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Efficiency in Multiobjective Programming with Generalized Invexity

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Abstract

This paper is concerned with the multiobjective programming problems including inequality constraints. By utilizing the directional derivatives in the direction $\eta(x,\bar{x})$*, the new classes of generalized invexity functions are introduced. Using the new concepts, the sufficient optimality conditions are obtained. It is proved that the feasible solutions of the multiobjective programming problems are an efficient solution (or a weakly efficient solution) for the multiobjective programming problems.*

***Keywords**: Multiobjective programming,* $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ *pseudoinvex-I, efficient solution, optimality*

1. Introduction

Convexity plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality sees [1, 2, 3]. To relax convexity assumptions impose on the functions in theorems on sufficient optimality, various generalized convexity notions have been proposed. Gao [4] introduced B - (p, r) - V - type I Functions and Antczakk [5] given a class of $B-(p,r)-$ invex functions. For details, the readers are advised to consult other similar literatures [6-8].

On the other hand, the sufficient optimality conditions for multiobjective optimization problems have particularly grown and became one of the most interesting topics in optimization. In particular, the sufficient optimality conditions and duality results for the nondifferentiable multiobjective programming problems were obtained by Kim [9-10] and Jayswal [11] under the generalized convex functions. Also, we can see the references in Ref. [12-13].

In this paper, motivated by the above work, we first introduce the new class of generalized invexity functions namely $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-I ($FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II et al.) by using the directional derivatives in the direction $\eta(x,\bar{x})$. Then several sufficient optimality conditions for efficient solutions (weakly efficient solutions) of nondifferentiable nultiobjective programming problems are derived and proved under the assumptions of the new generalized convexities.

2. Notations and Preliminaries

In this paper, we consider the following nondifferentiable multiobjective programming problem with constraints:

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$$\text{(MP)}\quad \begin{aligned} &\text{Mininmize } f(x) = (f_1(x), f_2(x), \cdots, f_k(x)) \\ &\text{subject to } g(x) = (g_1(x), g_2(x), \cdots, g_m(x)) \leqq 0 \\ &\qquad x \in X \end{aligned}$$

Where $X \subseteq R^n$ is a nonempty open set, $f_i : X \to R (i = 1, 2, \cdots, k)$ and $g_j : X \to R (j = 1, 2, \cdots, m)$. Following, let us denote $I = \{1, 2, \cdots, k\}$ and $J = \{1, 2, \cdots, m\}$. Let $D = \{x \in X \mid g_j(x) \leqq 0, j \in J\}$ denote the set of all feasible solutions in the multiobjective programming problem (MP). Further, we denote by $J(x) = \{j \in J \mid g_j(x) = 0\}$ the index set of all active constraints of (MP) at an arbitrary feasible solution x, and $\overline{J}(x) = \{j \in J \mid g_j(x) < 0\}$.

The following convention for equalities and inequalities will be used the paper.

For any $x = (x_1, x_2, \cdots, x_n)^T$, $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, we define:

$$x = y \Leftrightarrow x_i = y_i, \ \forall \ i = 1, 2, \cdots, n,$$

$$x < y \Leftrightarrow x_i < y_i, \ \forall \ i = 1, 2, \cdots, n,$$

$$x \leqq y \Leftrightarrow x_i \le y_i, \ \forall \ i = 1, 2, \cdots, n,$$

$$x \le y \Leftrightarrow x \leqq y, \text{ there exists } i \text{ such that } x_i < y_i.$$

Hereafter, we introduce some notions and definitions.

In the following definitions, $\eta(x, \overline{x}) : X \times X \to R^n$ is a vector valued function, with $\eta(x, \overline{x})$ nonzero.

Definition2.1. The directional derivative of f_i at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, denoted $f_i'(\overline{x}; \eta(x, \overline{x}))$ is given by

$$f_i'(\overline{x}; \eta(x, \overline{x})) = \lim_{\lambda \to 0^+} \frac{f_i(\overline{x} + \lambda \eta(x, \overline{x})) - f_i(\overline{x})}{\lambda}, \ i \in \mathrm{I}.$$

Similarly, $g_j'(\overline{x}; \eta(x, \overline{x}))$ is denoted for $j \in J$.

Definition2.2. [14] $w : X \to R$ is said to be semidirectionally differentiable at $\overline{x} \in X$, if there exists a nonempty subset $S \subset R^n$, such that $w'(\overline{x}; d)$ exists finite foe all $d \in S$. And w is said to be semidirectionally differentiable at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, if its directional derivative $w'(\overline{x}; \eta(x, \overline{x}))$ exists finite for all $x \in X$.

By an extension of the previous definition, we say that a vector function $f = (f_1, f_2, \cdots, f_k) : X \to R^n$ is semidirectionally differentiable at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, if each $f_i, i = 1, 2, \cdots, k$ is semidirectionally differentiable at $\overline{x} \in X$ in this direction. And simply, f is semidirectionally differentiable at $\overline{x} \in X$, if there exist a direction verifying the previous assertion.

Definition2.3. A feasible point $\overline{x}$ is said to be an efficient solution for (MP), if and only if there exists no another $x \in D$, such that

$$f(x) \le f(\overline{x}).$$

Definition2.4. A feasible point $\overline{x}$ is said to be a weakly efficient solution for (MP), if and only if there exists no another $x \in D$, such that

$$f(x) < f(\overline{x}).$$

Let $f_i (i \in I)$ and $g_j (j \in J)$ be semidirectionally differentiable at $\overline{x} \in X$ in the direction $\eta(x, \overline{x})$, where $\eta : X \times X \to R^n$, denote $\alpha, \beta : X \times X \to R_+ \setminus \{0\}$,

$\rho = (\rho_1, \rho_2, \cdots, \rho_k) \in R^k$, $\tau_{J(\bar{x})} = \{\tau_j \in R,\ j \in J(\bar{x})\}$, $\theta: X \times X \to R^n$. Following we introduce new definitions for the pair of involved vector functions in (MP).

Definition2.5. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) < 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 < 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.6. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \le 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 < 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.7. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoquasi-invex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \le 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 < 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 \leqq 0. \end{cases}$$

Definition2.8. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoquasi-invex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \le 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 \le 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 \leqq 0. \end{cases}$$

Definition2.9. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ quasipseudo-invex-I (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \le 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 \leqq 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 < 0. \end{cases}$$

Definition2.10. (f, g) is said to be $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ quasipseudo-invex-II (with respect to η) at $\bar{x} \in X$, if there exist $\alpha, \beta, \rho, \tau_{J(\bar{x})}$ and θ ,such that, for all $x \in X$, the following inequalities hold:

$$f(x) - f(\bar{x}) \le 0 \Rightarrow \begin{cases} \alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 \leqq 0, \\ \beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 \le 0. \end{cases}$$

3. Sufficient Optimality Conditions

In this section, we establish the sufficiency of the Fritz John optimality conditions.

Theorem 3.1. Let $\bar{x}$ be a feasible solution for (MP). Suppose that (f, g) is $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I at $\bar{x}$ with respect to η. If there exist $\lambda \in R^k, \mu \in R^m$, with $(\lambda, \mu) \geq 0$, such that

$$\lambda^T f'(\bar{x};\eta(x,\bar{x})) + \mu^T g'(\bar{x};\eta(x,\bar{x})) \geqq 0, \quad \forall x \in D \tag{1}$$

$$\mu^T g(\bar{x}) = 0 \tag{2}$$

And

$$\frac{\lambda^T \rho}{\alpha(x,\bar{x})} + \frac{\mu^T_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x,\bar{x})} \geqq 0 \tag{3}$$

Then $\bar{x}$ is a weakly efficient solution of (MP).

Proof: We proceed by contradiction. Suppose that $\bar{x}$ is not a weakly efficient solution of (MP). Then there exists a feasible solution x of (MP), such that

$$f(x) - f(\bar{x}) < 0 \tag{4}$$

Since (f, g) is $FJ - d - \alpha_\beta - \rho_\tau - \theta -$ pseudoinvex-I at $\bar{x}$ with respect to η, the inequality (4) follows :

$$\alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 < 0$$

$$\beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 < 0$$

That is

$$\alpha(x,\bar{x}) f_i'(\bar{x};\eta(x,\bar{x})) + \rho_i \left\|\theta(x,\bar{x})\right\|^2 < 0, \; \forall i \in I \tag{5}$$

$$\beta(x,\bar{x}) g_j'(\bar{x};\eta(x,\bar{x})) + \tau_j \left\|\theta(x,\bar{x})\right\|^2 < 0, \; \forall j \in J(\bar{x}) \tag{6}$$

On the other hand, there exists $(\lambda, \mu) \geq 0$ such that $\bar{x}$ verifies (1)-(2), we known $\mu_{J(\bar{x})} \geqq 0$ and $\mu_{J \setminus J(\bar{x})} = 0$, then the inequality (1) implies

$$\lambda^T f'(\bar{x};\eta(x,\bar{x})) + \mu^T_{J(\bar{x})} g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) \geqq 0, \quad \forall x \in D$$

That is

$$\sum_{i \in I} \lambda_i f_i'(\bar{x};\eta(x,\bar{x})) + \sum_{j \in J(\bar{x})} \mu_j g_j'(\bar{x};\eta(x,\bar{x})) \geqq 0 \tag{7}$$

Since $(\lambda, \mu_{J(\bar{x})}) \geq 0$ and from (5)-(6) and (3), with $\alpha(x,\bar{x}) > 0, \beta(x,\bar{x}) > 0$, we get

$$\begin{aligned}
&\sum_{i \in I} \lambda_i f_i'(\bar{x};\eta(x,\bar{x})) + \sum_{j \in J(\bar{x})} \mu_j g_j'(\bar{x};\eta(x,\bar{x})) \\
&< -\left(\frac{\sum_{i \in I} \lambda_i \rho_i}{\alpha(x,\bar{x})} + \frac{\sum_{j \in J(\bar{x})} \mu_j \tau_j}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2 \\
&= -\left(\frac{\lambda^T \rho}{\alpha(x,\bar{x})} + \frac{\mu^T_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2 \leqq 0
\end{aligned}$$

Which contradicts (7). Therefore $\bar{x}$ is a weakly efficient solution of (MP).

Theorem 3.2. Let $\bar{x}$ be a feasible solution for (MP). Suppose that (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II at $\bar{x}$ with respect to η . If there exist $\lambda\in R^k,\mu\in R^m$, with $(\lambda,\mu)\geq 0$, such that

$$\lambda^T f'(\bar{x};\eta(x,\bar{x}))+\mu^T g'(\bar{x};\eta(x,\bar{x}))\geqq 0,\quad \forall x\in D$$

$$\mu^T g(\bar{x})=0$$

And

$$\frac{\lambda^T\rho}{\alpha(x,\bar{x})}+\frac{\mu^T_{J(\bar{x})}\tau_{J(\bar{x})}}{\beta(x,\bar{x})}\geqq 0$$

Then $\bar{x}$ is an efficient solution of (MP).

Proof: Urging by contradiction, there exists a feasible solution x of (MP), such that

$$f(x)-f(\bar{x})\leq 0 \tag{8}$$

Since (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoinvex-II at $\bar{x}$ with respect to η , the inequality (8) follows:

$$\alpha(x,\bar{x})f'(\bar{x};\eta(x,\bar{x}))+\rho\left\|\theta(x,\bar{x})\right\|^2<0$$

$$\beta(x,\bar{x})g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x}))+\tau_{J(\bar{x})}\left\|\theta(x,\bar{x})\right\|^2<0$$

That is

$$\alpha(x,\bar{x})f_i'(\bar{x};\eta(x,\bar{x}))+\rho_i\left\|\theta(x,\bar{x})\right\|^2<0,\ \forall i\in I$$

$$\beta(x,\bar{x})g'_j(\bar{x};\eta(x,\bar{x}))+\tau_j\left\|\theta(x,\bar{x})\right\|^2<0,\ \forall j\in J(\bar{x})$$

From $(\lambda,\mu)\geq 0$ and (2), we known $(\lambda,\mu_{J(\bar{x})})\geq 0$, with $\alpha(x,\bar{x})>0,\beta(x,\bar{x})>0$, then above two inequalities follow that

$$\begin{aligned}&\sum_{i\in I}\lambda_i f_i'(\bar{x};\eta(x,\bar{x}))+\sum_{j\in J(\bar{x})}\mu_j g'_j(\bar{x};\eta(x,\bar{x}))\\ &<-\left(\frac{\sum_{i\in I}\lambda_i\rho_i}{\alpha(x,\bar{x})}+\frac{\sum_{j\in J(\bar{x})}\mu_j\tau_j}{\beta(x,\bar{x})}\right)\left\|\theta(x,\bar{x})\right\|^2\\ &=-\left(\frac{\lambda^T\rho}{\alpha(x,\bar{x})}+\frac{\mu^T_{J(\bar{x})}\tau_{J(\bar{x})}}{\beta(x,\bar{x})}\right)\left\|\theta(x,\bar{x})\right\|^2\leqq 0\end{aligned}$$

This implies that

$$\lambda^T f'(\bar{x};\eta(x,\bar{x}))+\mu^T_{J(\bar{x})}g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x}))+\mu^T_{J\backslash J(\bar{x})}g'_{J\backslash J(\bar{x})}(\bar{x};\eta(x,\bar{x}))<0$$

That is

$$\lambda^T f'(\bar{x};\eta(x,\bar{x}))+\mu^T g'(\bar{x};\eta(x,\bar{x}))<0$$

Which stand in contradiction to the condition (1). Therefore $\bar{x}$ is an efficient solution of (MP).

Theorem3.3. Let $\bar{x}$ is a feasible solution for (MP). Suppose that (f, g) is $FJ - d - \alpha_{\beta} - \rho_{\tau} - \theta -$ pseudoquasi-invex-I at $\bar{x}$ with respect to η. If there exist $\lambda \in R^{k}$, $\mu \in R^{m}$, with $\lambda \geq 0, \mu \geqq 0$, such that conditions (1)-(3) hold. Then $\bar{x}$ is an efficient solution for (MP).

Proof: We proceed by contradiction. Suppose that $\bar{x}$ is not an efficient solution for (MP). Then there exists a feasible solution x of (MP), such that

$$f(x) - f(\bar{x}) \leq 0$$

Since (f, g) is $FJ - d - \alpha_{\beta} - \rho_{\tau} - \theta -$ pseudoquasi-invex-I at $\bar{x}$ with respect to η, the above inequality yield

$$\alpha(x, \bar{x}) f'(\bar{x}; \eta(x, \bar{x})) + \rho \left\|\theta(x, \bar{x})\right\|^{2} < 0$$

$$\beta(x, \bar{x}) g'_{J(\bar{x})}(\bar{x}; \eta(x, \bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x, \bar{x})\right\|^{2} \leqq 0$$

That is

$$\alpha(x, \bar{x}) f_i'(\bar{x}; \eta(x, \bar{x})) + \rho_i \left\|\theta(x, \bar{x})\right\|^{2} < 0, \ \forall i \in I$$

$$\beta(x, \bar{x}) g'_j(\bar{x}; \eta(x, \bar{x})) + \tau_j \left\|\theta(x, \bar{x})\right\|^{2} \leqq 0, \ \forall j \in J(\bar{x})$$

By $\lambda \geq 0, \mu \geqq 0$ (the condition (2) implies $\mu_{J(\bar{x})} \geqq 0$), with $\alpha(x, \bar{x}) > 0, \beta(x, \bar{x}) > 0$, the above two inequalities imply

$$\begin{aligned}
&\sum_{i \in I} \lambda_i f_i'(\bar{x}; \eta(x, \bar{x})) + \sum_{j \in J(\bar{x})} \mu_j g'_j(\bar{x}; \eta(x, \bar{x})) \\
&< -\left(\frac{\sum_{i \in I} \lambda_i \rho_i}{\alpha(x, \bar{x})} + \frac{\sum_{j \in J(\bar{x})} \mu_j \tau_j}{\beta(x, \bar{x})} \right) \left\|\theta(x, \bar{x})\right\|^{2} \\
&= -\left(\frac{\lambda^{T} \rho}{\alpha(x, \bar{x})} + \frac{\mu^{T}_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x, \bar{x})} \right) \left\|\theta(x, \bar{x})\right\|^{2}
\end{aligned}$$

From the condition (2), we known $\mu = 0$, with the condition (1), the above inequality yields

$$\begin{aligned}
0 &\leqq \sum_{i \in I} \lambda_i f_i'(\bar{x}; \eta(x, \bar{x})) + \sum_{j \in J} \mu_j g'_j(\bar{x}; \eta(x, \bar{x})) \\
&= \lambda^{T} f'(\bar{x}; \eta(x, \bar{x})) + \mu^{T} g'(\bar{x}; \eta(x, \bar{x})) \\
&< -\left(\frac{\lambda^{T} \rho}{\alpha(x, \bar{x})} + \frac{\mu^{T}_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x, \bar{x})} \right) \left\|\theta(x, \bar{x})\right\|^{2}
\end{aligned}$$

That is

$$\frac{\lambda^{T} \rho}{\alpha(x, \bar{x})} + \frac{\mu^{T}_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x, \bar{x})} < 0$$

Which stands in contradiction to the condiction (3). Therefore $\bar{x}$ is an efficient solution for (MP).

Theorem3.4. Let $\bar{x}$ be a feasible solution for (MP). Suppose that (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoquasi-invex-II at $\bar{x}$ with respect to η . If there exist $\lambda \in R^k$, $\mu \in R^m$, with $\lambda > 0, \mu \geqq 0$, such that conditions (1)-(3) hold. Then $\bar{x}$ is an efficient solution for (MP).

Proof: We proceed by contradiction. Suppose that $\bar{x}$ is not an efficient solution for (MP). Then there exists $x \in D$, such that

$$f(x) - f(\bar{x}) \leq 0$$

Since (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ pseudoquasi-invex-II at $\bar{x}$ with respect to η , the above inequality follows

$$\alpha(x,\bar{x}) f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 \leq 0$$

$$\beta(x,\bar{x}) g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 \leqq 0$$

That is

$$\alpha(x,\bar{x}) f_i'(\bar{x};\eta(x,\bar{x})) + \rho_i \left\|\theta(x,\bar{x})\right\|^2 \leqq 0, \ \forall i \in I$$

And

$$\alpha(x,\bar{x}) f_l'(\bar{x};\eta(x,\bar{x})) + \rho_l \left\|\theta(x,\bar{x})\right\|^2 < 0, \text{ for some } l \in I, l \neq i$$

$$\beta(x,\bar{x}) g_j'(\bar{x};\eta(x,\bar{x})) + \tau_j \left\|\theta(x,\bar{x})\right\|^2 \leqq 0, \ \forall j \in J(\bar{x})$$

By $\lambda \geq 0, \mu \geqq 0$ (the condiction (2) implies $\mu_{J(\bar{x})} \geqq 0$), with $\alpha(x,\bar{x}) > 0, \beta(x,\bar{x}) > 0$ and (3), the above inequalities follow that

$$\sum_{i\in I} \lambda_i f_i'(\bar{x};\eta(x,\bar{x})) + \sum_{j\in J(\bar{x})} \mu_j g_j'(\bar{x};\eta(x,\bar{x}))$$
$$< -\left(\frac{\sum_{i\in I} \lambda_i \rho_i}{\alpha(x,\bar{x})} + \frac{\sum_{j\in J(\bar{x})} \mu_j \tau_j}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2$$
$$= -\left(\frac{\lambda^T \rho}{\alpha(x,\bar{x})} + \frac{\mu_{J(\bar{x})}^T \tau_{J(\bar{x})}}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2 \leqq 0$$

On the other hand, the condiction (2) implies $\mu = 0$, then the above inequality yields

$$\sum_{i\in I} \lambda_i f_i'(\bar{x};\eta(x,\bar{x})) + \sum_{j\in J(\bar{x})} \mu_j g_j'(\bar{x};\eta(x,\bar{x})) + \sum_{j\in J\setminus J(\bar{x})} \mu_j g_j'(\bar{x};\eta(x,\bar{x}))$$
$$= \lambda^T f'(\bar{x};\eta(x,\bar{x})) + \mu^T g'(\bar{x};\eta(x,\bar{x})) < 0$$

Which stands in contradiction to the condiction (1). Therefore $\bar{x}$ is an efficient solution for (MP).

Theorem3.5. Let $\bar{x}$ is a feasible solution for (MP). Suppose that (f,g) is $FJ-d-\alpha_\beta-\rho_\tau-\theta-$ quasipseudo-invex-I at $\bar{x}$ with respect to η . If there exist $\lambda \in R^k$, $\mu \in R^m$, with $\lambda \geqq 0, \mu \geq 0$, such that conditions (1)-(3) hold. Then $\bar{x}$ is an efficient solution for (MP).

Proof: We proceed by contradiction. Suppose that $\bar{x}$ is not an efficient solution for (MP). Then there exists $x \in D$, such that

$$f(x) - f(\bar{x}) \leq 0$$

Because (f, g) is $FJ - d - \alpha_{\beta} - \rho_{\tau} - \theta -$ quasipseudo-invex-I at $\bar{x}$ with respect to η, the above inequality follows

$$\alpha(x,\bar{x})\, f'(\bar{x};\eta(x,\bar{x})) + \rho \left\|\theta(x,\bar{x})\right\|^2 \leqq 0$$

$$\beta(x,\bar{x})\, g'_{J(\bar{x})}(\bar{x};\eta(x,\bar{x})) + \tau_{J(\bar{x})} \left\|\theta(x,\bar{x})\right\|^2 < 0$$

That is

$$\alpha(x,\bar{x})\, f_i'(\bar{x};\eta(x,\bar{x})) + \rho_i \left\|\theta(x,\bar{x})\right\|^2 \leqq 0,\ \forall i \in I$$

$$\beta(x,\bar{x})\, g'_j(\bar{x};\eta(x,\bar{x})) + \tau_j \left\|\theta(x,\bar{x})\right\|^2 < 0,\ \forall j \in J(\bar{x})$$

By $\lambda \geqq 0, \mu \geq 0$ (the condiction (2) implies $\mu_{J(\bar{x})} \geqq 0$), with $\alpha(x,\bar{x}) > 0, \beta(x,\bar{x}) > 0$ and (3), the above two inequalities follow that

$$\begin{aligned}
&\sum_{i \in I} \lambda_i f_i'(\bar{x};\eta(x,\bar{x})) + \sum_{j \in J(\bar{x})} \mu_j g'_j(\bar{x};\eta(x,\bar{x})) \\
&< -\left(\frac{\sum_{i \in I} \lambda_i \rho_i}{\alpha(x,\bar{x})} + \frac{\sum_{j \in J(\bar{x})} \mu_j \tau_j}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2 \\
&= -\left(\frac{\lambda^T \rho}{\alpha(x,\bar{x})} + \frac{\mu^T_{J(\bar{x})} \tau_{J(\bar{x})}}{\beta(x,\bar{x})} \right) \left\|\theta(x,\bar{x})\right\|^2 \leqq 0
\end{aligned}$$

This stands in contradiction to the condiction (1). Therefore $\bar{x}$ is an efficient solution for (MP).

Theorem3.6. Let $\bar{x}$ is a feasible solution for (MP). Suppose that (f, g) is $FJ - d - \alpha_{\beta} - \rho_{\tau} - \theta -$ quasipseudo-invex-II at $\bar{x}$ with respect to η. If there exist $\lambda \in R^k$, $\mu \in R^m$, with $\lambda \geqq 0, \mu_{J \setminus J(\bar{x})} = 0$, and $\mu_{J(\bar{x})} > 0$, such that conditions (1)-(3) hold. Then $\bar{x}$ is an efficient solution for (MP).

Proof: It is similar to the proof of theorem 3.4.

4. Discussion and Conclusion

In this paper, we study the multiobjective programming problems and introduce the new class of generalized invexity functions. Then the sufficient optimality conditions are obtained and proved under the new generalized invexity assumptions for the objective and constraint functions of the multiobjective programming. The results should be further opportunities for exploiting this structure of the multiobjective programming problems.

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