.

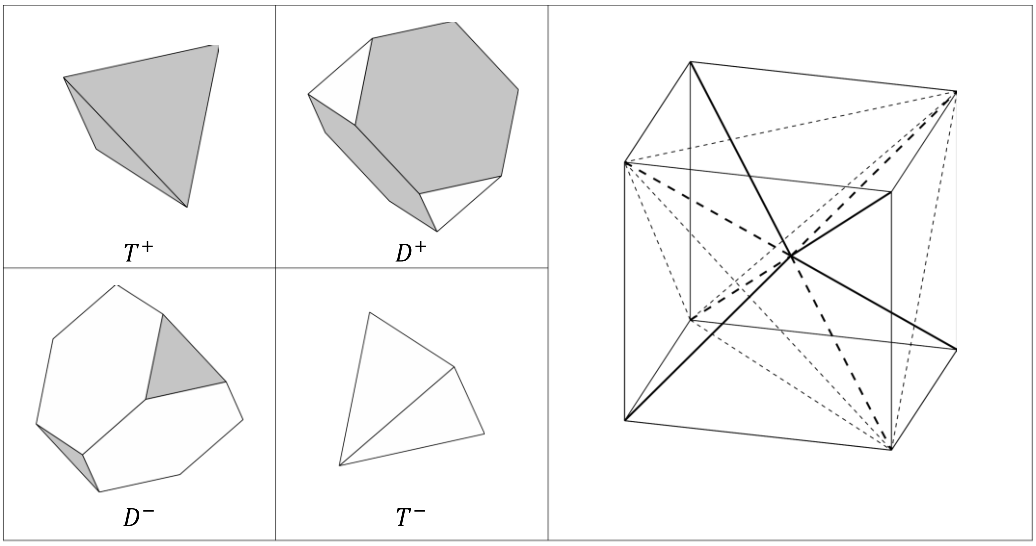


Figure 1. Left: The Four Great Enablers*(GEs)*:*D*$^+$is the Truncation of*T*$^+$,*D*$^-$of*T*$^-$;Right: α(dashed) and β(solid) √3 Axes Within *T*$^+$, Within Its Cube

Table 1. Matrices of*GE:GE* Pairs Fromtop left √1, Top Rightα√3, Bottom Left β√3, and Bottom Right Both α and β√3 Axes

	down, from above	\	\	/	/
up, from below	$\sqrt{1}$	T^+	D^+	T^-	D^-
/	T^+			/	/
/	D^+			/	/
\	T^-	\	\		
\	D^-	\	\		

α	down, from above	△	⬡		▽
up, from below	$\sqrt{3}$	T^+	D^+	T^-	D^-
	T^+				
△	D^+	△			
▽	T^-				▽
⬡	D^-		⬡		

β	down, from above		▽	△	⬡
up, from below	$\sqrt{3}$	T^+	D^+	T^-	D^-
▽	T^+		▽		
⬡	D^+				⬡
	T^-				
△	D^-			△	

$\sqrt{3}$	T^+	D^+	T^-	D^-
T^+		▽		
D^+	△			⬡
T^-				▽
D^-		⬡	△	

These matrices clearly demonstrate that GE: GE axial matings always occur in pairs, *e.g.* on the$\sqrt{1}$ axis,T^{+} mates with T^{-} and D^{-}; on the $\sqrt{3}_{\beta}^{\alpha}$ axes, T^{+} mates with D^{+} and T^{-}.We shall show that this axial mating with a pair of polytopes applies in general.

For the GEs, firstly, the $\sqrt{2}$ diagonal edge elements of a polyhedron on each $\sqrt{1}$ axis alternate in orientation from top/side to bottom/opposite side.Secondly, the facial elements on each pair of coaxial α and $\beta\sqrt{3}$ axes change between triangles and hexagons, both of which have an associated orientation. This orientation is obvious in the case of the triangles (for convention, I show this as up- or down-ward pointing, but of course in a honeycomb, these lie in multiple directions). In the case of the hexagons,I indicatethis in notation by the appendage of an extended triangle; for vertices, I append a small line.

The matrices reveal the quite highly constrained proper relations betweenGEandGE. A GE can mate with the same polyhedron of opposite sign, or with the other GEpolyhedron of opposite sign along the $\sqrt{1}$ axes; but it cannot properly mate with either GE of its own sign along those axes (*i.e.* it can't mate with itself, or with the other polyhedron of the same sign). AGE can only meet with one of its opposite sign, or with the other GE of the same sign as itself along the α and $\beta\sqrt{3}$ axes; but it cannot properly meet with itself, or with the other GEpolyhedron of the opposite sign along those axes. Figure 2 shows arbitrary builds on each of the GEs, while Figure 3 shows the axial mating patterns.

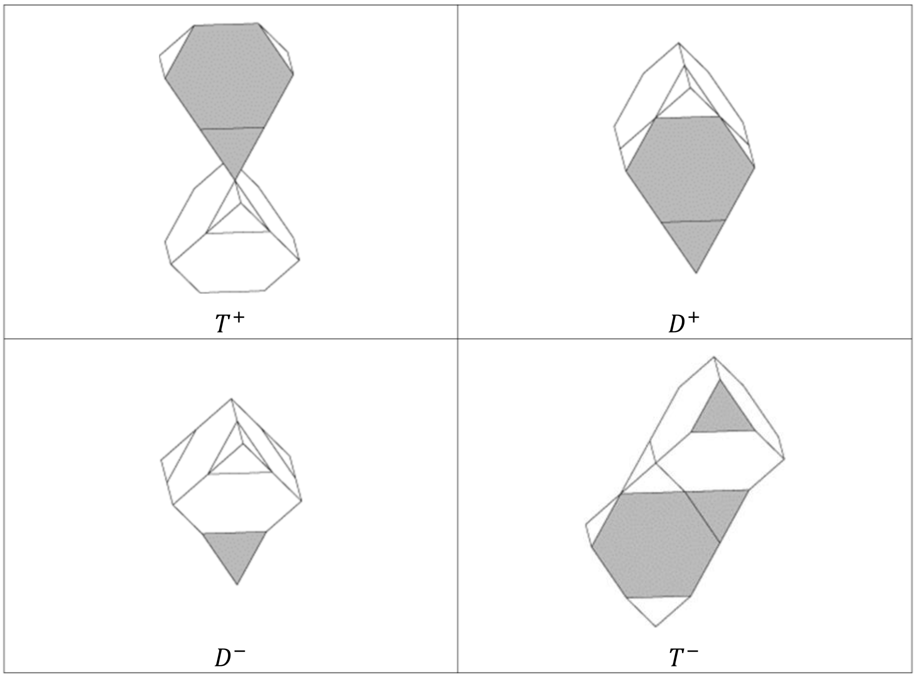


Figure 2. Arbitrary Builds of the Other Three*GEs* on Top Left*T*+, Top Right*D*+, Bottom Left*D*–, and Bottom Right*T*–, Respectively

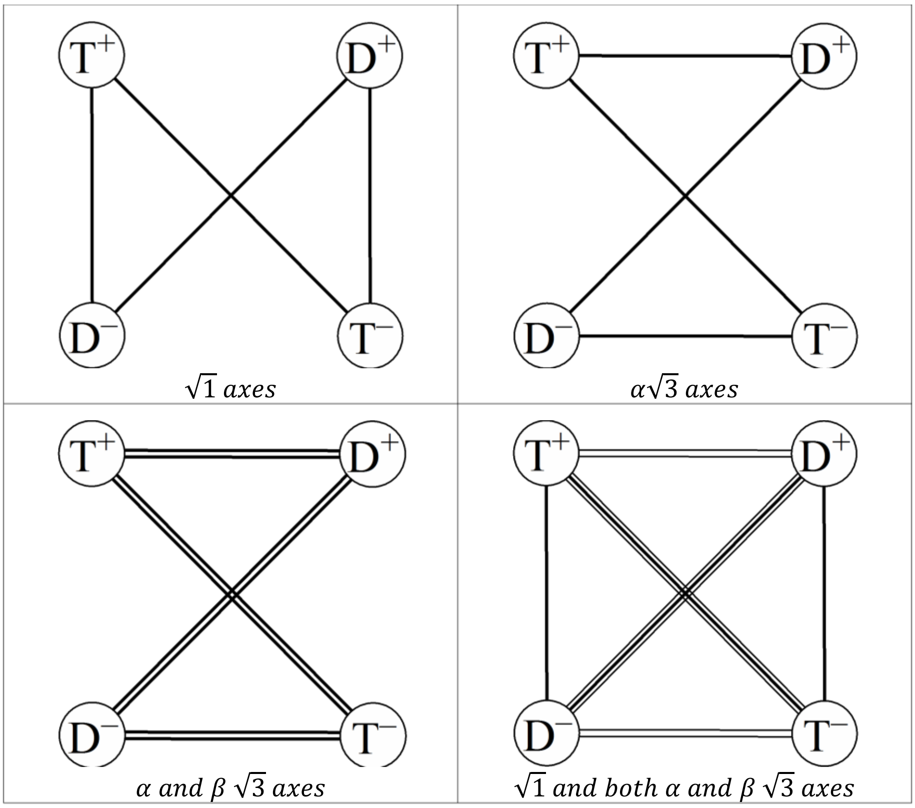


Figure 3. GE Pairings: on √1; α √3; α and β √3; and √1 and Both √3 Axes

The solitary {2,3,3|2,3,3} honeycomb(of tetrahedra and truncated tetrahedra) meets these constraints.In my earlier paper on the polyhedral honeycombs [7], I describe this particular honeycomb as a four-way alternation, or mix-and-match. This singular honeycomb provides four permutations, according to which GE associates with which reference tetrahedral lattice, *i.e.*:

$$\begin{Vmatrix} D^- & T^- \\ D^+ & T^+ \end{Vmatrix}, \quad \begin{Vmatrix} D^+ & D^- \\ T^+ & T^- \end{Vmatrix},$$

$$\begin{Vmatrix} T^+ & D^+ \\ T^- & D^- \end{Vmatrix}, \quad \begin{Vmatrix} T^- & T^+ \\ D^- & D^+ \end{Vmatrix}.$$

4. How Then Can*GE:PP*pairs Mate?

I now turn to the potential relations between GEs and PPs. GEs only develop transverse diagonal edges on the $\sqrt{1}$ axes,but no PP does (for OH and TO, the transverse edge is on the $\sqrt{2}$ axis, and the $\sqrt{1}$ axis element of the TO is the rotated square, which is merely bounded by non-axial diagonal edges). Thus they cannot mate on the$\sqrt{1}$ axes. The GEs do not develop symmetry on $\sqrt{2}$ axes, so they cannot mate on those axes.Therefore, we need only examine how GEs and PPs relate on the$\sqrt{3}$ axes.

We find that again, the potential matings of polytopes are constrained. On the α and $\beta\sqrt{3}$ axes, Tscan mate only with CBs or VTs, by vertex; or SRs or OHs, by downward pointing triangle. Dscan mate only with TCs and COs, by upward pointing triangle; or with GRs and TOs, by hexagon. Each GE can mate with one set of four of the PPs:

$$T\text{: }\{\ CB, VT; SR, OH\ \}$$

$$D: \{\, GR, TO; TC, CO \,\}$$

Tables 2 and 3 show the pairings, and arrays. Note the arrowed expansion sequences.

Table 2. Matrix of*GE:PP* for $\boldsymbol{\alpha + \beta}\sqrt{}$3 Axes

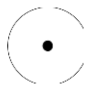
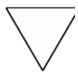
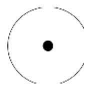
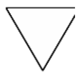
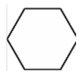
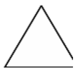
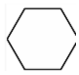
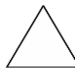
$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
T					←– –	— –		
D			←– –	— –				

Table 3. Corresponding Matrix of {2,3,3 | 2,3,4} Arrays

$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
T			$\begin{vmatrix} T^{+} & CB \\ SR & T^{-} \end{vmatrix}$		←– – – –		$\begin{vmatrix} T^{+} & VT \\ OH & T^{-} \end{vmatrix}$	
D	$\begin{vmatrix} D^{+} & TC \\ GR & D^{-} \end{vmatrix}$		←– – – –		$\begin{vmatrix} D^{+} & CO \\ TO & D^{-} \end{vmatrix}$			

Comparison of Tables 2 and 3 shows that the pairings of mateable polyhedra for the $\sqrt{3}\,PP$ axes(shown in Table 2) correlate with the possible {2,3,3|2,3,4} arrays (shown in Table 3), namely:

$$\begin{vmatrix} T^{+} & VT \\ OH & T^{-} \end{vmatrix}, \begin{vmatrix} T^{+} & CB \\ SR & T^{-} \end{vmatrix}, \begin{vmatrix} D^{+} & CO \\ TO & D^{-} \end{vmatrix}, \text{ and} \begin{vmatrix} D^{+} & TC \\ GR & D^{-} \end{vmatrix},$$

together with their permutations.

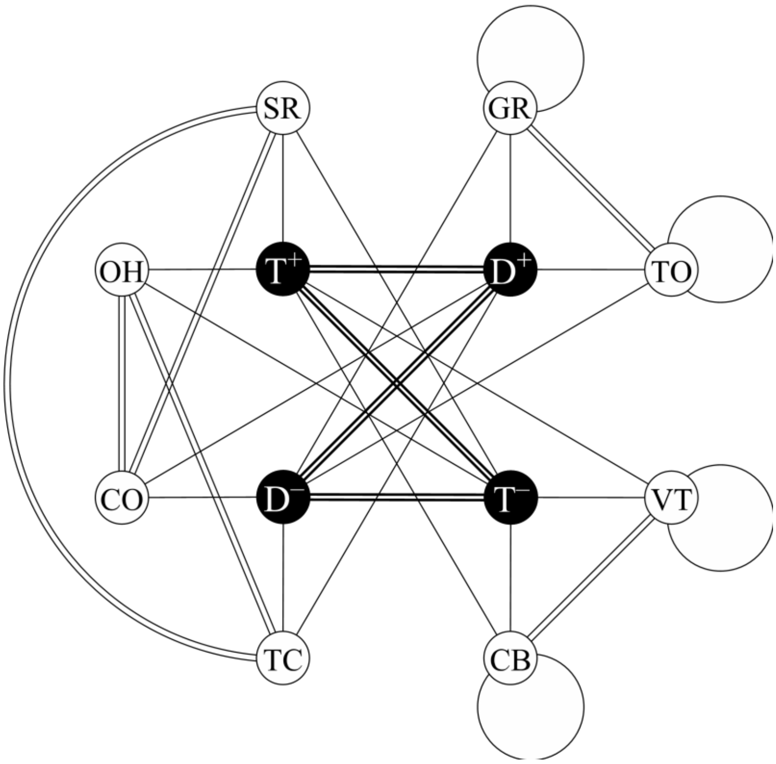


Figure 4. *GE:GE, GE:PP* and *PP:PP* Pairings on Both α and β √3 Axes.

Notes to Figure 4: The four GEs are shown in inner square arrangement, with the eight PPs in outer octagonal arrangement. Lines, circular and semi-circular arcs represent directed matings. GE:GE matings are shown in heavier line width. The circular arcs appended to GR, TO, VT and CB represent self-reflexive matings, their ends depicting both mater and "matee". To aid clarity, the two lines connecting SR and TC are shown as semicircular arcs at left. The two groups of PPs are clearly distinct, one group at left, the other at right.

GE:GE Matings: As described earlier, each GE mates (and is mated with) its polyhedron of opposite sign, and the other polyhedron of the same sign, but does not mate with the other polyhedron of opposite sign.

PP:PP Matings: In the non-reflective group of four at left, each PP mates and is mated with two other PPs, but not with the fourth. In the reflective group of four at right, each PP mates and is mated with one other PP of that group, and with itself.

GE:PP Matings: Each PP mates and is mated with two GEs that are the same polyhedra but of opposite signs (*i.e.* positive and negative). Within each group of four PPs, two PPs mate or are mated with one GE, and are mated or mate with the other GE that is the same polyhedron, but of different sign. The other two PPs in that group of four mate or are mated with the □ □that is the other polyhedron and of opposite sign, and are mated or mate with that same other polyhedron of the "opposite to the opposite", *i.e.* same sign. Each □ □mates or is mated with one pair of □ □ □from one group of four, and is mated or mates with the opposite pair on the octagon from the other group of four.

5. How Can*PP:PP*pairs Mate?

I employ a similar procedure to compare pairs of □ □along the $\sqrt{1}, \sqrt{2}, \sqrt{3}$ axes, todetermine their relationships of one to another on the basis of whether they are compatible or not, *i.e.* can they mate together, at either a vertex, a (transverse or occasionally axial) edge, or an (axial or occasionally transverse) face? In the case of □ □: □ □,which is to say for$\{2,3,4|2,3,4\}$ symmetry, this can be proximal, where they make actual contact (by vertex, edge or face); but it can also be distal, through a secondary neutral intermediary (which could be an axial edge, neutral polygonal face, or neutral polyhedron - a prism). While this could also be explored for the □ □: □ □and the □ □: □ □cases, it is not applicable to the all-space-filling periodic honeycombs.By the same token, nor is the potential use of antiprismatic neutral elements, particularly the □ □, to deal with the alternation in orientation of triangular faces, though rather intriguing arrays can be imagined from the [□ □|□ □] that are not all-space-filling *e.g.* $[TC|OH]$, $[SR|OH]$; see also [10].

This comparison is striking, and demonstrates the same behavior observed earlier for the $GE{:}GE$ and $GE{:}PP$ matings.*For each axial case, the matings of polytopes form natural pairs, and these pairs differ for each axis.* This formal behavior is beautiful to appreciate.

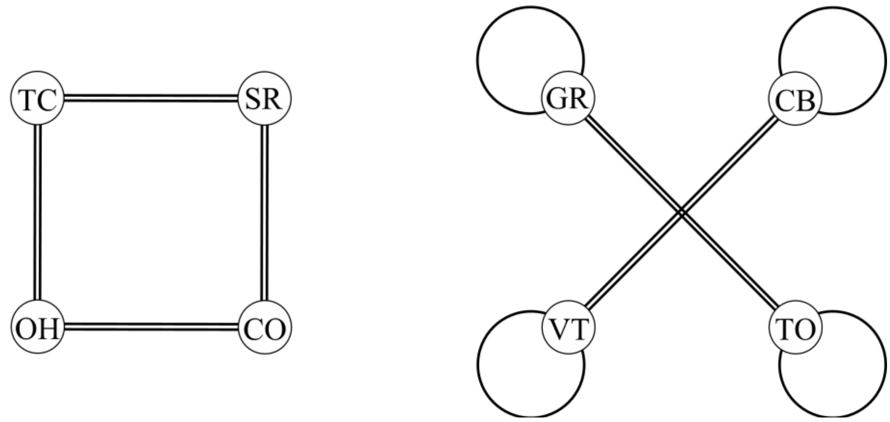


Figure 5. Two Groups of Four*PPs*with √3 Axis Matings: Left: Non-Self-Reflective *PPs* Share Double Relations with Each of Two Neighbors (*e.g. OH:CO, CO:OH, CO:SR, SR:CO);* and Right: Self-Reflective *PPs* Where Larger Circle Denotes Self-reflective Partnership (*e.g. GR:GR),* and Diagonals Denote Double Relations with Opposites (*e.g. GR:TO, TO:GR).*

Figure 5 shows that in this $PP{:}PP$ case, further complexity develops on the $\sqrt{3}$ axes.(Note that in this figure, the connections are directed; so in the right figure of self-reflective PPs, though it might appear there are four lines of connection (2 straight diagonals, 2 ends of the self-reflective circle), two (one shown as straight, one circular) are outward (mater) and two are inward ("matee"), as each PP mates with just two PPs: itself and its diagonal opposite).

These natural pairs for the various axes are:

$\sqrt{1}$ pairs :$\{ (\mathrm{GR},\mathrm{TC}),(\mathrm{CB},\mathrm{SR}),(\mathrm{TO},\mathrm{CO}),(\mathrm{VT},\mathrm{OH}) \}$

$\sqrt{2}$ pairs :$\{ (\mathrm{GR},\mathrm{SR}),(\mathrm{TC},\mathrm{CB}),(\mathrm{TO},\mathrm{OH}),(\mathrm{CO},\mathrm{VT}) \}$

$\sqrt{3}$ selfreflective pairs : $\{(\mathrm{GR},\mathrm{TO}),(\mathrm{CB},\mathrm{VT}) \}$
$= \{ \mathrm{GR{:}\,GR},\mathrm{TO{:}\,TO},\mathrm{CB{:}\,CB},\mathrm{VT{:}\,VT},\mathrm{GR{:}\,TO},\mathrm{TO{:}\,GR},\mathrm{CB{:}\,VT},\mathrm{VT{:}\,CB} \}$

$\sqrt{3}$ nonreflective pairs : $\{ \mathrm{OH{:}\,CO},\mathrm{OH{:}\,TC},\mathrm{SR{:}\,CO},\mathrm{SR{:}\,TC} \}$

$$= \{ \mathrm{CO}: (\mathrm{OH}, \mathrm{SR}), \mathrm{OH}: (\mathrm{TC}, \mathrm{CO}), \mathrm{TC}: (\mathrm{SR}, \mathrm{OH}), \mathrm{SR}: (\mathrm{CO}, \mathrm{TC}) \}$$

Table 6below will later show that each pair correlates one-to-one with its corresponding {2,3,4|2,3,4} array.

For the$\sqrt{1}$ and $\sqrt{2}$ axes, each PP is self-reflective - it mates with itself, as well as with just one other, its pair. However,for the $\sqrt{3}$ axes, four of the PPs are self-reflective - each can mate with itself, while it can also mate with one other. But the other four PPs have triangular faces.These may alternate in orientation (point up or down); in these cases each PP cannot mate with itself, as the direction of apex flips between upper and lower. So placing these four in square array, Figure 5 shows each mates with its two neighbors, but not with its opposite. In $\sqrt{3}$ matrices, common mating conditions, situated in overlapping squares, accord with the expansion/contraction sequences of arrays discussed in my earlier paper (which I recommend be read together with this paper) [7].

Table 4. Matrices of *PP:PP* for Top Left: √1, Top Right: √2, and Bottom Left: √3 Axes.Bottom Right: Axes That Facilitate Mating for the *PPs,* Where √1,2,3 Represents √1, √2, √3; and √1,2 Represents √1, √2.

$\sqrt{1}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	⯃	⯃						
TC	⯃	⯃						
CB			□	□				
SR			□	□				
TO					◇	◇		
CO					◇	◇		
VT							⊙	⊙
OH							⊙	⊙

$\sqrt{2}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	□			□				
TC	┼							
CB		┼	┼					
SR	□			□				
TO					╳			╳
CO						⊙	⊙	
VT						⊙	⊙	
OH					╳			╳

$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	⬡				⬡			
TC				✡				✡
CB			⊙				⊙	
SR		✡				✡		
TO	⬡				⬡			
CO				✡				✡
VT			⊙				⊙	
OH		✡				✡		

$axes$	GR	TC	CB	SR	TO	CO	VT	OH
GR	$\sqrt{1,2}$	$\sqrt{1}$		$\sqrt{2}$	$\sqrt{3}$			
TC	$\sqrt{1}$	$\sqrt{1,2}$	$\sqrt{2}$	$\sqrt{3}$				$\sqrt{3}$
CB		$\sqrt{2}$	$\sqrt{1,2}$	$\sqrt{1}$			$\sqrt{3}$	
SR	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$		$\sqrt{3}$		
TO	$\sqrt{3}$				$\sqrt{1,2}$	$\sqrt{1}$		$\sqrt{2}$
CO				$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$	$\sqrt{2}$	$\sqrt{3}$
VT			$\sqrt{3}$			$\sqrt{2}$	$\sqrt{1,2}$	$\sqrt{1}$
OH		$\sqrt{3}$			$\sqrt{2}$	$\sqrt{3}$	$\sqrt{1}$	$\sqrt{1,2}$

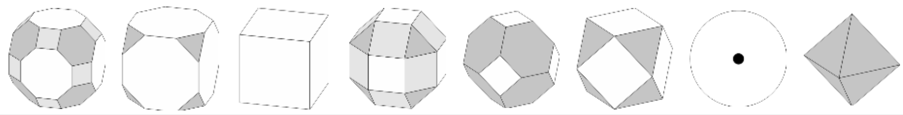
Table 5. Matrices of √1, √2, √3 Axes that Facilitate Mating

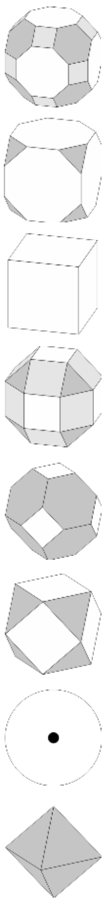
$axes$	TC	SR	CO	OH
TC	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$		$\sqrt{3}$
SR	$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$	
CO		$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$	$\sqrt{3}$
OH	$\sqrt{3}$		$\sqrt{3}$	$\sqrt{1}, \sqrt{2}$

$axes$	GR	CB	TO	VT
GR	$\sqrt{1}, \sqrt{2}, \sqrt{3}$		$\sqrt{3}$	
CB		$\sqrt{1}, \sqrt{2}, \sqrt{3}$		$\sqrt{3}$
TO	$\sqrt{3}$		$\sqrt{1}, \sqrt{2}, \sqrt{3}$	
VT		$\sqrt{3}$		$\sqrt{1}, \sqrt{2}, \sqrt{3}$

for the Two Groups of *PPs.*

Table 6. Matrix of the {2,3,4 | 2,3,4} Arrays, with Overlapping Squares of Expansion Sequences





$\sqrt{3}$	GR	TC	CB	SR	TO	CO	VT	OH
GR	$[GR \mid GR]$				$[GR \mid TO]$			
TC				$[TC \mid SR]$				$[TC \mid OH]$
CB			$[CB \mid CB]$				$[CB \mid VT]$	
SR		$[SR \mid TC]$				$[SR \mid CO]$		
TO	$[TO \mid GR]$				$[TO \mid TO]$			
CO				$[CO \mid SR]$				$[CO \mid OH]$
VT			$[VT \mid CB]$				$[VT \mid VT]$	
OH		$[OH \mid TC]$				$[OH \mid CO]$		

Might I suggest to the reader this table and Table 4are well worth the contemplation.

Conclusion

Inspired by my recognition of an adequate order to properly describe the regular and semi-regular polyhedra, this paper continues my ongoing research into a comprehensive order that would properly account for the all-space-filling polyhedral honeycombs, by investigating how pairs of the constituent polyhedra can combine.Having identified four Great Enablers,of positive and negative tetrahedra and truncated tetrahedra; and eight Primary Polytopes, of vertex, truncated octahedron, small rhombic cuboctahedron, great rhombic cuboctahedron, cuboctahedron, cube and truncated cube;I consider how $GE{:}\,GE$, $GE{:}\,PP$, and $PP{:}\,PP$ pairs combine or mate with one another, proximally or distally, along their$\sqrt{1}$, $\sqrt{2}$, or $\sqrt{3}$ axes, and how these diverse matings relate to specific honeycombs. It becomes evident that the matings are highly constrained.In Section 2, I describe the possible axial relations of polytope pairs. Section 3 details how$GE{:}\,GE$ matings correlate with the singular $\{2,3,3|2,3,3\}$honeycomb (the various honeycombs being detailed in my earlier paper [7], which should be tread with this paper). In Section 4, I show how$GE{:}\,PP$ matings correlate with the four $\{2,3,3|2,3,4\}$honeycombs.Section 5 explains how$PP{:}\,PP$ matings correlate with the ten $\{2,3,4|2,3,4\}$honeycombs.

I show that matings always occur in pairs; that is to say- and having regard to the honeycombs - for a particular axis, a given polytope can mate with just one polytope, and separately,with just one other polytope.For the given polytope, these pairs of mateable polytopes vary by axis. These pairs also vary by symmetry group –for any one symmetry group and axis, a constituent polytope pairs with just two others, and that association pattern is unique to the symmetry group and axis. In the case of the $PP{:}\,PP$ matings of the $\{2,3,4|2,3,4\}$ symmetries, in general one of these matings is with itself, the exceptions being$\sqrt{3}$ axis matings.The characteristics of these$\sqrt{3}$ axis matings then enable the $PPs$to be formally differentiated into two groups of four, which I arrange as two squares. $PP{:}\,PP$ pairings of the first group behave in a similar manner to $GE{:}\,GE$ and $GE{:}\,PP$ pairings, with PPs pairing with themselves and with their opposites. Conversely, those of the second group do not. Instead, each PP pairs with its two neighbors, but not with itself or its opposite. For the $GE{:}\,PP$ and $PP{:}\,PP$ pairings, the expansion/contraction sequences I discuss in my earlier paper [7] are evident in the $\sqrt{3}$ matrices; in particular for the $PP{:}\,PP$ pairings, the sequences are evident as overlapping squares in Table 4 (bottom left) and in Table 6 (in this paper).

I therefore move beyond the mere recognition of sets of GEs and PPs, to the appreciation of a profound inner order that relates the individual elements, according to their potential to mate with one another, and that correlates these matings with the proper honeycombs that they form. This research effort respects prior efforts [4-6], but seeks to surpass them. The challenge is to evince an adequate formal representation of the profound harmony that one can at present merely glimpse, a new order that in a future paper I hope to directly address.

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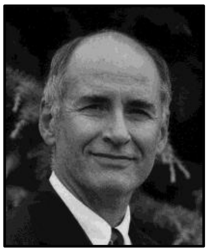

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