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Moduli Space of Time-like Pseudo Circles in $\mathbf{R}_1^3$

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Abstract

It is of great importance to classify all kinds of hypersurface in different space forms. In this paper, we focus on the hypersurfaces foliated by time-like pseudo circles. In order to complete the classification, we study the moduli space $\mathbf{Q}_2^3$ of time-like pseudo circles in $\mathbf{R}_1^3$. Firstly, We build the moduli space $\mathbf{Q}_2^3$ of time-like pseudo circles in $\mathbf{R}_1^3$ which is definitely a Riemannian manifold. Secondly, we build Riemannian metric, Riemannian connections in $\mathbf{Q}_2^3$ and prove that those are Möbius invariant. Thirdly, up to Möbius transformation, all the geodesics in $\mathbf{Q}_2^3$ are determined to form a one-parameter family of time-like pseudo circles on a generalized helicoid in space form $\mathbf{M}_1^3(1)$, $\mathbf{M}_1^3(-1)$, $\mathbf{M}_1^3(0)$, resp. Moreover, we show that mean curvature of those hypersurfaces are zero in three space forms respectively. Finally by software Mathematica and Jreality, we show some special hypersurfaces foliated by time-like pseudo circles.

1. Introduction

Let $\mathbf{R}_2^5$ be the 5-dimensional Lorentz space, equipped with the inner product

$$\langle X, Y\rangle = X_1Y_1 + X_2Y_2 + X_3Y_3 - X_4Y_4 - X_5Y_5,\ X, Y \in \mathbf{R}_2^5.$$

By $O(3,2)$ we denote the Lorentz group in $\mathbf{R}_2^5$ which preserves the light-cone

$$\mathbf{C}_2^4 = \{X \in \mathbf{R}_2^5 | \langle X, X\rangle = 0\}.$$

Let $\mathbf{S}_2^4$ be the de-sitter hypersphere $\mathbf{S}_2^4 = \{X \in \mathbf{R}_2^5 | \langle X, X\rangle = 1\}$. It is easy to check that there is a $1-1$ correspondence between $\mathbf{S}_2^4$ and the moduli space $\mathbf{M}$ of pseudo spheres

$$S_1^2(p, r) = \{X \in \mathbf{R}_1^3 | \langle X - p, X - p\rangle = 1\}$$

and Lorentz planes

$$L(\omega, \delta) = \{X \in \mathbf{R}_1^3 | \langle X, \omega\rangle = \delta, \langle \delta, \delta\rangle = 1\},$$

which is shown as:

$$\gamma: \mathbf{M} \to \mathbf{S}_2^4$$

$$S_1^2(p, r) \to \frac{1}{r}\left(\frac{1-\langle p,p\rangle + r^2}{2}, p, \frac{1+\langle p,p\rangle - r^2}{2}\right) \tag{1}$$

$$L(\omega, \delta) \to (-\delta, \omega, \delta)$$

Any points in $\mathbf{R}_1^3$ can also be injected into $\mathbf{R}_2^5$ by (2)

$$\gamma: \mathbf{R}_1^3 \to \mathbf{C}_2^4$$

$$x \to \left(\frac{1-\langle x,x\rangle}{2}, x, \frac{1+\langle x,x\rangle}{2}\right) \in \mathbf{R}_2^5 \tag{3}$$

It follows the fact that $x \in S_1^2(p, r)$ if and only if $\langle \gamma(x), X\rangle = 0$. And two pseudo spheres e_1, e_2 have an angle θ of intersection if and only if $e_1, e_2 \in \mathbf{S}_2^4$ satisfy $\langle e_1, e_2\rangle = \cos\theta$. In particular, two pseudo spheres intersect orthogonally if and only if $\langle e_1, e_2\rangle = 0$.

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Define time-like hyperbolic curves in $\mathbf{R}_1^3$ as time-like pseudo circles. Given any two pseudo spheres which intersect orthogonally, the intersection must be a time-like pseudo circle; while conversely, given any time-like pseudo circle, it has to be the orthogonal intersection of two pseudo spheres.

Let $\mathbf{Q}_2^3$ be the moduli space of time-like pseudo circles in Lorentz space $\mathbf{R}_1^3$. In this paper, we show that $\mathbf{Q}_2^3$ is a complex 3-manifold, equipped with a Mobius invariant Hermit metric h of type $(1,2)$. So the geodesics with respect to the Lorentz metric $g=\mathrm{Re}(h)$ on form a one-parameter family of time-like pseudo circles in $\mathbf{R}_1^3$, which is so-called generalized helicoid in a space form with zero mean curvature.

In this paper, our main theorems are:

Theorem 1.1. Geodesics on $(\mathbf{Q}_2^3,\ g)$ are Möbius equivalent to the following:

(1) The one-parameter family of parallel straight lines, time-like pseudo circles origin-centered in a certain plane in $\mathbf{R}_1^3$, or rotating time-like pseudo circles with same origin;

(2) The one-parameter family of time-like pseudo circles lying in a generalized helicoid of space form $\mathbf{M}_1^3(1),\ \mathbf{M}_1^3(-1), \mathbf{M}_1^3(0)$.

Theorem 1.2. Mean curvatures of three generalized helicoids are zero in corresponding space forms.

2. Moduli Space of Time-like Pseudo Circles in $\mathbf{R}_1^3$

Let $\mathbf{C}_2^5$ be the complex 5-space, equipped with inner product

$$\langle Z,W\rangle = Z_1W_1+Z_2W_2+Z_3W_3-Z_4W_4-Z_5W_5,\ Z,W\in\mathbf{C}_2^5,$$

and N_2^4 be a complex submanifold in $\mathbf{C}_2^5$ defined by

$$N_2^4 \triangleq \left\{Z\in\mathbf{C}_2^5 \middle| \langle Z,Z\rangle=0,\ \langle Z,\bar{Z}\rangle>0\right\}.$$

We define the horizontal subspace H_Z in $T_ZN_2^4$ by

$$H_Z \triangleq \left\{W\in\mathbf{C}_2^5 \middle| \langle W,Z\rangle=0,\ \langle W,\bar{Z}\rangle=0\right\}.$$

Then we have a complex orthogonal decomposition

$$\mathbf{C}_2^5=\mathbf{C}\bar{Z}\cup T_ZN_2^4=\mathbf{C}\bar{Z}\cup\mathbf{C}Z\cup H_Z.$$

Let $\mathbf{Q}_2^3$ be the complex 3-manifold defined by $\mathbf{Q}_2^3\triangleq\{[Z]|\langle Z,Z\rangle=0,\ \langle Z,\bar{Z}\rangle>0\}$, where $[Z]$ is the equivalent class of $[Z]\in N_2^4$ for the equivalent relation $Z{\sim}W$ if and only if $Z=kW,\ k\in\mathbf{C}\backslash\{0\}$.

Then we have complex line bundle $\pi: N_2^4\to\mathbf{Q}_2^3$ with $d\pi_Z^{-1}(0)=\mathbf{C}Z, d\pi_Z: H_Z\cong T_{[Z]}\mathbf{Q}_2^3$ being an isomorphism. The complex structure J for $\mathbf{Q}_2^3$ is determined by $d\pi\circ i=J\circ d\pi$.

For any $[Z]\in\mathbf{Q}_2^3$ we may assume that $Z=e_1+ie_2$ and $\langle Z,Z\rangle=2$. So we get

$$\langle e_1,e_1\rangle=\langle e_2,e_2\rangle=1\ ,\ \langle e_1,e_2\rangle=0$$

Thus $\{e_1,e_2\}$ are two orthogonal pseudo spheres in $\mathbf{R}_1^3$, and the intersection of them gives a time-like pseudo circle in $\mathbf{R}_1^3$. Conversely, let γ be and time-like pseudo circle in $\mathbf{R}_1^3$ as orthogonal intersections of two pseudo spheres $\{e_1,e_2\}$, then $Z=e_1+ie_2$ satisfies $\langle Z,Z\rangle=0,\ \langle Z,\bar{Z}\rangle=2$ and thus $[Z]\in\mathbf{Q}_2^3$. For another pair of pseudo spheres$\{\widetilde{e_1},\widetilde{e_2}\}$, also orthogonally intersecting into γ, there must have $\tilde{Z}=\widetilde{e_1}+i\widetilde{e_2}=e^{i\alpha}Z$, then $[Z]=[\tilde{Z}]$.

Specifically, without loss of generality, we set the Lorentz plane which lies on be $\{(t,0,s)|t,s\in\mathbf{R}\}$. Select three points randomly on γ:

$$x_1=(\cosh u,0,\sinh u),$$
$$x_2=(\cosh(u+1),0,\sinh(u+1))$$
$$x_3=(\cosh(u+2),0,\sinh(u+2)).$$

Due to (3), we have

$$\gamma(x_1)=(0,\cosh u,0,\sinh u,1),$$
$$\gamma(x_2)=(0,\cosh(u+1),0,\sinh(u+1),1),$$

$$\gamma(x_3) = (0, \cosh(u+2), 0, \sinh(u+2), 1).$$

Consider the linear space $\mathbf{V} = \mathrm{Span}\{\gamma(x_1),\ \gamma(x_2),\ \gamma(x_3)\}$, we claim that $\dim\mathbf{V} = 3$. Otherwise there exists $a, b, c \in \mathbf{R}$ but not all being zero satisfying $a\gamma(x_1) + b\gamma(x_2) - c\gamma(x_3) = 0$. Noticing $c = a + b$, we have

$$a\cosh(u+1) + b\cosh(u+1) = (a+b)\cosh(u+2),$$
$$a\sinh(u+1) + b\sinh(u+1) = (a+b)\sinh(u+2).$$

Therefore, $a^2 + b^2 + 2ab\cosh 1 = (a+b)^2 \xrightarrow{yields} \cosh 1 = 1$. Contradiction!

Noticing that $\{\gamma(x_1),\ \gamma(x_2),\ \gamma(x_3),\ X_1,\ X_2\}$ and $\{\gamma(x_1),\ \gamma(x_2),\ \gamma(x_3),\ X_1',\ X_2'\}$ are two basis of $\mathbf{R}_2^5$ in which $\{X_1, X_2\}$ and $\{X_1', X_2'\}$ are both orthogonal subsets, we get $Z' = X_1' + iX_2' = e^{i\alpha}Z \xrightarrow{yields} [Z] = [\tilde{Z}]$. It follows that the complex 3-manifold $\mathbf{Q}_2^3$ defined by (4) is exactly the moduli space of time-like pseudo circles in $\mathbf{R}_1^3$. The action of Möbius group on which is equivalent to the action of $O^+(3,2)$ on $\mathbf{Q}_2^3$, which is subgroup of $O(3,2)$.

A Hermit metric on $\mathbf{Q}_2^3$ can be defined globally by $h = h_Z$, shown as

$$h_Z = \frac{1}{\langle Z,\bar{Z}\rangle}\langle dZ - \frac{\langle dZ,\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Z, d\bar{Z} - \frac{\langle d\bar{Z},Z\rangle}{\langle Z,\bar{Z}\rangle}\bar{Z}\rangle,$$

which makes $d\pi_Z$ an isometric map. Its real part $g = \mathrm{Re}(h)$ is a Möbius invariant Lorentz metric of type $(2,4)$ with Levi-Civita connection reads

$$\nabla_X Y = d\pi_Z\left(X\big(Y(Z)\big) - \frac{\langle X(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Y(Z) - \frac{\langle Y(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}X(Z) + \frac{\langle X(Z),Y(Z)\rangle}{\langle Z,\bar{Z}\rangle}\bar{Z}\right) \quad (4)$$

Moreover, we claim that it is independent of the choice of the local section Z, which means it is globally defined. In fact, for any $X, Y \in T_Z\mathbf{Q}_2^3$, and any smooth function f, we have

$$\nabla_X Y - \nabla_Y X = d\pi\big([X,Y](Z)\big) = [X,Y],$$

$$\nabla_{fX} Y = d\pi\left(fX\big(Y(Z)\big) - \frac{\langle fX(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Y(Z) - \frac{\langle Y(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}fX(Z) + \frac{\langle fX(Z),Y(Z)\rangle}{\langle Z,\bar{Z}\rangle}\bar{Z}\right) = f\cdot\nabla_X Y,$$

$$\nabla_X(fY) = d\pi\left(X\big(fY(Z)\big) - \frac{\langle X(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}fY(Z) - \frac{\langle fY(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}X(Z) + \frac{\langle X(Z),fY(Z)\rangle}{\langle Z,\bar{Z}\rangle}\bar{Z}\right) = f\cdot\nabla_X Y + X(f)\cdot Y.$$

That makes ∇ a Riemannian connection. Secondly, we will show its compatibility with metric g. Define $X^* = X(Z) - \frac{\langle X(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Z \in H_Z$, then $d\pi(X^*) = d\pi\big(X(Z)\big) = X$, $h(X,Y) = h(X^*, \overline{Y^*})$. Thus,

$$\nabla^*{}_W X(Z) = W\big(X(Z)\big) - \frac{\langle W(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}X(Z) - \frac{\langle X(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}W(Z) + \frac{\langle W(Z),X(Z)\rangle}{\langle Z,\bar{Z}\rangle}\bar{Z}$$
$$- \frac{\langle W\big(X(Z)\big),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Z + \frac{2\langle W(Z),\bar{Z}\rangle\langle X(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}Z,$$

$$\xrightarrow{\text{yields}} \langle \nabla^*{}_W X(Z), Y^*(\bar{Z})\rangle = \langle W(X^*(Z)), Y^*(\bar{Z})\rangle - \frac{\langle W(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}\langle X^*(Z)), Y^*(\bar{Z})\rangle,$$

$$\langle \nabla^*{}_W Y(Z), X^*(\bar{Z})\rangle = \langle W(Y^*(Z)), X^*(\bar{Z})\rangle - \frac{\langle W(Z),\bar{Z}\rangle}{\langle Z,\bar{Z}\rangle}\langle Y^*(Z)), X^*(\bar{Z})\rangle.$$

By $\langle Y^*(\bar{Z}), Z\rangle = \langle Y^*(\bar{Z}), \bar{Z}\rangle = 0$, we get

$$W\big(g(X,Y)\big) = -\frac{\langle W(Z),\bar{Z}\rangle + \langle W(\bar{Z}),Z\rangle}{2\langle Z,\bar{Z}\rangle^2}(\langle X^*(Z)), Y^*(\bar{Z})\rangle + \langle Y^*(Z)), X^*(\bar{Z})\rangle)$$
$$+ \frac{1}{2\langle Z,\bar{Z}\rangle}\big(\langle W\big(X^*(Z)\big), Y^*(\bar{Z})\rangle + \langle X^*(Z), W(Y^*(\bar{Z}))\rangle$$
$$+ \langle W\big(Y^*(Z)\big), X^*(\bar{Z})\rangle + \langle Y^*(Z), W(X^*(\bar{Z}))\rangle\big)$$
$$= \frac{1}{2\langle Z,\bar{Z}\rangle}(\langle \nabla^*{}_W X(Z), Y^*(\bar{Z})\rangle + \langle \nabla^*{}_W X(\bar{Z}), Y^*(Z)\rangle + \langle \nabla^*{}_W Y(Z), X^*(\bar{Z})\rangle$$
$$+ \langle \nabla^*{}_W Y(\bar{Z}), X^*(Z)\rangle)$$

$$= g(\nabla_W X, Y) + g(\nabla_W Y, X).$$

Finally, we assume that Z^*be another section of π. There must exist some smooth function λ such that $Z^* = \lambda Z$ on the intersection of both Z and $\overline{Z^*}$. Therefore,

$$\nabla_X Y = d\pi_{Z^*}\left(X\big(Y(Z^*)\big) - \frac{\langle X(Z^*), \overline{Z^*}\rangle}{\langle Z^*, \overline{Z^*}\rangle} Y(Z^*) - \frac{\langle Y(Z^*), \overline{Z^*}\rangle}{\langle Z^*, \overline{Z^*}\rangle} X(Z^*) + \frac{\langle X(Z^*), Y(Z^*)\rangle}{\langle Z^*, \overline{Z^*}\rangle}\overline{Z^*}\right)$$

$$= d\pi_{\lambda\cdot Z}\Bigg(\lambda \cdot X\big(Y(Z)\big) + X(\lambda)Y(Z) + Y(\lambda)X(Z) + XY(\lambda)\cdot Z - \lambda \cdot \frac{\langle X(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} Y(Z)$$

$$- X(\lambda)Y(Z) - Y(\lambda)\cdot\left(\frac{\langle X(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} + \frac{X(\lambda)}{\lambda}\right) Z - \lambda \cdot \frac{\langle Y(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} X(Z)$$

$$- Y(\lambda)X(Z) - X(\lambda)\cdot\left(\frac{\langle Y(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} + \frac{Y(\lambda)}{\lambda}\right) Z + \lambda \cdot \frac{\langle X(Z), Y(Z)\rangle}{\langle Z, \bar{Z}\rangle}\bar{Z}\Bigg)$$

$$= d\pi_{\lambda\cdot Z}\left(\lambda \cdot X\big(Y(Z)\big) - \frac{\langle X(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} Y(Z) - \frac{\langle Y(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} X(Z) + \frac{\langle X(Z), Y(Z)\rangle}{\langle Z, \bar{Z}\rangle}\bar{Z}\right) + d\pi_{\lambda\cdot Z}(\lambda \cdot Z)$$

$$= d\pi_{\lambda\cdot Z}\left(\lambda \cdot X\big(Y(Z)\big) - \tfrac{\langle X(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} Y(Z) - \tfrac{\langle Y(Z), \bar{Z}\rangle}{\langle Z, \bar{Z}\rangle} X(Z) + \tfrac{\langle X(Z), Y(Z)\rangle}{\langle Z, \bar{Z}\rangle}\bar{Z}\right),$$

which indicates that this connection is globally defined. ■

3. Geodesics on Moduli Space $\mathbf{Q}_2^3$

In this section we determine all the geodesics on $(\mathbf{Q}_2^3,\ g)$.

Let $\gamma(t)$ be a geodesic on $(\mathbf{Q}_2^3,\ g)$, $Z: U \to N_2^4$ be a local section of $\pi: N_2^4 \to \mathbf{Q}_2^3$. Then $c(t) = Z \circ \gamma(t)$is a curve in N_2^4. For another local section $\tilde{Z}$, we have $c_*(t) = k(t)c(t)$ for some smooth function $k(t) \neq 0$. Since

$$\langle c_*'(t), \overline{c_*(t)}\rangle = \overline{k(t)}\big(k'(t)\langle c(t), \overline{c(t)}\rangle,\ k(t)\langle c'(t), \overline{c(t)}\rangle\big). \tag{5}$$

We may take an $k(t) \neq 0$ adapted such that $c'(t) \in H_{c(t)}$, or equivalently, $\langle c'(t), \overline{c(t)}\rangle = 0$. Therefore we call such $c(t)$ the horizontal lift of $\gamma(t)$, which from (5) is uniquely determined up to a constant $k \in \mathbf{C}\backslash\{0\}$.

For a horizontal lift, (5) yields

$$\nabla_{\gamma'(t)}\gamma'(t) = d\pi_{c(t)}\left(c''(t) + \tfrac{\langle c'(t), c'(t)\rangle}{\langle c(t), \overline{c(t)}\rangle}\overline{c(t)}\right) = 0.$$

Since $d\pi_Z^{-1}(0) = \mathbf{C}c(t)$, we can find $\mu(t)$ such that

$$c''(t) + \tfrac{\langle c'(t), c'(t)\rangle}{\langle c(t), \overline{c(t)}\rangle}\overline{c(t)} + \mu(t)c(t) = 0.$$

Multiplying this equation with $\overline{c(t)}$, we can solve $\mu(t)$ and get the following equation:

$$c''(t) + \tfrac{\langle c'(t), \overline{c'(t)}\rangle}{\langle c(t), \overline{c(t)}\rangle}c(t) + \tfrac{\langle c'(t), c'(t)\rangle}{\langle c(t), \overline{c(t)}\rangle}\overline{c(t)} = 0. \tag{6}$$

Using $\langle c'(t), \overline{c(t)}\rangle = 0$ and (6), it is easy to verify that

$$\begin{cases} \langle c'(t), \overline{c(t)}\rangle + \langle \overline{c'(t)}, c(t)\rangle = 0 \xrightarrow{\text{yields}} \langle c(t), \overline{c(t)}\rangle = \text{const}; \\ 2\langle c''(t), c'(t)\rangle = 0 \xrightarrow{\text{yields}} \langle c'(t), c'(t)\rangle = 0; \\ \langle \overline{c''(t)}, c'(t)\rangle + \langle c''(t), \overline{c'(t)}\rangle = 0 \xrightarrow{\text{yields}} \langle c'(t), \overline{c'(t)}\rangle = 0. \end{cases} \tag{7}$$

Taking another horizontal lift $kc(t)$ for some constant $k \in \mathbf{C}\backslash\{0\}$, we may assume $\langle c(t), \overline{c(t)}\rangle = \langle c(0), \overline{c(0)}\rangle = 2$. Such horizontal lift $c(t)$ is determined up to a change $c(t) \to e^{i\theta}c(t)$ for some constant $\theta \in \mathbf{R}$. We write $c(t) = u(t) + iv(t)$,.thus we get

$$\begin{cases} u''(t) + \frac{1}{2}\left(\langle c'(0), \overline{c'(0)}\rangle + \langle c'(0), c'(0)\rangle\right)u(t) = 0 \\ v''(t) + \frac{1}{2}\left(\langle c'(0), \overline{c'(0)}\rangle - \langle c'(0), c'(0)\rangle\right)u(t) = 0 \end{cases}, \quad (8)$$

with initial value

$$\begin{cases} \langle u(0), u(0)\rangle = \langle v(0), v(0)\rangle = 1 \\ \langle u(0), v(0)\rangle = 0 \end{cases}. \quad (9)$$

Noticing that $\{u(0), v(0)\}$ is orthogonal to $\{u'(0), v'(0)\}$, we have to consider the two cases that $u'(0)$ and $v'(0)$ are linearly dependent or independent.

3.1. $\{u'(0), v'(0)\}$ are Linearly Dependent

Select certain θ such that $v'(0) = ku'(0)$ be a real vector, or $v'(0) = 0$. With arc length parametric transformation $t \to at, t \in \mathbf{R}$ such that $K = \langle u'(0), v'(0)\rangle = 1, 0$ or -1, we get

(1) $$\begin{cases} u''(t) + Ku(t) = 0 \\ v''(t) = 0 \end{cases},$$

3.1.1. $K = 1$. We get the solution to (10), which is:

$$\begin{cases} u(t) = u(0)\cos t + u'(0)\sin t \\ v(t) = v(0) \end{cases}.$$

Take a Möbius transformation such that

$$u(0) = (0, -1,0,0,0),\ u'(0) = (0,0,1,0,0),\ v(0) = (1,0,0,0,0).$$

Then $u(t)$ represents one-parameter Lorentz planes $L\left((-\cos t, \sin t, 0), 0\right), t \in \mathbf{R}$ in $\mathbf{R}_1^3$, with $v(t)$ representing a pseudo sphere $S_1^2\left((0,0,0), 1\right)$ in $\mathbf{R}_1^3$. The intersection is one-parameter time-like pseudo circles in rotating Lorentz planes with same center and radius:

$$(\cosh s \sin t, \cosh s \cos t, \sinh s), s, t \in \mathbf{R}.$$

3.1.2. $K = 0$. We get the solution to (10), which is:

$$\begin{cases} u(t) = u(0) + u'(0)t \\ v(t) = v(0) \end{cases}$$

Take a Möbius transformation such that

$$u(0) = (0,0,1,0,0),\ u'(0) = (-1,0,0,0,1),\ v(0) = (0,1,0,0,0).$$

Then $u(t)$ represents one-parameter Lorentz planes $L\left((0,1,0), t\right), t \in \mathbf{R}$ in $\mathbf{R}_1^3$, with $v(t)$ representing Lorentz plane $L\left((1,0,0), 0\right)$ in $\mathbf{R}_1^3$. The intersection is one-parameter time-like parallel lines in Lorentz plane:

$$(0, t, s),\ s, t \in \mathbf{R}.$$

3.1.3. $K = -1$. We get the solution to (10), which is:

$$\begin{cases} u(t) = u(0)\cosh t + u'(0)\sinh t \\ v(t) = v(0) \end{cases}$$

Take a Möbius transformation such that

$$u(0) = (1,0,0,0,0),\ u'(0) = (0,0,0,0,-1),\ v(0) = (0,1,0,0,0).$$

Then $u(t)$ represents one-parameter pseudo spheres $S_1^2\left((0,0,0), e^t\right),\ t \in \mathbf{R}$ in $\mathbf{R}_1^3$, with $v(t)$ representing Lorentz plane $L\left((1,0,0), 0\right)$ in $\mathbf{R}_1^3$. The intersection is one-parameter time-like pseudo circles in Lorentz plane with the same center but different radius:

$$e^t(0, \cosh s, \sinh s), s, t \in \mathbf{R}.$$

3.2. $\{u'(0), v'(0)\}$ are Linearly Independent

Take a Möbius transformation $c(t) \to e^{i\theta}c(t)$ such that

$$u'(0) \to \frac{1}{\sqrt{2}}\left(u'(0) - v'(0)\right), v'(0) \to \frac{1}{\sqrt{2}}\left(u'(0) + v'(0)\right).$$

Applying (10) we get

(2) $$\begin{cases} u''(t) + \langle u'(0), u'(0)\rangle u(t) = 0 \\ v''(t) + \langle v'(0), v'(0)\rangle v(t) = 0 \end{cases},$$

Consider the solution to (11) in three cases:

3.2.1. Span $\{\boldsymbol{u}'(\boldsymbol{0}), \boldsymbol{v}'(\boldsymbol{0})\}$ is of type $(\boldsymbol{0}, \boldsymbol{2})$. With arc length parametric: transformation $t \to \alpha t, t \in \mathbf{R}$ such that $\langle v'(0), v'(0)\rangle = -1, \langle u'(0), u'(0)\rangle = -\alpha^2$ for some constant α ∈ **R**, we get the solution to (11),

$$\begin{cases} u(t) = u(0)\cosh \alpha t + u'(0)\sinh \alpha t \\ v(t) = v(0)\cosh t + v'(0)\sinh t \end{cases}.$$

By Mobius transformation, we get

$u(0) = (1,0,0,0,0),\ v(0) = (0,1,0,0,0),\ u'(0) = (0,0,0,0,-\alpha),\ v'(0) = (0,0,0,1,0)$.

Then $u(t)$ represents one-parameter pseudo spheres $S_1^2\big((0,0,0), e^{\alpha t}\big),\ t \in \mathbf{R}$ and $v(t)$ represents Lorentz spaces $L\big((\cosh t, 0, \sinh t), 0\big), t \in \mathbf{R}$ in $\mathbf{R}_1^3$. Therefore, the intersection will be surface in $\mathbf{R}_1^3$, which writes as

$$e^{\alpha t}(\sinh s \sinh t, \cosh s, \sinh s \cosh t), s, t \in \mathbf{R}.$$

Noticing $e^{\alpha t}\cosh s > 0$, consider metric $\bar{g} = \frac{dx^2+dy^2-dz^2}{y^2}$ in $\mathbf{R}_1^3$, we calculate the Christoffel symbols: $\bar{\Gamma}_{12}^1 = \bar{\Gamma}_{22}^2 = \bar{\Gamma}_{33}^2 = \bar{\Gamma}_{23}^3 = -\frac{1}{y}, \bar{\Gamma}_{11}^2 = \frac{1}{y}$, with others being zero. Then we get a 3-dimensional manifold $\mathbf{M}_1^3$ with constant section curvature -1. For fixed t_0, the curve

$$\gamma(s) \triangleq e^{\alpha t_0}(\sinh s \sinh t_0, \cosh s, \sinh s \cosh t_0),\ s \in \mathbf{R}$$

is a geodesic on surface above, which is a generalized helicoid in $\mathbf{M}_1^3(-1)$. With Mathematica and Jreality, we get pictures of that surface shown as below:

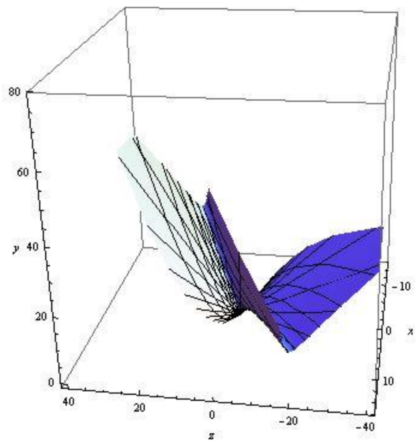

Figure 1. Surface Drawn by Mathematica

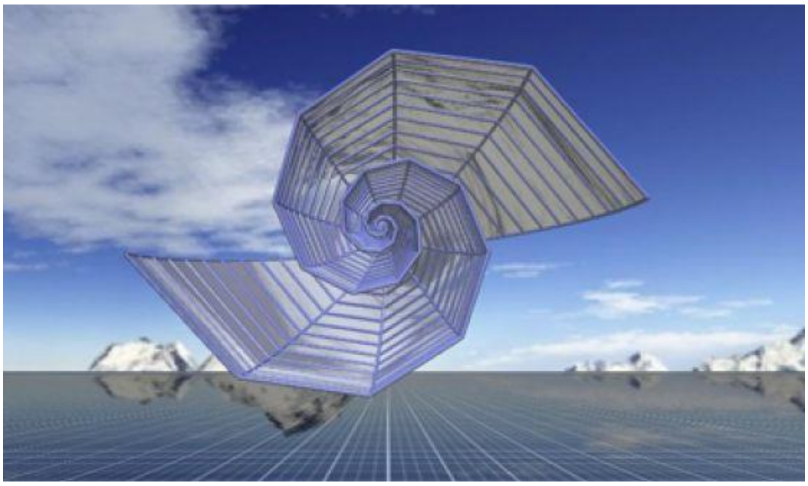

Figure 2. Surface Drawn by Jreality

3.2.2. Span $\{\boldsymbol{u}'(\mathbf{0}), \boldsymbol{v}'(\mathbf{0})\}$ is of type $(\mathbf{0}, \mathbf{1})$. With arc length parametric: transformation $\langle v'(0), v'(0)\rangle = 0, \langle u'(0), u'(0)\rangle = -\alpha^2$ for some constant $\alpha \in \mathbf{R}$, we get the solution to (11),

$$\begin{cases} u(t) = u(0)\cosh \alpha t + \frac{u'(0)}{\alpha}\sinh \alpha t \\ v(t) = v(0) + v'(0)t \end{cases}$$

By Mobius transformation, we get
$u(0) = (0,1,0,0,0)$, $v(0) = (0,0,1,0,0)$, $u'(0) = (0,0,0,\alpha,0)$, $v'(0) = (-1,0,0,0,1)$.

Then $u(t)$ represents Lorentz planes $L\big((\cosh \alpha t, 0, \sinh \alpha t), 0\big), t \in \mathbf{R}$ and $v(t)$ represents Lorentz spaces $L\big((0,1,0), 0\big)$ in $\mathbf{R}^3_1$. Therefore, the intersection will be generalized helicoid in $\mathbf{M}^3_1(0)$ due to the metric $\bar{g} = dx^2 + dy^2 - dz^2$ in $\mathbf{R}^3_1$,

$$(s \sinh \alpha t, t, s \cosh \alpha t), s, t \in \mathbf{R}.$$

With Mathematica and Jreality, we get pictures of that surface shown as below:

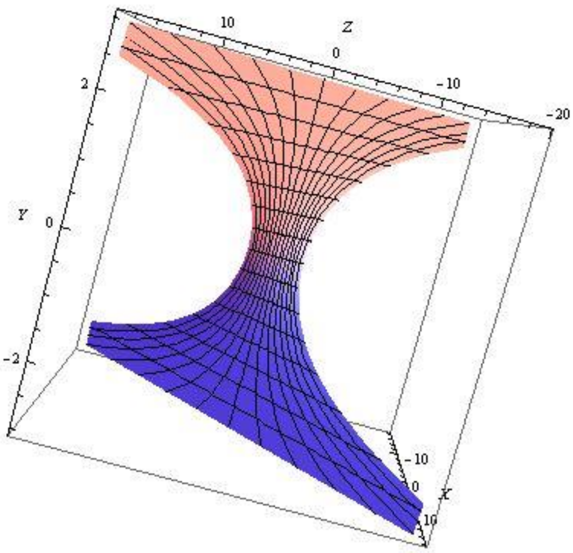

Figure 3. Surface Drawn by Mathematica

3.2.3. Span $\{\boldsymbol{u}'(\mathbf{0}), \boldsymbol{v}'(\mathbf{0})\}$ is of type $(\mathbf{1},\mathbf{1})$. With arc length parametric: transformation $t \to \alpha t, t \in \mathbf{R}$ such that $\langle v'(0), v'(0)\rangle = 1, \langle u'(0), u'(0)\rangle = -\alpha^2$ for some constant $\alpha \in \mathbf{R}$, we get the solution to (11),

$$\begin{cases} u(t) = u(0)\cos\alpha t + u'(0)\sin\alpha t \\ v(t) = v(0)\cosh t + v'(0)\sinh t \end{cases}$$

By Mobius transformation, we get

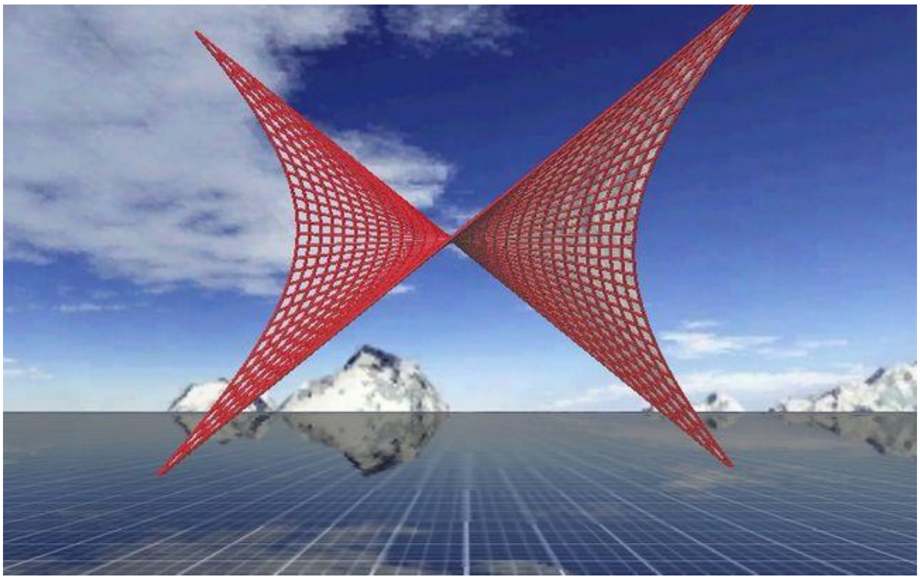

Figure 4. Surface Drawn by Jreality

$u(0) = (0,-1,0,0,0)$, $v(0) = (1,0,0,0,0)$, $u'(0) = (0,0,\alpha,0,0)$, $v'(0) = (0,0,0,0,-1)$.

Then $u(t)$ represents Lorentz spaces $L\big((-\cos\alpha t, \sin\alpha t, 0), 0\big), t \in \mathbf{R}$ and $v(t)$ represents pseudo spheres $S_1^2\big((0,0,0), e^t\big)$, $t \in \mathbf{R}$. Therefore, the intersection will be a generalized helicoid in $\mathbf{M}_1^3(1)$ due to the metric $\overline{g} = \frac{dx^2+dy^2-dz^2}{z^2}$ in $\mathbf{R}_1^3$,

$$e^t(\cosh s \sin\alpha t, \cosh s \cos\alpha t, \sinh s), \quad s, t \in \mathbf{R}.$$

With Mathematica and Jreality, we get pictures of that surface shown as below:

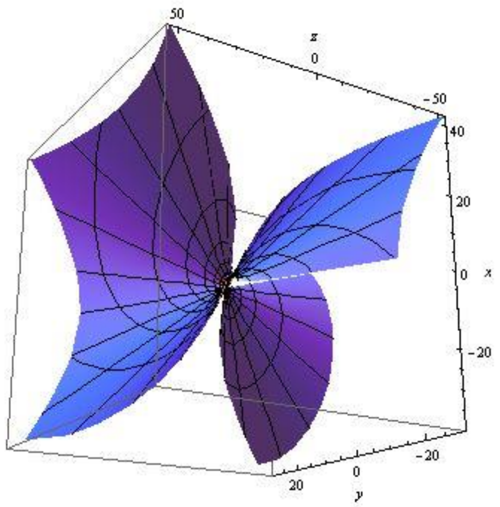

Figure 5. Surface Drawn by Mathematica

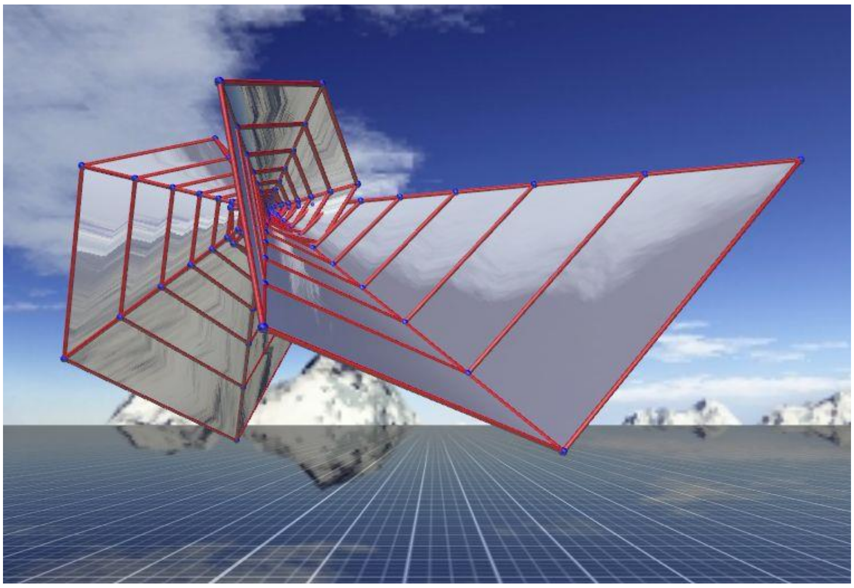

Figure 6. Surface Drawn by Jreality

This completes the proof of **Theorem 1.1.**

Proof of ***Theorem 1.2.*** Three cases will be discussed as following:

Firstly, Consider surface $\gamma(s,t) = e^{\alpha t}(\sinh s \sinh t, \cosh s, \sinh s \cosh t), s,t \in \mathbf{R}$, with corresponding space form $\mathbf{M}_1^3(-1)$ and metric $\overline{g} = \frac{dx^2+dy^2-dz^2}{y^2}$. Assume $\gamma: M \to \mathbf{R}_1^3$ with locally by $\gamma^i = x^i \circ \gamma$, which inducts Riemannian metric $g = \frac{\alpha^2+\sinh^2 s}{\cosh^2 s} dt^2 + \frac{-1}{\cosh^2 s} ds^2$. Write $\frac{\partial}{\partial t} = \frac{\partial}{\partial u_1}, \frac{\partial}{\partial s} = \frac{\partial}{\partial u_2}$, then the Christoffel symbols of g are

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{22}^1 = 0, \Gamma_{11}^2 = (1-\alpha^2)\tanh s, \Gamma_{12}^1 = \frac{1-\alpha^2}{\alpha^2+\sinh^2 s}\tanh s, \Gamma_{22}^2 = -\tanh s.$$

By Gauss Formula, we get

$$\overline{D}_{\frac{\partial}{\partial u_i}} \gamma_*\left(\frac{\partial}{\partial u_j}\right) = \gamma_*\left(D_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j}\right) + h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \sum_{k,c} \Gamma_{ij}^k \frac{\partial f^c}{\partial u_k}\frac{\partial}{\partial x_c} + h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

$$\xrightarrow{yields} h\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right) = \sum_c \left(\frac{\partial^2 \gamma^c}{\partial u^i \partial u^j} + \sum_{a,b} \frac{\partial \gamma^a}{\partial u^i}\frac{\partial \gamma^b}{\partial u^j}\overline{\Gamma}_{ab}^c - \sum_k \Gamma_{ij}^k \frac{\partial \gamma^c}{\partial u^k}\right)\frac{\partial}{\partial x^c}.$$

By direct calculation $c = 1,2,3$, we get

$$h\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = 0 \xrightarrow{\text{yields}} H = \frac{1}{2} g^{ij} h_{ij} = 0.$$

(2) Consider surface $\gamma(s,t) = (s\sinh \alpha t, t, s\cosh \alpha t), s,t \in \mathbf{R}$, with corresponding space form $\mathbf{M}_1^3(0)$ and its metric $\bar{g} = dx^2 + dy^2 - dz^2$, then we get

$$\gamma_s = (\sinh \alpha t, 0, \cosh \alpha t), \gamma_t = (\alpha s \cosh \alpha t, 1, \alpha s \sinh \alpha t)$$

$$\xrightarrow{yields} E = -1, F = 0, G = 1 + \alpha^2 s^2.$$

Moreover,

$$n = (-\cosh \alpha t, 1, -\sinh \alpha t), \gamma_{ss} = 0, \gamma_{tt} = (\alpha^2 s \sinh \alpha t, 0, \alpha^2 s \cosh \alpha t)$$

$$\xrightarrow{yields} L = N = H = 0.$$

(3) Consider surface $\gamma(s,t) = e^t(\cosh s \sin \alpha t, \cosh s \cos \alpha t, \sinh s)$, $s,t \in \mathbf{R}$, space form $\mathbf{M}_1^3(1)$ and its metric $\overline{g} = \frac{dx^2+dy^2-dz^2}{z^2}$. Assume the injection γ which inducts Riemannian metric $g = \frac{1+\alpha^2\cosh^2 s}{\sinh^2 s} dt^2 + \frac{-1}{\sinh^2 s} ds^2$. Write $\frac{\partial}{\partial t} = \frac{\partial}{\partial u_1}, \frac{\partial}{\partial s} = \frac{\partial}{\partial u_2}$, the Christoffel symbols of g can be written as

$\Gamma_{11}^{1}=\Gamma_{12}^{2}=\Gamma_{22}^{1}=0,\ \Gamma_{11}^{2}=-(1+\alpha^2)\coth s\,,\Gamma_{12}^{1}=-\frac{1+\alpha^2}{1+\alpha^2\cosh^2 s}\coth s\,,\Gamma_{22}^{2}=-\coth s.$

By Gauss formula,

$$\overline{D}_{\frac{\partial}{\partial u_i}}\gamma_*\left(\frac{\partial}{\partial u_j}\right)=\gamma_*\left(D_{\frac{\partial}{\partial u_i}}\frac{\partial}{\partial u_j}\right)+h\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\sum_{k,c}\Gamma_{ij}^{k}\frac{\partial f^c}{\partial u_k}\frac{\partial}{\partial x_c}+h\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)$$
$$\xrightarrow{yields} h\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_j}\right)=\sum_c\left(\frac{\partial^2\gamma^c}{\partial u^i\partial u^j}+\sum_{a,b}\frac{\partial\gamma^a}{\partial u^i}\frac{\partial\gamma^b}{\partial u^j}\overline{\Gamma}_{ab}^{c}-\sum_k\Gamma_{ij}^{k}\frac{\partial\gamma^c}{\partial u^k}\right)\frac{\partial}{\partial x^c}.$$

Then we have $h\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=h\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right)=0\xrightarrow{\text{yields}}H=\frac{1}{2}g^{ij}h_{ij}=0.$ ■

Problem 3.1. Determine all Willmore surfaces foliated by time-like pseudo circles in $\mathbf{R}_1^3$.

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