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A Interlaced Filling Algorithm in Deterministic Constructing Compressed Sensing Matrix

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Abstract

The sensing matrix has an important influence on the original signal sampling and reconstruction algorithm in the compressed sensing theory. A complete random sensing matrix has the drawbacks of large storage and high complexity in its implementation. In this paper, we propose an interlaced filling algorithm to construct the sensing matrix, which has a quasi-cyclic structure for efficient hardware implementation. The new sensing matrix has small coherence, which provides assurance for the recovery of sparse signal. Meanwhile, some experimental comparison with the other sensing matrix is accomplished. The simulation results demonstrate that the proposed sensing matrix not only obtains better performance but also owns easy hardware implementation.

Keywords: *Compressed Sensing; Sensing Matrices; Interlaced Filling; Coherence*

1. Introduction

Traditional approaches to sampling signal follow the celebrated Nyquist theorem: the sampling rate must be at least twice the maximum frequency present in the signal. Unfortunately, in many important and emerging applications, the sampling rate is so high that we deal with far too many samples that need to be transmitted, stored, and processed. Compressed sensing (CS) [1-3] is an innovative signal processing mode, and its theory shows that a finite-dimensional signal having a sparse or compressible representation can be recovered from a small set of linear measurements.

At present, the core problem of CS research mainly includes two aspects: signal measurement and reconstruction algorithm. Signal measurement is done by sensing matrix. An important and meaningful criteria called restricted isometry property (RIP) for was introduced by Candes and Tao [4]. A number of random matrices, such as Gaussian matrices and Fourier matrices satisfy RIP with preponderant probability. In spite of the theoretical advantages, a random matrix has the defects of high complexity and large storage to its implementation. Deterministic sensing matrices can avoid these drawbacks. For the deterministic sensing matrices constructions, binary 0-1 matrices have been brought to a remarkable attraction [5-9]. Recently, Dimakis [10] showed that parity-check matrices of "good" channel codes can be used as probably "good" measurement matrices under basis pursuit.

The main contribution of this paper is to use interlaced filling algorithm to construct the sensing matrix, which has a quasi-cyclic structure for efficient hardware implementation. The experimental results show that the performance of the proposed sensing matrix outperforms the Gaussian random matrix, a sparse random matrix, Toeplitz matrix and Bernoulli matrix. The paper is organized as below. In Section 2, the theory of compressed sensing is recalled. Section 3 shows the interlaced filling algorithm is used to construct the sensing matrix and analysis recovery conditions by coherence. In Section 4, experiments are carried out in order to simulate the performance of the proposed sensing matrix. Finally, the conclusion is given in Section 5.

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2. The Theory of Compressive Sensing

Compressed Sensing is a quite new framework that enables to get exact and approximate reconstruction of sparse or almost sparse signals from incomplete measurements. CS theory considers a *k*-sparse signal $x \in \mathbb{R}^N$ with *k* nonzero elements, and then the system can get the measurements $y \in \mathbb{R}^M$ from linear projection in the noiseless setting.

$$y = Hx \tag{1}$$

Where $H \in \mathbb{R}^{M \times N}$ is the sensing matrix with $M \ll N$. The solution to this system by solving the following l_0-minimization problem

$$\min \|x\|_0 \qquad s.t. \quad Hx = y \tag{2}$$

However, (2) is NP-hard problem in general, which is a non-convex optimization problem. There are two kinds of solution to recover the *k*-sparse signal *x*. The first method is to found convex relationship to (2), and *x* can recovered via l_1-minimization

$$\min \|x\|_1 \qquad s.t. \quad Hx = y \tag{3}$$

This solution to recover the k-sparse signal x can be completed by the basis pursuit (BP) algorithm [11]. The second method is greedy algorithms for l_0-minimization, such as orthogonal matching pursuit (OMP) [12], which can exactly recover *x*.

The construction of the sensing matrix is part of the main concerns in CS. In order to select the appropriate matrix, some criteria have been proposed. An insightful and useful criteria RIP was proposed [4-13]. In addition, Xu [14] proposed a sufficient and necessary condition of exactly recovering, named the null space property (NSP). Although NSP and RIP all provide guarantees for the recovery of *k*-sparse signal, but they are very hard computable. In many cases it is preferable to use properties of *H* that are easily computable to provide more concrete recovery conditions. The coherence of a matrix is one such property.

The coherence of a matrix *H*, denoted $\mu(H)$, is the largest absolute inner product between any two columns h_i, h_j of *H* and is then defined as:

$$\mu(H) \triangleq \max_{1 \le i \ne j \le N} \frac{\left|\langle h_i, h_j \rangle\right|}{\|h_i\|_2 \|h_j\|_2} \tag{4}$$

Where $\langle h_i, h_j \rangle \triangleq h_i^T h_j$ denotes inner product of vectors. The following proposition bounds the value of the coherence for an arbitrary matrix. Let *H* be a matrix of size $M \times N$ with $M < N$, whose columns are normalized so that $\|h_i\| = 1$ for all *i*. Then the mutual coherence of A satisfies

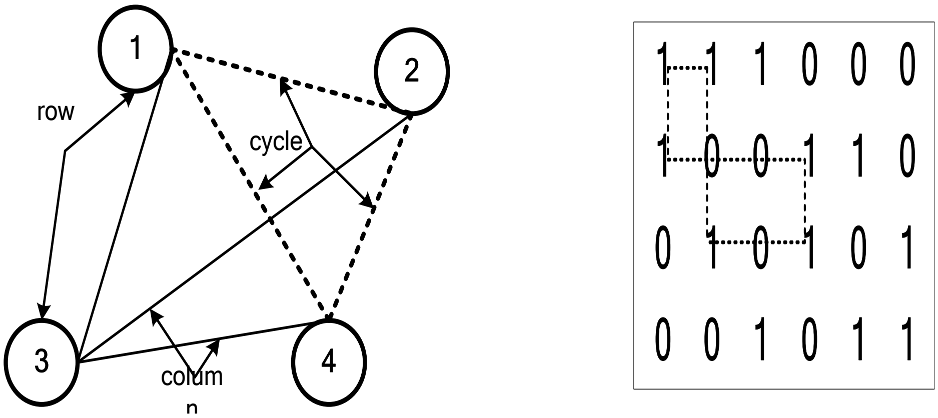
$$\sqrt{\frac{N-M}{M(N-1)}} \le \mu(H) \le 1 \tag{5}$$

The lower bound in (5) is known as the Welch bound [15]. If $N \gg M$, the lower bound is approximately $\mu(H) \ge \sqrt{1/M}$.

3. The Construction of Sensing Matrix via Interlaced Filling Algorithm

3.1. A Representation of Matrix by a Distance Graph

The matrix can be represented with distance graph [16], where vertices are rows and edges represent columns. If the two vertices, h_x and h_y, are connected in the graph, then $H_{xr} = H_{yr} = 1$ for a column r. That is, the "1" entries in the matrix indicates that the two vertices are connected in the graph. The number of rows is equal to the number of vertices in the graph whereas the number of columns is equal to the number of edges. Figure 1 (a), shows a distance graph of four vertices and a minimum cycle length of three, a corresponding matrix is showed as in part (b) of the Figure1. In the graph, each vertex is a row of a matrix, and each edge represents a column of the matrix. In general the size of the matrix is given by $M \times Mk/2$, where M is the number of vertices; k is the degree of each vertex (row weight) and $Mk/2$ is the number of edges.



(a) A Distance Graph (b) The Corresponding Matrix

Figure 1. A Matrix Derived from a Distance Graph

The cycle in a distance graph is composed of a path of edges or vertices starting from a vertex h_x and ending at h_x. The length of girth, g, is the smallest cycle in the graph. A cycle of length of g in the graph corresponds to a cycle of length $2g$ in the matrix. Therefore, the graph cycle represents half of the matrix cycle. A cycle of three in the distance graph is shown Figure 1 (a) in dotted lines between vertices 1, 2 and 4. It forms a cycle of six between rows 1, 2, 3 and columns 2, 4 in matrix form as shown part (b) of the Figure 1.

3.2. The Construction of Sensing Matrix

Many random matrices, such as Gaussian matrices and Fourier matrices have been verified to satisfy RIP with overwhelming probability. However, there is no guarantee that the random matrix of the specific implementation methods. Moreover, storing a random matrix may require lots of storage space. The deterministic construction of sensing matrices is necessary. There are many works on deterministic constructions. This paper presents an interlaced filling method to construct the sensing matrix, which has a quasi-cyclic structure to efficient hardware implementation. Vertexes or rows are divided into the same size group to obtain a block structure in the form of sub-matrices. In order to obtain the cyclic structure in sub-matrix, Vertexes in the group are connected according to their position. The basic idea of the algorithm is as follows:

1. We assume that the number of vertices of the distance graph is *M*, the row weight of the matrix is *k*, and the column weight is *j*. Divide rows into equal groups $\left(G_1, G_2, \cdots, G_j\right)$, each group contains *p* rows, where $p = M/j$. r_x is the row x. $\bigcup_{r_x}$ is a set of rows within a distance of g from r_x.

2. Divide $\left(G_1, G_2, \cdots, G_{j-1}, G_j\right)$ into sub-groups with j row-groups with each row-group appearing k times. The number of sub-groups is k, $\left(GP_1, GP_2, \cdots, GP_{k-1}, GP_k\right)$.

3. For $t = 1$ to k

{ (1) Select $GP_1 \in GP_t$ as a reference group

(2) Select $r_1 \in GP_1$ as a reference row

(3) Sequentially search $j-1$ row $\left(r_2 \cdots r_j\right)$,where $\left(r_2 \cdots r_j\right) \in GP_t$ (Such conditions must be met: the distance between each row and the reference row r_1 is at least *g*, and the distance between each other is at least g). If such a row can be found, then r_1 is connected to r_x (the corresponding position in the matrix is filled by "1", and execute (4), else the algorithm fails.

(4) For $z = 1$ to $p-1$

{ if r_1 is connected to r_x , r_{1+z} is connected to r_{x+z} .

else the algorithm fails.}

}.

4. Use obtained distance graph to form deterministic sensing matrix.

Figure 2, shows row connections for the matrix *H* (20, 2, 4) with girth of eight, which uses our proposed algorithm. The total numbers of rows of the matrix *H* is 10, and the number row-groups, *j*, is 2. The size of each group, $p = M/j$, is 5. Group 1 has rows 1 to 5 $\left(r_1, r_2, r_3, r_4, r_5\right)$ row, and group 2 rows 5 to 8 $\left(r_6, r_7, r_8, r_9, r_{10}\right)$ row. The number of sub-groups, *k*, is 4. The 4 sub-groups are [1 2], [1 2], [1 2] and [1 2] with each group appearing four times. An interlaced filling method for a row satisfying the distance is used in this case. Group 1 and row 1 are always be chosen as the reference group and row. During the first filling process, row 6 is found to satisfy the distance of four (desired girth) from row 1. Therefore, the 1 rows and 6 rows of the 1 column of the matrix are filled with "1" entries. The rest of group 1 rows, rows 2 to 5, are then filled to rows 7 to 10 with "1" entries. In the second search, row 7 is the first to satisfy the distance. It is filled to row 1 with the rest of group 1 filled to the rest of group 2. This process is repeated in the third and fourth filled processes as shown in Figure 2. We can see from the distance graph, the number of vertices is 10, a vertex degree equal to four and a girth of four. Figure 3 is the corresponding matrix representation. From the matrix can be seen, the top five rows contain four identity sub-matrices as the first group is not searched or shifted. The bottom 4×4 cyclic shifted sub-matrices represent group 2 searching.

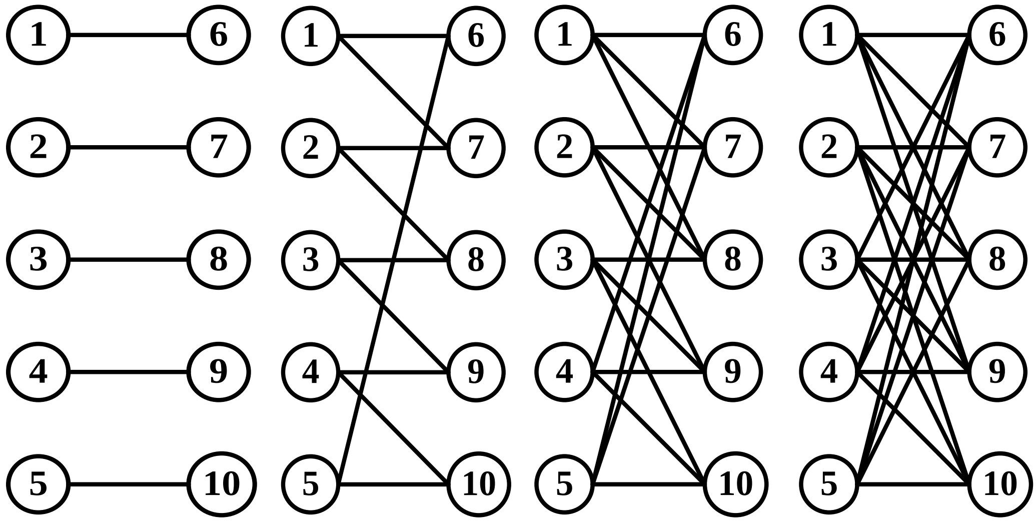


Figure 2. The Distance Graph Representation of (20, 2, 4) Code with Girth-8

1					1					1					1				
	1					1					1					1			
		1					1					1					1		
			1					1					1					1	
				1					1					1					1
1									1				1				1		
	1				1									1				1	
		1				1				1									1
			1				1				1				1				
				1				1				1				1			

Figure 3. The Matrix Representation of (20, 2, 4) Code with Girth-8

The proposed sensing matrices H can be represented by circulant permutation matrices I as follows:

$$H = \begin{bmatrix} I^{p_{00}} & I^{p_{01}} & \cdots & I^{p_{0(w_c-1)}} \\ I^{p_{10}} & I^{p_{11}} & \cdots & I^{p_{1(w_c-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ I^{p_{(w_v-1)0}} & I^{p_{(w_v-1)1}} & \cdots & I^{p_{(w_v-1)(w_c-1)}} \end{bmatrix} \quad (6)$$

Where $p_{ij} \in \{0,1,\dots,s-1\}$, s executes the right cyclic shift operation for times, $0 \le i \le w_v$, $0 \le i \le w_c$, $I^{p_{ij}}$ represents identity matrix cyclically shifted the columns to the right p_{ij} positions. The index matrix *P* is defined by

$$P = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0(w_c-1)} \\ p_{10} & p_{11} & \cdots & p_{1(w_c-1)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{(w_v-1)0} & p_{(w_v-1)1} & \cdots & p_{(w_v-1)(w_c-1)} \end{bmatrix} \quad (7)$$

The design of index matrix *P* is widely investigated, such as array codes. In order to avoid 4-cycle, (6) is expressed as (8). Any rows have overlapping "1" in more than one position.

$$H = \begin{bmatrix} I & I & \cdots & I \\ I & I^{p_{11}} & \cdots & I^{p_{1(w_c-1)}} \\ \cdots & \cdots & \cdots & \cdots \\ I & I^{p_{(w_v-1)1}} & \cdots & I^{p_{(w_v-1)(w_c-1)}} \end{bmatrix} \quad (8)$$

A deterministic construction of compressed sensing matrix is constructed with the interlaced filling algorithm. The algorithm divides rows into equal group sizes to obtain the sub-matrix or block structure, which has a quasi-cyclic structure to efficient hardware implementation. In order to get the structure of the circular matrix, the rows are divided into equal groups. Rows are filled in their numerical order to obtain a cyclic structure in a group. Two rows that are composed of columns must be satisfied with the distance between them to get the desired cycle. The quasi-cyclic structure of the sensing matrix is easier to implement in hardware.

3.2. The Coherence and Spark of Constructed Sensing Matrix

To measure the performance of sensing matrices, RIP and NSP are the widely used criterions. But generally, there is no effective way to verify whether a sensing matrix satisfies RIP and NSP or not. As discussed in [17], any k-sparse signal can be exactly recovered from the measurement via BP algorithm or OMP, provided

$$k < \frac{1}{2}\left[1 + \frac{1}{\mu(H)}\right] \quad (9)$$

Coherence is an important criterion to guarantee exact signal recovery, and one would design such that is minimized. For an $M \times N$ ($M < N$) sensing matrix H built from proposed algorithm, we define

$$\mu(H) = \frac{1}{w_v} \quad (10)$$

Where w_v is the uniform column weight of the sensing matrix *H*.

Suppose *H* has *N* columns $h_1, h_2, \dots h_N$, then $\|h_i\|_2 = \sqrt{w_v}$ for $1 \le i \le N$. Since the proposed sensing matrix is free of cycles of length 4,so it is easy to see that any two distinct columns of H has only one same '1' in all lows. When the maximum inner product of any two columns is 1, we have

$$\mu(H) = \frac{1}{w_v} \tag{11}$$

According to (10), the *spark* of the sensing matrix H, where *spark* (H) is defined to be the smallest number of columns of H that are linearly dependent [17], can be calculated.

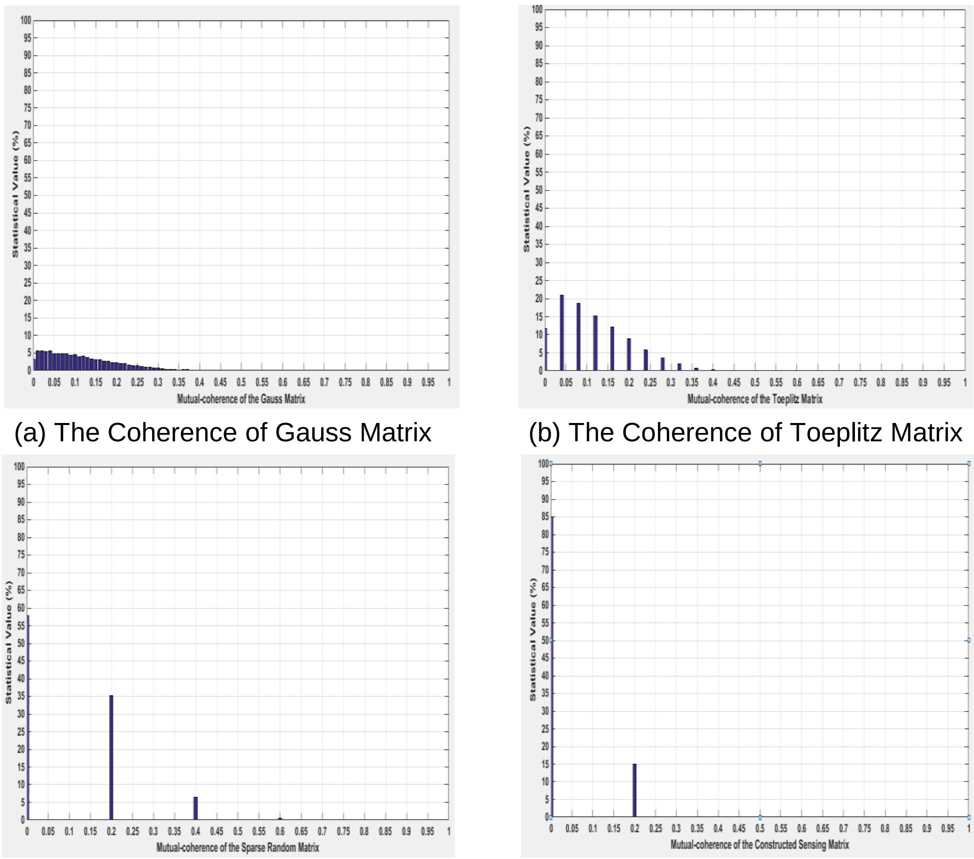
$$spark\left(H\right) \geq 1 + \frac{1}{\mu\left(H\right)} = 1 + w_v \tag{12}$$

According to (9) and (10), we can have

$$k < \frac{1}{2}\left(1 + w_v\right) \tag{13}$$

If k-sparse signal x is satisfied (13), the signal x can be solved the l_0-minimization problem.

To show the intuition for mutual coherence of the proposed sensing matrix, Five types of matrices are generated as contrast: Gauss matrix, Toeplitz matrix and sparse random matrix with size $m \times n$, where $m = 150$, and $n = 300$. Assumed the column weight $w_v = 5$ for the proposed sensing matrix and sparse random matrix. The mutual coherence of these matrices are plotted in Figure 4. It can be seen that the mutual coherence of the proposed sensing matrix is superior to other matrices.



(a) The Coherence of Gauss Matrix
(b) The Coherence of Toeplitz Matrix
(c) The Coherence of Sparse Matrix
(d) The Coherence of Proposed Matrix

Figure 4. The Coherence of Various Sensing Matrices

4. The Results and Analysis of Experiments

In this section, some simulation experiments are carried out to validate the performance of the proposed sensing matrix. In order to compare result, Gauss matrix and sparse random matrix also are designed as the sensing matrix. The experimental procedure is organized as follows. We give a *k*-sparse signal *x* with length-N, and *k* is randomly generated. Then, Gauss matrix, sparse random matrix, Toeplitz matrix and the proposed matrix are chosen as the sensing matrix to measure the signal *x* separately. Among them, the Gauss matrix is drawn with each entry $H_{i,j} \sim N(0,1)$ is constructed and the columns of the matrix are normalized to unit magnitude. Sparse random matrix is constructed with a random way and its column weight is 5 and 6 respectively. We adopt interlaced filling algorithm to construct the sensing matrix. In the process of experiment, the size of the proposed sensing matrix is 180×360 ($M = 180, N = 360$) and 240×480 ($M = 240,\ N = 480$), and its column weight is 5 and 6 respectively. The girth length of the proposed sensing matrix is 6. According to (1), the measurements *y* can be calculated. The recovery signal of *x* from *y* is done by the OMP algorithm.

The experiment was repeated 1000 times in each sparse level, and the probability of accurate recovery was calculated. In order to calculate the exact recovery of the frequency, the threshold $\|x' - x\|_2 < 10^{-6}$ was used in the experimental process. The nonzero value of k-sparse signals is get according to a standard Gaussian distribution.

As observed from Figure 5, and Figure 6, our proposed sensing matrix with interlaced filling algorithm can improve significantly the exact recovery probabilities in both data models compared to Gauss matrix, sparse random matrix and Toeplitz matrix. When the sparsity is 50, the reconfiguration failure starts to emerge for the proposed sensing matrix from the reconstructed curve of Figure 6. However, for the same size of sparse random matrix and Toeplitz matrix, when the sparsity is 30, the reconfiguration failure starts to emerge.

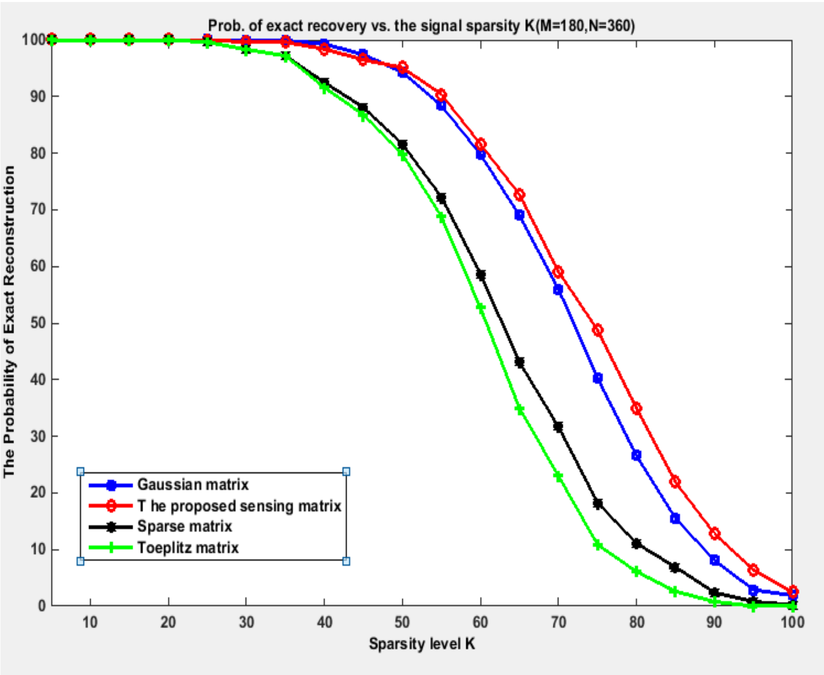


Figure 5. The Reconstitution Performance of Various Sensing Matrices (Column Weight is 5)

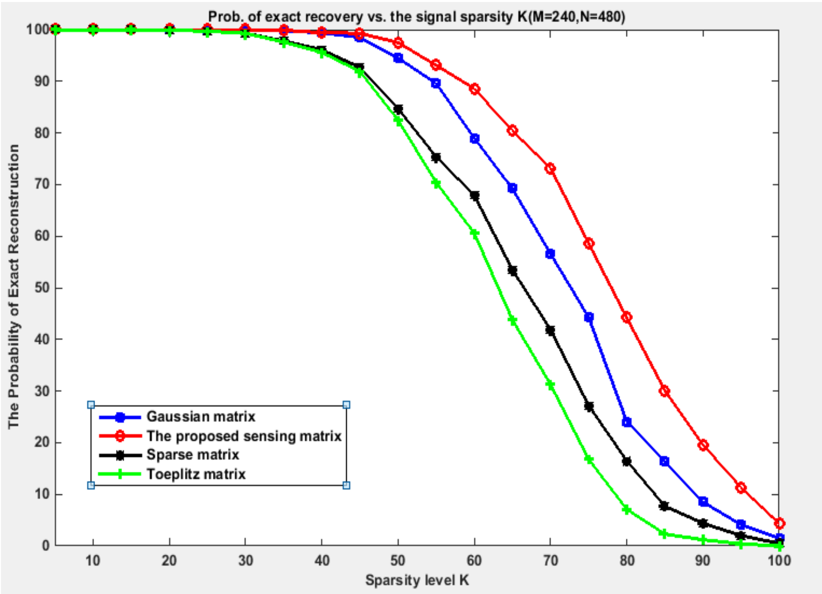


Figure 6. The Reconstitution Performance of Various Sensing Matrices (Column Weight is 6)

To test the two-dimensional image signal of the proposed sensing matrix, the Lena image (128×256, $M=128$, $N=256$) is selected as the testing object. Compression ratio ($M\ /\ N$) is 0.5, and the reconstruction algorithm still uses the OMP algorithm. Gauss matrix, sparse random matrix, Toeplitz matrix, Bernoulli matrix and the proposed matrix are chosen as the sensing matrix. The column weight of the proposed sensor matrix is 4, and the girth length is 10 In order to compare the reconstruction performance of the matrix, Peak Signal to Noise Ratio (PSNR) and Mean Square Error (MSE) are used to measure the effect of reconstruction. When x is the original signal and $x^{'}$ is the reconstructed signal, the calculation method of PSNR and MSE are as follows:

$$MSE=\frac{\sum_{i=0}^{N-1}\sum_{j=0}^{M-1}\left(x-x^{'}\right)^2}{M\times N} \tag{14}$$

$$PSNR=10\lg\left(\left|\frac{255\times255}{MSE}\right|\right) \tag{15}$$

As can be seen from Table 1, the proposed sensor matrix compared with the other matrix, the value of PSNR has increased 4dB, and MSE also has a certain degree of improvement. In Figure 7, the effect of different matrices in the same compression ratio is presented for the reconstruction of 2D image. Among them, the reconstruction effect of the proposed sensing matrix is better than that of the other sensing matrices.

Table 1. The Performance Comparison of Different Sensing Matrices

The sensing matrix	Peak Signal to Noise Ratio (PSNR)	Mean Square Error (MSE)
Gauss matrix	26.3914	149.2601
sparse random matrix	26.5893	142.6090
Toeplitz matrix	26.8502	134.2939
Bernoulli matrix	26.4264	148.0606
the proposed matrix	30.5869	56.8060

(a) Original Image (b) Gauss Matrix (c) Sparse Random Matrix

(d) Toeplitz Matrix (e) Bernoulli Matrix (f) the Proposed Matrix

Figure 7. The Reconstruction Lena Images Using Five Sensing Matrix

5. Conclusion

In this thesis, we introduce a novel approach to construct deterministic sensing matrices by interlaced filling algorithm. The coherence is used to analyze the optimal performance of the constructed sensing matrix. In addition, we set up the relationship between the column weight of a matrix and the coherence, and obtained the increase of the column weight to reduce the coherence. As compared with the existing constructions, the proposed sensing matrix is not only obtains better performance but also has easy hardware implementation. In future work, we will create deterministic sensing matrices from improved interlaced filling algorithm.

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