International Journal of Signal Processing, Image Processing and Pattern Recognition
Vol.9, No.9, (2016), pp.357-368
http://dx.doi.org/10.14257/ijsip.2016.9.9.33

Alternative Multiplicative Iterative Method for Projection Matrix Design in Compressive Images

Tian Shujuan[1,2,3], Fan Xiaoping[1], Xie Jingxiong[2,3], Li Zhetao[2,3], Choi Youngjune[4,2] and Sekiya Hiroo[5]

[1]*School of Information Science and Engineering, Central South University, Changsha, 410075, China*
[2]*The College of Information Engineering, Xiangtan University, Xiangtan, 411105, China*
[3]*Key Laboratory of Intelligent Computing & Information, Ministry of Education, Xiangtan University, Xiangtan, 411105, China*
[4]*Department of Information and Computer Engineering, Ajou University, Suwon, 443749, Korea*
[5]*Graduate School of Advanced Integration Science, Chiba University, Chiba, 263-8522, Japan*
sjtianwork@xtu.edu.cn

Abstract

Projection matrix plays an important role in compressive sensing (CS). Small mutual coherence between a projection matrix and a sparsifying matrix is considered to enhance reconstruction performance in CS. The equiangular tight frame (ETF) was demonstrated with minimum mutual coherence in previous works. However, ETF does not exist for any dimensions. A practical solution is to make the production of a projection matrix and a sparsifying matrix become the nearest one to ETF. Here, the optimization objective is regarded as the minimization of error between them. First, a maximum-likelihood estimation model is presented to crack the minimization problem. Then an alternative multiplicative iteration method is employed to guarantee that the error will converge to the minimum efficiently. Experimental results show that the proposed method obtains smaller mutual coherence with better reconstruction performance compared to existing methods.

Keywords: *compressed sensing; projection matrix; alternative multiplicative; average mutual coherence*

1. Introduction

Around 2006, compressed sensing (CS) was presented by David Donoho, Emmanuel Candès and Terence Tao [1-3]. It has been a new data acquisition approach for the sparse signal [4-7]. It argues that a small number of linear measurements of sparse signals contain enough information for reconstruction [8]. Generally, exact reconstruction is difficult, even impossible. However, projection matrix optimization is considered as a promising way to improve the possibility of reconstruction. The purpose of optimization is to decrease the mutual coherence between a projection matrix and a sparsifying matrix. The equiangular tight frame (ETF) [9] was demonstrated with minimum mutual coherence in previous works. However, ETF matrix does not exist for any dimensions. Hence, projection matrix optimization is transferred into the problem of making the production of a projection matrix and a sparsifying matrix, called sensing matrix, to be the nearest one to the ETF.

ISSN: 2005-4254 IJSIP

In order to decrease mutual coherence and find a matrix to be the nearest one to the ETF, an alternative multiplicative iterative (Alt Mul) method is proposed. This work is inspired by error estimation and gradient iteration. The contributions are as follows:

(1) We transform the matrix approximation problem into an optimization problem. By way of maximum-likelihood estimation model and probability density function, the objective of approximation between Gram matrix and ETF matrix is transformed to find the extreme minimum of likelihood function.

(2) We present an alternative multiplicative iteration method to simplify the process of finding the extreme minimum of likelihood function. It accelerates convergence and improves computational efficiency.

The remainder of the paper is organized as follows. Section 2, summarizes related work about projection matrix optimization. Section 3, introduces the background and problem description. Section 4, briefly describes the proposed method. In section 5, we make extensive comparisons of the proposed method with existing ones through simulation. Section 6, concludes our work.

2. Related Works

Many approaches have shown that the optimized random projection matrixes will improve the performance in CS [10-15]. In [10], Elad proposed an algorithm to iteratively decrease the average mutual coherence. Its goal was to minimize t-average coherence with respect to projection matrix, assuming that the dictionary matrix and the parameter t were known and fixed. They used a shrinkage operation followed by singular value decomposition (SVD) to gradually minimize it. The reconstruction performance has been significantly improved. However, this method has high computational complexity because it needs many iterative shrinkage steps [10]. Besides, this method may create some large mutual coherence values which ruin the guarantees of the reconstruction algorithms.

In [11], Xu *et. al.,* proposed a method to optimize the projection matrix from the perspective of ETF design. The objective was to find an equivalent dictionary whose Gram matrix was as close as possible to an ETF's. It didn't have exact solutions because of heavy complexity, so an alternative minimization method was used to find a feasible solution. This method overcame the time-consuming drawbacks of Elad's method, and avoided creating a function to shrink the large values. However, it didn't always guarantee the new Gram matrix to be the nearest one to the ETF with shrinkage operation.

In [12], Duarte *et. al.* proposed a method based on eigenvalue decomposition and K-SVD to make any subset columns in a sensing matrix as orthogonal as possible. In other words, it made the Gram matrix as equivalent as possible to the identity matrix. This noniterative method obtained low computational complexity, while the reconstruction results were not very good. The reason is that the sensing matrix was overcomplete and could not be considered as an orthogonal basis.

Based on Duarte's work, an improved method in [13] was formulated in terms of finding those projection matrices such that the Frobenius norm of the difference between the Gram matrix of the equivalent dictionary and the identity matrix is minimized. It derived an analytical solution in closed form for designing an optimal projection matrix. The solution had degrees of freedom. Furthermore, it also proposed to minimize coherence between the atoms of the equivalent dictionary among the solution set. And its experimental results outperformed the existing work in recovering a signal.

In [14], Schnass *et. al.,* presented a sensing dictionary to identify the atoms combining the measurement signal in orthogonal matching pursuit (OMP) and

threshold algorithms. A sensing matrix was constructed by using alternative projection between the set of Gram type matrix and the set of ideal Gram matrix. However, this method was sensitive to the choice of a projection matrix.

Recently, in [15], a low-rank model was proposed to make the Gram matrix near the ETF for the projection design. This method employed an optimization problem to guarantee both nearness and low-rank properties. It overcame the shortcoming of Elad's and Xu's methods by using SVD to force the rank to m, but it did not guarantee the new low-rank Gram matrix to be the nearest one to the ETF by shrinkage operation. Furthermore, the first-order algorithm proposed in [16] was deployed to obtain the nearest low-rank correlation matrix. It worked well in image denoising. However, it had high computational complexity.

It seems to be a dilemma that iteration methods result in heavy complexity while non-iteration methods cannot guarantee reconstruction performance. This motivates us to design the projection matrix with better performance and to make the optimization process more efficient simultaneously, especially for the 2D image signals.

3. Preliminaries and Problem Formulation

In this section, we introduce the background of compressed sensing and model the problem. The objective is to obtain an optimized projection matrix which has the minimum mutual coherence with a sparsifying matrix by making the Gram matrix to be the nearest one to the ETF.

3.1. Compressed Sensing Background

The model of CS can be described as a linear sampling operator by a projection matrix $\mathbf{\Phi}$ yielding a measurement vector $\mathbf{y}$:

$$\mathbf{y} = \mathbf{\Phi x}, \tag{1}$$

Where $\mathbf{y} \in \Re^{m\times 1}, \mathbf{\Phi} \in \Re^{m\times n}$ $(m \ll n)$, and $\mathbf{x}$ is a n-length original signal. Assume that the original signal $\mathbf{x}$ is sparse in a domain of $\mathbf{\Psi}$, that is

$$\mathbf{x} = \mathbf{\Psi\theta}, \tag{2}$$

where $\mathbf{\theta}$ is a K-sparse vector, $\mathbf{\Psi} \in \Re^{n\times n}$ is reversible sparsifying matrix. Taking equation (2) into equation (1), $\mathbf{y}$ can be rewritten as:

$$\mathbf{y} = \mathbf{\Phi\Psi\theta} = \mathbf{D\theta}, \tag{3}$$

where $\mathbf{D} \in \Re^{m\times n}$ is the product of $\mathbf{\Phi}$ and $\mathbf{\Psi}$. It is called sensing matrix.

Although sensing matrix is an underdetermined matrix, Donoho argued that it is possible to recover $\mathbf{\theta}$ exactly from the measurement $\mathbf{y}$ provided that sensing matrix possesses mutual incoherence property [1]. It means that the mutual coherence of sensing matrix plays a decisive role in reconstruction.

The mutual coherence of sensing matrix $\mathbf{D}$, usually denoted by $\mu\{\mathbf{D}\}$, is defined as the absolute value and normalized inner products between different columns in $\mathbf{D}$ or the off-diagonal entries of Gram matrix $\mathbf{G} = \tilde{\mathbf{D}}^{\mathrm{H}}\tilde{\mathbf{D}}$, where $\tilde{\mathbf{D}}$ is column-normalized of $\mathbf{D}$, $\tilde{\mathbf{D}}^{\mathrm{H}}$ is the hermitian transposition of $\tilde{\mathbf{D}}$.

Some quantitative indexes such as maximal mutual coherence in paper [17], average mutual coherence in paper [18] and t-average mutual coherence in paper [19], and β-power average coherence in paper [19] are present to measure sensing matrix's mutual coherence [20].

3.2. Problem Description

Generally, the sparsifying matrix is a fixed dictionary and known. Hence, a sensing matrix's mutual coherence depends mainly on projection matrix. Although random matrix is an ideal candidate for projection matrix, we have no option but to substitute random matrix with pseudo random matrix. This makes the mutual incoherence of sensing matrix higher than the minimum. A feasible method is to design the projection matrix with minimum mutual coherence property.

The equiangular tight frame (ETF) is demonstrated with minimum mutual coherence. Each pair of columns has the same coherence, that is $\sqrt{n-m \big/ m(n-1)}$. Unfortunately, ETF matrix only exists for rare combinations of m and n.

Therefore, the problem is to obtain a Gram matrix to be the nearest one to the ETF. In other words, it is to optimize Gram matrix $\mathbf{G}$ and make it as close as possible to an ETF $\mathbf{H}$. The approximation process is to minimize the difference between $\mathbf{G}$ and $\mathbf{H}$. Here, we regard the difference as an error matrix $\mathbf{E}$, that is:

$$\mathbf{E} = \mathbf{G} - \mathbf{H}$$

So, the objective of optimization process is to converge $\mathbf{E}$ to $\mathbf{0}$.

4. Alternative Multiplicative Iterative Method for Projection Design

In this section, we first present maximum likelihood estimation to describe the approximation error between Gram matrix $\mathbf{G}$ and ETF $\mathbf{H}$. Then, we show how to find the minimum value of likelihood function $L(\mathbf{\Phi}, \mathbf{H})$ with alternative multiplicative iterative method. Finally, we summarize the proposed method in pseudo-coding.

4.1. Model Description and Maximum-Likelihood Estimation

The optimization of projection matrix $\mathbf{\Phi}$ is to minimize the difference between Gram matrix $\mathbf{G}$ and ETF $\mathbf{H}$. We regard the difference as an error matrix $\mathbf{E}$. Hence, the error matrix between $\mathbf{G}$ and $\mathbf{H}$ describes as followed:

$$\mathbf{E} = \left\| \mathbf{G} - \mathbf{H} \right\|_F^p, \tag{4}$$

where $\mathbf{G} = \mathbf{\Psi}^T \mathbf{\Phi}^T \mathbf{\Phi} \mathbf{\Psi}$, $\mathbf{\Psi}$ is a fixed sparsifying matrix and known, $\left\| \square \right\|_F^p$ indicates the frobenius-norm, p is a natural number. Hence, the objective of optimization is to make $\mathbf{E}$ equal to $\mathbf{0}$.

To simplify the model, we take p equals to 2 as an example to describe the proposed method. Then, Equation (4) becomes:

$$\mathbf{E} = \left\| \mathbf{\Psi}^T \mathbf{\Phi}^T \mathbf{\Phi} \mathbf{\Psi} - \mathbf{H} \right\|_F^2, \tag{5}$$

or $\mathbf{E}_{i,j} = (\mathbf{G}_{i,j} - \mathbf{H}_{i,j})^2$,

where i, j denote the (i, j) element of a matrix, and $\mathbf{E}_{i,j}, \mathbf{G}_{i,j}$ and $\mathbf{H}_{i,j}$ denote the elements of matrix $\mathbf{E}, \mathbf{G}, \mathbf{H}$. Because $\mathbf{\Psi}$ is known, the error matrix is determined by $\mathbf{\Phi}$ and $\mathbf{H}$.

We present the maximum likelihood estimation model as follows to obtain $\mathbf{\Phi}$ and $\mathbf{H}$:

$$\{\mathbf{\Phi},\mathbf{H}\} = \arg\min_{\mathbf{\Phi},\mathbf{H}} \{-\log \rho(\mathbf{E}\,|\,\mathbf{\Phi},\mathbf{H})\}. \tag{6}$$

where $\rho(\mathbf{E}\,|\,\mathbf{\Phi},\mathbf{H})$ is the probability density function.

Generally, the noise is Gaussian distribution. We regard the element in error matrix as same independent Gaussian distribution $(0,\sigma^2)$. Therefore, the $\rho(\mathbf{E}_{i,j}\,|\,\mathbf{\Phi},\mathbf{H})$ can be described as below:

$$\rho(\mathbf{E}_{i,j}\,|\,\mathbf{\Phi},\mathbf{H}) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{\mathbf{E}_{i,j}^2}{2\sigma^2}\right), \tag{7}$$

$$\rho(\mathbf{E}\,|\,\mathbf{\Phi},\mathbf{H}) = \prod_{i,j}\rho(\mathbf{E}_{i,j}\,|\,\mathbf{\Phi},\mathbf{H}).$$

Note that in Equation (6), (7), we gain the likelihood function $L(\mathbf{\Phi},\mathbf{H})$:

$$L(\mathbf{\Phi},\mathbf{H}) = \frac{1}{2\sigma^2}\sum_{i,j}(\mathbf{G}_{i,j}-\mathbf{H}_{i,j})^2 + \sum_{i,j}\log\sqrt{2\pi}\sigma. \tag{8}$$

Eventually, the minimization of Equation (5) is transformed to find the extreme minimum value of Equation (8).

4.2. Alternative Multiplicative Iterative Method

In order to obtain the extreme value of $L(\mathbf{\Phi},\mathbf{H})$, we have to compute the partial derivative of $L(\mathbf{\Phi},\mathbf{H})$, $\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}}$ and $\frac{\partial L}{\partial \mathbf{H}_{l,k}}$. Once the partial derivative of $L(\mathbf{\Phi},\mathbf{H})$ is obtained, the extreme minimum value of Equation (8) is obtained. To accelerate convergence and improve efficiency, negative gradient iteration is present.

Before we compute $\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}}$, we must first obtain $\frac{\partial \mathbf{G}_{i,j}}{\partial \mathbf{\Phi}_{l,k}}$. The Equation (9) shows the details.

$$\begin{aligned}\frac{\partial \mathbf{G}_{i,j}}{\partial \mathbf{\Phi}_{l,k}} &= \frac{\partial\left(\sum_l\left(\sum_k \mathbf{\Phi}_{l,k}\mathbf{\Psi}_{k,i}\right)\cdot\left(\sum_k \mathbf{\Phi}_{l,k}\mathbf{\Psi}_{k,j}\right)\right)_{i,j}}{\partial \mathbf{\Phi}_{l,k}} \\ &= \mathbf{\Psi}_{k,i}\left(\sum_k \mathbf{\Phi}_{l,k}\mathbf{\Psi}_{k,j}\right) + \mathbf{\Psi}_{k,j}\left(\sum_k \mathbf{\Phi}_{l,k}\mathbf{\Psi}_{k,i}\right) \\ &= \mathbf{\Psi}_{k,i}(\mathbf{\Phi}\mathbf{\Psi})_{l,j} + \mathbf{\Psi}_{k,j}(\mathbf{\Phi}\mathbf{\Psi})_{l,i}\end{aligned}. \tag{9}$$

Then we derive $\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}}$ and $\frac{\partial L}{\partial \mathbf{H}_{l,k}}$ as below:

$$\begin{aligned}\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}} &= \sum_{i,j}\left(\mathbf{\Psi}_{k,i}\left(\mathbf{\Phi\Psi}\right)_{l,j}+\mathbf{\Psi}_{k,j}\left(\mathbf{\Phi\Psi}\right)_{l,i}\right)\left(\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}-\mathbf{H}_{i,j}\right)\\ &= \sum_{i,j}\mathbf{\Psi}_{k,i}\left(\mathbf{\Phi\Psi}\right)_{l,j}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}-\sum_{i,j}\mathbf{\Psi}_{k,i}\left(\mathbf{\Phi\Psi}\right)_{l,j}\mathbf{H}_{i,j}\\ &\quad+\sum_{i,j}\mathbf{\Psi}_{k,j}\left(\mathbf{\Phi\Psi}\right)_{l,i}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}-\sum_{i,j}\mathbf{\Psi}_{k,j}\left(\mathbf{\Phi\Psi}\right)_{l,i}\mathbf{H}_{i,j}\end{aligned}, \tag{10}$$

$$\frac{\partial L}{\partial \mathbf{H}_{l,k}} = \left\{\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{l,k}-\mathbf{H}_{l,k}\right\}. \tag{11}$$

To further simplify $\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}}$, we need to determine the four items in Equation (10). For the first item,

$$\begin{aligned}&\sum_{i,j}\mathbf{\Psi}_{k,i}\left(\mathbf{\Phi\Psi}\right)_{l,j}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}=\sum_{i,j}\left(\mathbf{\Phi\Psi}\right)_{l,j}\mathbf{\Psi}_{k,i}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}\\ &=\sum_{j}\left(\mathbf{\Phi\Psi}\right)_{l,j}\left(\sum_{i}\mathbf{\Psi}_{k,i}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}\right)=\sum_{j}\left(\mathbf{\Phi\Psi}\right)_{l,j}\left(\mathbf{\Psi\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{k,j}.\\ &=\left(\mathbf{\Phi\Psi\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi\Psi}^T\right)_{l,k}\end{aligned} \tag{12}$$

Similarly, we obtain the other three items in Equation (10):

$$\sum_{i,j}\mathbf{\Psi}_{k,i}\left(\mathbf{\Phi\Psi}\right)_{l,j}\mathbf{H}_{i,j}=\left(\mathbf{\Phi\Psi H}^T\mathbf{\Psi}^T\right)_{l,k}, \tag{13}$$

$$\sum_{i,j}\mathbf{\Psi}_{k,j}\left(\mathbf{\Phi\Psi}\right)_{l,i}\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}\right)_{i,j}=\left(\mathbf{\Phi\Psi\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi\Psi}^T\right)_{l,k}, \tag{14}$$

$$\sum_{i,j}\mathbf{\Psi}_{k,j}\left(\mathbf{\Phi\Psi}\right)_{l,i}\mathbf{H}_{i,j}=\left(\mathbf{\Phi\Psi H\Psi}^T\right)_{l,k}. \tag{15}$$

Taking Equations (12), (13), (14), and (15) into (10), (11), we get

$$\frac{\partial L}{\partial \mathbf{H}_{l,k}}=\left(\mathbf{\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi}-\mathbf{H}\right)_{l,k}, \tag{16}$$

$$\frac{\partial L}{\partial \mathbf{\Phi}_{l,k}}=(2\mathbf{\Phi\Psi\Psi}^T\mathbf{\Phi}^T\mathbf{\Phi\Psi\Psi}^T-\mathbf{\Phi\Psi H}^T\mathbf{\Psi}^T-\mathbf{\Phi\Psi H\Psi}^T)_{l,k}, \tag{17}$$

To accelerate convergence, negative gradient direction iteration was employed to obtain $\mathbf{H}$ and $\mathbf{\Phi}$ as follows:

$$(\mathbf{H}^{q+1})_{l,k}=(\mathbf{H}^q)_{l,k}+\lambda_1(\mathbf{\Psi}^T(\mathbf{\Phi}^q)^T(\mathbf{\Phi}^q)\mathbf{\Psi}-(\mathbf{H}^q))_{l,k}, \tag{18}$$

$$\begin{aligned}(\mathbf{\Phi}^{q+1})_{l,k}=(\mathbf{\Phi}^q)_{l,k}+\lambda_2(&2(\mathbf{\Phi}^q)\mathbf{\Psi\Psi}^T(\mathbf{\Phi}^q)^T(\mathbf{\Phi}^q)\mathbf{\Psi\Psi}^T-\\ &(\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)^T\mathbf{\Psi}^T-(\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)\mathbf{\Psi}^T)_{l,k}\end{aligned}, \tag{19}$$

Where λ_1 and λ_2 are constant, and q is the iteration index.

To improve efficiency, let $\lambda_2 = \frac{(\mathbf{\Phi}^q)_{l,k}}{\left((\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)^T\mathbf{\Psi}^T + (\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)\mathbf{\Psi}^T\right)_{l,k}}$, and then equation (19) can be written as below:

$$(\mathbf{\Phi}^{q+1})_{l,k} = 2(\mathbf{\Phi}^q)_{l,k}\frac{((\mathbf{\Phi}^q)\mathbf{\Psi}\mathbf{\Psi}^T(\mathbf{\Phi}^q)^T(\mathbf{\Phi}^q)\mathbf{\Psi}\mathbf{\Psi}^T)_{l,k}}{\left((\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)^T\mathbf{\Psi}^T + (\mathbf{\Phi}^q)\mathbf{\Psi}(\mathbf{H}^q)\mathbf{\Psi}^T\right)_{l,k}} \tag{20}$$

Therefore $\mathbf{\Phi}$ and $\mathbf{H}$ in Equation (6) will be obtained by formulations (18) and (20). Once optimized projection matrix $\mathbf{\Phi}$ is obtained, we obtain the minimization of Equation (5).

4.3. Summary of Proposed Method

The algorithm for optimizing $\mathbf{\Phi}$ with alternative multiplicative iterative method is described in Table 1.

Table 1. Alternative Multiplicative Iterative Algorithm for the Optimal Projection M.43 cmatrix $\mathbf{\Phi}$

Algorithm 1：Alternative multiplicative iterative algorithm
Input: Number of Measurements m, Length of signal k,Row of Basic Matrix n, Base Matrix $\mathbf{\Psi}$, Initial Projection Matrix $\mathbf{\Phi}$, Iteration In, Loop variable $q=0$ **Start Procedure:** 1.initialize $\mathbf{H}$ according to the definition in Ref [7] 2.update $\mathbf{H}$ by formulation (18) 3.update $\mathbf{\Phi}$ by formulation (20) 4. $q++$ 4.if $q > In$ then quit; else go to 1 **End Procedure** **Output:** the optimized projection matrix $\mathbf{\Phi}$

5. Simulation Results

In this section, we demonstrate the performance of the proposed method about projection matrix and its effect on the reconstruction process through experiments. The results show that the proposed method increases the performance in CS framework.

5.1. Mutual Coherence Histogram Comparison

In the first simulation, we use $k=400, n=200, m=30$, $\gamma=0.95$, $t=20\%$ to generate a random basis matrix $\mathbf{\Psi}_{200\times400}$ and projection matrix $\mathbf{\Phi}_{30\times200}$. Elad's algorithm [5]，Xu's algorithm [6], LZYCB's algorithm [8] and LRK algorithm [10] are simulated with the same $\mathbf{\Psi}$ and $\mathbf{\Phi}$. Figure 1, shows the distribution of absolute off-diagonal elements of Gram matrix $\mathbf{G}$. As seen from Figure 1, after applying the proposed method, the distribution of the absolute values makes the equivalent dictionary as close to an ETF as possible. This improvement will further reduce the numbers of sample for recovery and enhance the recovery performance.

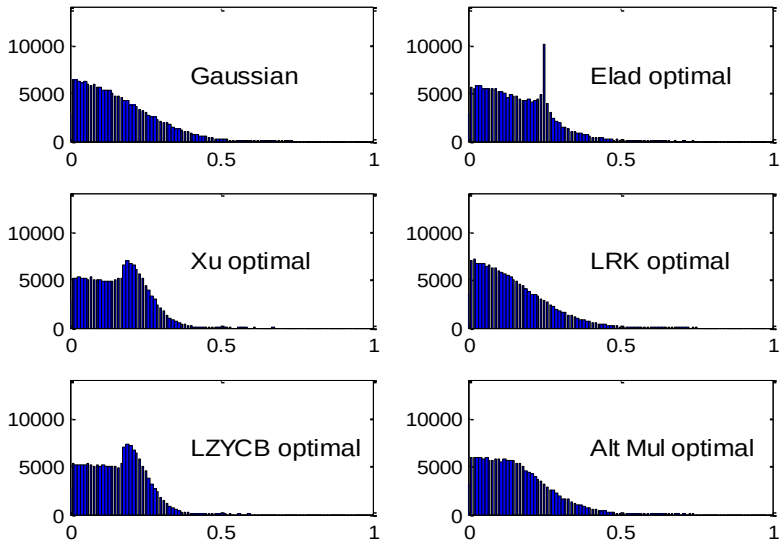

Figure 1. Distribution of the Off-Diagonal Entries of the Gram Matrix Obtained Using Random, Elad, Xu, LRK, LZYCB and Alternative Multiplicative Iterative Methods

5.2. Average Mutual Coherence Comparison

The second simulation illustrates the change of the average mutual coherence with variable measurement m. Figure 2, shows that the average mutual coherence decreases with the increase of measurements. The reduction of average mutual coherence implies that the performance of the optimal method does not depend heavily on the projection matrix. Figure 2, shows that the proposed method achieves the lowest average mutual coherence compared with other methods.

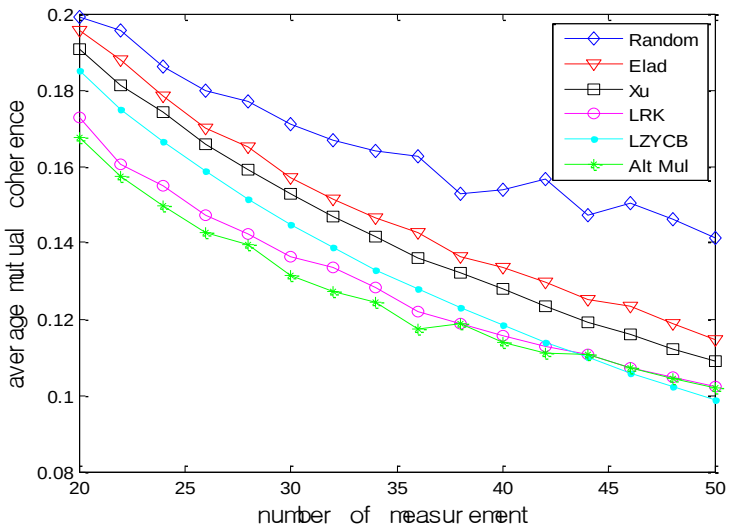

Figure 2. Average Mutual Coherence for $m \times n$ Matrices, with $m \in [20, 50]$ and $n = 80, k = 120$

5.3. Image Reconstruction Performance Comparison

Through 2D image experiments, we compared the performance of the proposed method with others. All these methods use OMP (known as a greedy iterative method) in the reconstruction process. We take three performance indexes into consideration: 1)

qualitative analysis is employed to compare the real original images with the recovered images; 2) PSNR (Peak Signal to Noise Ratio) as quantitative analysis is employed to compare the original and recovered images; 3) time-consumption is measured to compare the efficiency of different methods. The experiment steps are as follows:

1. For the given images $\mathbf{X}$, we generate sparse basis matrix $\mathbf{\Psi}$ and obtain sparse images $\hat{\mathbf{X}}$, here $\mathbf{X} = \mathbf{\Psi}\hat{\mathbf{X}}$.
2. We use the alternative multiplicative iterative method, LRK method, LZYCB's method, Xu's method and Elad's method to generate a projection matrix $\mathbf{\Phi}$ to measure the sparse images $\hat{\mathbf{X}}$ and obtain the measurement $\mathbf{Y}$, $\mathbf{Y} = \mathbf{\Phi}\hat{\mathbf{X}}$.
3. For each measurement $\mathbf{Y}$, we apply independently the OMP reconstruction method to reconstruct images. And we evaluate the time cost in step 2 and step 3.
4. We compute the PSNR between original images and reconstructed images.
5. Repeat steps 1 to 4 for a new measurement m.

We use wavelet basis (using bior1.1) as the sparse basis, and compare the performance of Lena、Ship、Fingerprint、hill and pepper images. The left column in Figure 3 is original images. The second column to sixth column are reconstructed images with OMP and measured by alternative multiplicative matrix, LRK matrix, LZYCB's matrix, Xu's matrix and the Elad's matrix, respectively.

(a) original (b) Alt Mul rec (c) LRK rec (d) LZYCB rec (e) Xu rec (f) Elad rec

Figure 3. The Original Images and Reconstructed Images with OMP

Figure 3, shows that each algorithm performs well. Furthermore, the proposed method outperforms other methods. For example, there are many thick lines in the third column and the forth column; however, these lines are greatly weakened in the second column.

The proposed method works adaptively with different images. In Figure 4, we use the proposed projection matrix to measure different images and reconstruct them with OMP algorithm. As shown in Figure 4, for the same image the PSNR increases with the increase of measurements. The proposed algorithm works well in these images except the Fingerprint. We find that it has nothing to do with the algorithm because there are more high frequency signals in Fingerprint. Detailed sequences discarded in the process of transformation and quantization result in poor performance. The third row in Figure 3, also shows similar results as Figure 4.

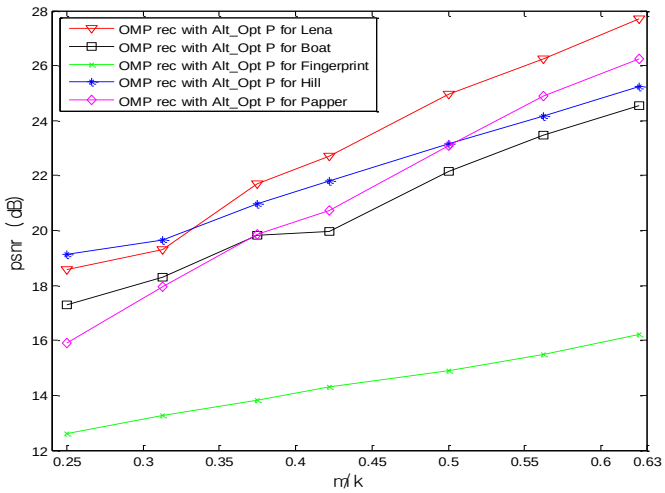

Figure 4. The PSNR of Different Signals with Proposed Method

Here, we choose Lena for further analysis. Now we show the results of Lena images with different sampling rates in Figure 5. Figure 5, shows that the PSNR increases with the sampling rate, and the proposed method achieves the highest PSNR compared with other methods. One can also see from Figure 5, that the proposed method is better than LRK, LZYCB's, Xu' and Elad' algorithms in PSNR by 0.55db, 1.18db, 1.45db and 2.81, respectively.

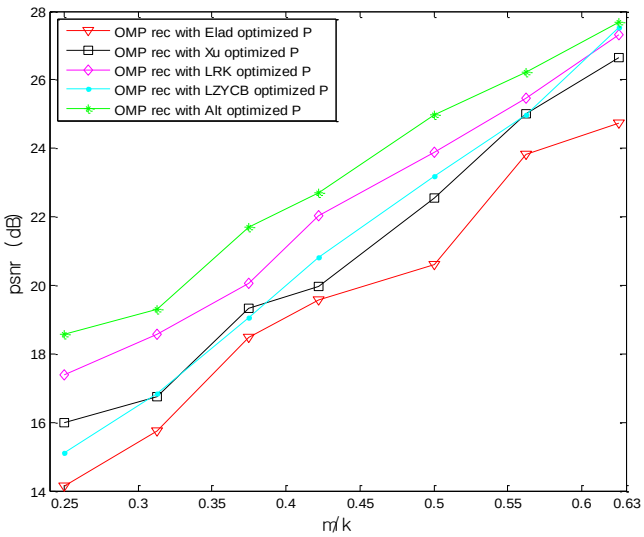

Figure 5. The PSNR of the Lena Image with Different Projection Matrix

We measure time consumption of different methods with the Lena at different sampling rates. The running time means the average execution time. We record the MATLAB2010b running time of this experiment by using a desktop computer with an Intel-core i3 CPU of 3.30GHz and 4GB RAM. It is seen from Figure 6, that more measurements increase the running time, as expected. However, less computation time is observed in the proposed method compared to Elad's and Xu's, LZYCB's, LRK's. The proposed algorithm achieves the shortest time because it derives a matrix closer to the optimality. Meanwhile, Elad's, Xu's, LZYCB's and LRK optimal algorithms spend more time on shrinking all elements for the projection matrix.

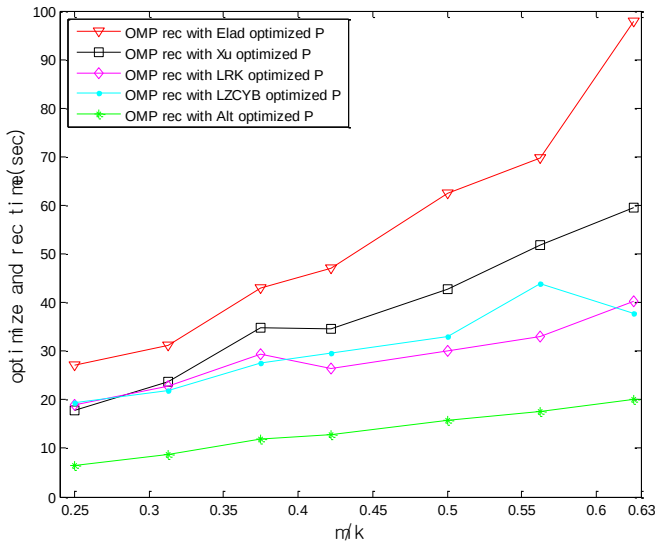

Figure 6. The Time Consumption of Different Sampling Rates

6. Conclusions

In this paper, we propose an alternative multiplicative iterative method for projection matrix design in compressed sensing. We employ maximum-likelihood estimation method to solve the problem of adjusting the Gram matrix to be the nearest one to the ETF. Results from experiments demonstrate that the proposed method makes significant improvements compared to Elad's, Xu's, LZYCB's and LRK methods for 2D image signal reconstruction with OMP. However, the existing design methods of sensing matrix are based on the mutual coherence theory. More better discriminant and design theories are to be further studied.

Acknowledgments

This work was supported in part by the National Natural Science Foundation of China under Grants No.61379115, No.61100215, No.61311140261, No.61372049 and No.61070180, Hunan Provincial National Natural Science Foundation of China with Grant No.2015JJ4047, No.12JJ9021 and No.13JJ8006. Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology under Grant No.2012R1A1A1017284 and the construct program of the key discipline in Hunan province.

References

[1] D. L. Donoho, "Compressed sensing", Information Theory, IEEE Transactions on, vol. 52, no. 4, **(2006)**, pp. 1289-1306.

[2] E. J. Candes and T. Tao, "Near-optimal signal recovery from random projections: Universal encoding strategies?", Information Theory, IEEE Transactions on, vol. 52, no. 12, **(2006)**, pp. 5406-5425.
[3] E. J. Candès, "Compressive sampling", In Proceedings on the International Congress of Mathematicians, Madrid, **(2006)** August 22-30.
[4] L. C. Jiao, S. Y. Yang, F. Liu and B. Hou, "Development and prospect of compressive sensing", Acta Electronica Sinica, vol. 39, no. 7, **(2011)**, pp. 1651-1662.
[5] E. J. Candès and M. B. Wakin, "An introduction to compressive sampling", Signal Processing Magazine, IEEE, vol. 25, no. 2, **(2008)**, pp. 21-30.
[6] N. Michelusi and M. Urbashi, "Adaptive distributed compressed sensing for dynamic high-dimensional hypothesis testing", Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on, Acoustics, **(2014)** May.
[7] J. Zniel and P. Schniter, "Dynamic Compressive Sensing of Time-Varying Signals Via Approximate Message Passing", Signal Processing, IEEE Transactions on, vol. 61, no. 21, **(2013)**, pp. 5270–5284.
[8] H. R. Yang, C. Zhang, D. W. Ding and S. Wei, "The theory of compressed sensing and reconstruction algorithm", Acta Electronica Sinica, vol. 39, no. 1, **(2011)**, pp. 142-148.
[9] M. A. Sustik, J. A. Tropp, I. S. Dhillon and R. W. Heath Jr, "On the existence of equiangular tight frames", Linear Algebra and its applications, vol. 426, no. 2, **(2007)**, pp. 619-635.
[10] M. Elad, "Optimized projections for compressed sensing", Signal Processing, IEEE Transactions on, vol. 55, no. 12, **(2007)**, pp. 5695-5702.
[11] J. Xu, Y. Pi and Z. Cao, "Optimized projection matrix for compressive sensing", EURASIP Journal on Advances in Signal Processing, vol. 2010, no. 43, **(2010)**.
[12] J. M. Duarte-Carvajalino and G. Sapiro, "Learning to sense sparse signals: Simultaneous sensing matrix and sparsifying dictionary optimization", Image Processing, IEEE Transactions on, vol. 18, no. 7, **(2009)**, pp. 1395-1408.
[13] G. Li, Z. Zhu, D. Yang, L. Chang and H. Bai, "On projection matrix optimization for compressive sensing systems", Signal Processing, IEEE Transactions on, vol. 61, no. 11, **(2013)**, pp. 2887-2898.
[14] K. Schnass and P. Vandergheynst, "Dictionary preconditioning for greedy algorithms", Signal Processing, IEEE Transactions on, vol. 56, no. 5, **(2008)**, pp. 1994-2002.
[15] Q. Zhang, Y. Fu, H. Li and R. Rong, "Optimized Projection Matrix for Compressed Sensing", Circuits, Systems, and Signal Processing, vol. 33, no. 5, **(2014)**, pp. 1627-1636.
[16] Z. Wen and W. Yin, "A feasible method for optimization with orthogonality constraints", Mathematical Programming, vol. 142, no. 1-2, **(2013)**, pp. 397-434.
[17] R. Gribonval and M. Nielsen, "Sparse representations in unions of bases", Information Theory, IEEE Transactions on, vol. 49, no. 12, **(2003)**, pp. 3320-3325.
[18] V. Abolghasemi, S. Ferdowsi and S. Sanei, "A gradient-based alternating minimization approach for optimization of the measurement matrix in compressive sensing",Signal Processing, IEEE Transactions on, vol. 92, no. 4, **(2012)**, pp. 999-1009.
[19] Z. Li, T. Pan and G. Zhu, "A construction algorithm of measurement matrix with low power average column coherence", Chinese Journal of Electronics, vol. 42, no. 7, **(2014)**, pp. 1360-1364.
[20] D. L. Donoho and M. Elad, "Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ1 minimization", Proceedings of the National Academy of Sciences, vol. 100, no. 5, **(2003)**, pp. 2197-2202.

