International Journal of Signal Processing, Image Processing and Pattern Recognition
Vol.8, No.9 (2015), pp.1-12
http://dx.doi.org/10.14257/ijsip.2015.8.9.01

A Study on Partial Differential Equation Model of Image Denoising Method

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Abstract

Image denoising is a basic problem in image processing, fourth order partial differential equation model is an image denoising method proposed for maintaining the balance of denoising and edge keeping. But when the image is severely contaminated, the denoising effect is not ideal, it may fuzzes the boundary at the same time, this paper puts forward an improved fourth order partial differential equation model, in order to overcome the contradiction between denoising and edge keeping, better denoising effect is obtained, and through the numerical simulation the results show that the method has good stability and practical value.

Keywords: *Image denoising; Partial differential equation; Edge keeping; Numerical simulation*

1. Introduction

Digital image molding and processing has become the current field of research and use of objects, the scientific research and is widely used in social life each domain, for work and life has brought great convenience and benefits. However, in the process of image collection and transmission, it is inevitable to be polluted by outside noise. Surrounding conditions, such as precision outside factors can affect the image of the real degree, the existence of various noise in the image, thus acquired a lot of images is derived from the original image and noise signal overlap, the noise will cover part of the image information, bring adverse effect for image processing of the follow-up work. To get high quality original image, image denoising is an important link in the process of image processing. The purpose of image denoising is to get the original image from the noise in the image restoration with removing the noise at the same time, and keep the important information of the original image as much as possible.

In recent years, the application of partial differential equations in image processing and depend on the curvature of the contour evolution at home and abroad by the attention of scholars. IEEE a special edition in 1998. Based on partial differential equations in image processing has become an important branch in the field of image processing, the relevant content is increasingly become a hot spot of relevant researchers. The research work can be traced back to the earliest Nagao and Rudin, study of image smoothing and image enhancement and Koenderink[1] for investigating the structure of image, image processing and other branches of mathematics, such as mathematical morphology (mathematics morphology), image (level set), horizontal image shape (shape) and also for the formation of the subject into a content.

To the introduction of the partial differential equation of research direction in the field of image processing is image filtering. As a pretreatment method, image filtering almost became the prelude to all image processing methods. As early as in 1984, the image signal was discovered Koenderink after gaussian filtering results and there is a certain relationship heat conduction equation. On several occasions, image filtering is as a

ISSN: 2005-4254 IJSIP

pretreatment method of image recognition, it needs to satisfy two conditions, contrast and affine invariant. The affine invariant can be decomposed into translation invariant and rotation invariant, Euclidean invariant and scale invariant etc. Meet the relevant terms of the invariance of filter has a bunch of partial differential equations and corresponding with the augmentation of the invariance constraints, the scope of the corresponding partial differential equations in narrow. Finally, the contrast and affine invariant partial differential equation is only one, the so-called AMSS equation. L.A. lvarez, such as the whole derivation for the clever organization, formed an axiom system. The publication of the reference [2], is considered to be image processing based on partial differential equation of a symbol of this discipline. Among them, the mathematical morphology operators have also been incorporated into the whole system is derived, thus these classic filter have been given a new meaning.

Image processing based on partial differential equation of belongs to the category of low level image processing, the processing result is often used as intermediate results for other image processing methods further. So the application of image processing based on partial differential equation of range covers almost the entire field of image processing, including image recognition, image segmentation, image reconstruction, image edge detection, image retrieval, medical image processing, color image processing, dynamic image analysis, etc.

At present, the image processing based on partial differential equation is derived for many branches, such as dynamic boundary, based on level set (line) of image processing, image form, image model, etc. Image processing based on partial differential equations in the use of the theory of partial differential equation as well as promote the development of the theory of partial differential equation. The mathematical model of the commonly used image as a function of bounded variation, so the continuity of the partial differential equation need not meet. Garden, a kind of weak solution viscosity solution of equation (viscosity solution) was introduced into the image processing method.

2. Related Works

For a lot of noise more outstanding points, they are some of the gray level is only for different noise. Their distribution, and spread them with diffusion equation can be (is actually changing the point for the noise pixel values, make its gray and gray contrast in the perimeter points, make the human eye is difficult to distinguish, or the machine can also be difficult to distinguish (within the permissible standard in artificially) or make noise greatly abate). Of course this diffusion is the use of the gray levels of the perimeter points to change the grey value of noise points, to take the perimeter points method also has a lot of (neighborhood domain method and determining the eight neighborhood). Of course, if the filter radius (window) is large, for larger noise can be more effective to remove, but has the potential to more severe fuzzy boundaries. For different noise, of course, need to use a different model, because up to now, no general denoising model. For noise image, first of all, probably to analyze its noise type, and then adopt corresponding model (for those who can know in advance which model noise effectively, but may not be from the nature of the mathematical properties of the model itself and noise to analyze which kind of model to which effective noise), and of course for mixed noise, can be used to combine multiple models, so far, what kinds of noise is not what the nature of models to standard or conclusion.

With minimal noise is commonly this neighborhood points within the larger point, the grey value of contrast and the font itself is gray contrast is bigger, but the noise is generally small. So by diffusion equation can be dropped the spread of noise (if the noise is large, then can for thinning image first), make it into the decreased peripheral within the scope of the gray contrast to the human eye cannot distinguish, so the noise is removed. Of course, on the edge of the font. Also can appear excessive proliferation (on the edge of

internal diffusion is not shown), so there will be a fuzzy boundaries, or even removed the boundary of the small, but it is allowed (because of denoising is to achieve human standard), but when the noise is very big or very serious noise pollution, often at this time and no effective method to voice, even for denoising, it will lead to serious distortion of the image.

Using anisotropic diffusion image noise removal is very popular. But in effect by the model of the image, the block effect is common. The second order partial differential equations have been as wai like denoising and useful tool in image scale space analysis. Including form like reference [3-4] in the form of the anisotropic diffusion equation. Although these techniques are shown to achieve denoising and boundaries to keep a good balance, but they tend to cause the image looks a blocky effect, such as the image in [4]. This effect cause visual discomfort, also may make the computer vision system errors to identify different block as the boundary, but in fact they belong to the image in the same region. The block effect larger extent, originated in the inner nature of these equations is the second order. Let's look at why the block effect in the process of diffusion.

Define u as image intensity function, t as time, g as diffusion coefficient, the anisotropic diffusion can be expressed as

$$\frac{\partial u}{\partial t} = div[g(|\nabla u|)\nabla u] \tag{1}$$

This equation is associated with the following energy function

$$E(u) = \int_{\Omega} f(|\nabla u|) d\Omega \tag{2}$$

Ω is support set of image, $f \geq 0$ is with the increasing function of the diffusion coefficient has the following relationship.

$$g(s) = \frac{f'(s)}{s} \tag{3}$$

Then anisotropic diffusion is considered to minimize the energy function of the energy dissipation process. We can clearly see from (2), the level of the image is the global minimum of energy function. Data analysis shows that, when there is no backward diffusion level image as the only minimum energy function, therefore the anisotropy will be carried out in accordance with the level of image function. Because of the anisotropic diffusion is to press in the smooth area design spread faster than not smooth area, block effect will appear in front of the diffusion of stage, even if all the block image will eventually shrink with a horizontal image.

In general, for a partial differential equation model of well-posed, do not need to decide when to stop, because for a well-posed model, the image after reach a certain number of iterations, can't again along with the minimization algorithm of iteration times have too big change, can achieve a relatively stable state. Usually only need to remove the relatively small number of iterations and larger images, and the comparison of the number of iterations used species (such as 10 times, 50 times) can be judgment. Don't need too much for some partial differential equation model of ill-posed, may decide to stop the types to take the number of iterations of more, but when close to the image appeared oscillation, can judge the phenomenon is more obvious, because when the number of iterations (few) image will appear when reach certain oscillation, so then can stop effect. Generally speaking, the image processing (noise) stop and there is no standard, just according to the requirement of the people and the artificial regulation, different people for different purposes, may stop working when asked will be different. Time parameters of the partial differential equation with the number of iterations and step length, step length and the values of noise removal effect may also be affected, sometimes need a lot of experiment to determine a relatively better step length.

For second order partial differential equation itself determines that they produce blocky effect, at the same time of smooth the image denoising, can cause a false boundary effect

after image. Because for using mathematical model (partial differential model) to smooth the image denoising of the essence of which is to minimize the energy function image. Of course for fourth order partial differential equation model, the essence of the smooth image denoising is to minimize the energy function, the process can be regarded as process can be obtained from the minimum energy dissipation of energy, but for the second order partial differential equation model, due to the backward diffusion it, which makes it to minimize the energy function value is not the only, so there will be a massive effect, lead to the emergence of the false edge. For the fourth order partial differential equation model, it is to minimize the energy function is the only one. So that its boundary is the only, did not produce false boundaries, effectively avoid the block effect. For block at the various points inside the image, because the gray gradient between the block and block is different, so its Laplace values are also different, especially in the noise around the Laplace's value is bigger, so its diffusion to (this is different from the principle of second order partial differential model), when diffusion into the boundary, the Laplace's value is almost zero, then the energy function values close to the minimum, thus slow diffusion, the last stop quickly, effectively protecting the borders. Also you can see the boundary and interior point relative grayscale contrast increases, which have played an important role in enhancing border.

Fourth order partial differential equation model for denoising in balance and maintain boundaries to avoid a blocky effect. When images are serious noise pollution, the fourth order partial differential equation model can effectively remove noise or after denoising images appear blurred boundaries. Later in this article, therefore, puts forward a new kind of four order partial differential equation model, in order to overcome the shortage of the mentioned in the previous sections.

First of all let us consider the following definition in the continuous space of support set Ω of images [5-7]

$$E(u) = \int_{\Omega} f(\left|\nabla^2 u\right|)dxdy \tag{4}$$

∇^2 is laplace operator, here $f \geq 0$ and is Increasing function, Function on with $\left|\nabla^2 u\right|$ measure of image smooth operator is increasing. Thus minimizing function is equivalent to smooth the image. By minimizing the function, we can get the following gradient descent,

$$\frac{\partial u}{\partial t} = -\nabla^2[f'(\left|\nabla^2 u\right|)\frac{\nabla^2 u}{\left|\nabla^2 u\right|}] = -\nabla^2[g(\left|\nabla^2 u\right|)\nabla^2 u] \tag{5}$$

Take observed image as the initial conditions, When $t \to \infty$ gets problem solution, but the questionnaire may stop earlier, denoising and border to keep balance. In order to achieve the fourth order partial differential diffusion equation and its nature can refer to the [10].

Let us discuss an image, the strength of the function to meet a plane function is a graphic image, we can prove a plan like the consumption function $E(u)$ of a global minimum. Due to the negative of $f(\left|\nabla^2 u\right|)$, the function $E(u)$ is bounded,

$$E(u) \geq 0 \tag{6}$$

Because $f(\left|\nabla^2 u\right|)$ is increasing function of $\left|\nabla^2 u\right|$, so when $\left|\nabla^2 u\right| = 0$, achieve the global minimum. So when

$$\left|\nabla^2 u\right| \equiv 0, \forall (x, y) \in \Omega \tag{7}$$

a plane image obvious meets (5), can achieve the global minimum, so is $E(u)$'s global minimum.

If f is convex or equally if

$$f''(s) \geq 0, \forall s > 0 \tag{8}$$

plane image is $E(u)$'s only global minimum. Because the consumption function $E(u)$ under this condition is convex.

Above the fourth order partial differential equation, while the observed image to planar piecewise smooth image, but we think it is better to close to the original image. But the above four order partial differential equation is put forward by the tends to leave a larger or smaller isolated white and black noise, the noise intensity values than its surrounding pixels or big or small. Because of the image intensity function of Laplace value around the pixel is very large, and f is keep the boundary with the decrease of the rapid design, g is defined in (2) around the pixel value is small, so (2) on the right side of the part is very small. The multiplicative noise may be intact retention, as reserve boundary.

3. Improved Model of Fourth-order Differential Equations

Based on the above issues, here put forward a new kind of four order partial differential equations to overcome the above shortcomings [8-11],

$$\frac{\partial u}{\partial t} = -\nabla^2[g(|\nabla^2 u|)\nabla^2 u] - \lambda(u-1) \tag{9}$$

The right side of the equation, the first ensure the border to keep smooth, at the same time avoid the block effect. Here $g(s) = 1/(1+(s/k)^2)$ Is an option, it with K as the scale parameter to keep the border, after a second to ensure smooth image well close to the original image, at the same time better to remove noise and keep intact the border, here is a specialization of the weighting factor.

Numerical iterative format as follows, the diffusion equation (9) available numerical methods iterative approximation. Assume that a synchronous long and empty asked net grid h size, contact quantitative as we put the time and space： $t = n\Delta t, n = 0,1,2,\ldots$, $x = ih, i = 0,1,2,\ldots,I$， $y = jh, j = 0,1,2,\ldots,J$， here Ih×Jy is the size of support set of image. We can use the following central difference format to calculate the intensity function of Laplacian image:

$$\nabla^2 u_{i,j}^n = \frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n}{h^2} \tag{10}$$

and symmetric boundary conditions:

$$u_{-1,j}^n = u_{0,j}^n, u_{I+1,j}^n = u_{I,j}^n, i = 0,1,2,\ldots,I \tag{11}$$

and \

$$u_{i,-1}^n = u_{i,0}^n, u_{i,J+1}^n = u_{i,J}^n, j = 0,1,2,\ldots,J \tag{12}$$

Here define f_1 as follows:

$$f_1(\nabla^2 u) = f'(|\nabla^2 u|)\frac{\nabla^2 u}{|\nabla^2 u|} = g(|\nabla^2 u|)\nabla^2 u \tag{13}$$

means

$$(f_1)_{i,j}^n = f_1(\nabla^2 u_{i,j}^n)$$

（14）

Calculate laplace operator f_1 with

$$\nabla^2 (f_1)_{i,j}^n = \frac{(f_1)_{i+1,j}^n + (f_1)_{i-1,j}^n + (f_1)_{i,j-1}^n + (f_1)_{i,j-1}^n - 4(f_1)_{i,j}^n}{h^2} \tag{15}$$

and symmetric boundary conditions:

$$(f_1)_{-1,j}^n = (f_1)_{0,j}^n, (f_1)_{I+i,j}^n = (f_1)_{I,j}^n, i = 0,1,2,\ldots,I \tag{16}$$

and

$$(f_1)_{i,-1}^n = (f_1)_{i,0}^n, (f_1)_{i,J+1}^n = (f_1)_{i,J}^n, j = 0,1,2,\ldots,J \tag{17}$$

Partial differential equation for the final

$$u_{i,j}^{n+1} = u_{i,j}^n - \Delta t \nabla^2 (f_1)_{i,j}^n - \lambda (u_{i,j}^n - u_{i,j}^0) \tag{18}$$

here $u_{i,j}^0$ is the original image polluted by salt and pepper noise.

There is a problem of choosing suitable step length Δt, Δt can ensure the speed of convergence. Due to the nonlinear equations, while the right step is hard to get and very cost calculation in theory, we choose to step 0.25. In this paper, we set a threshold value K = 1.8, using the grid and h = 1 = 1.5.

4. Experiment and Analysis

To commonly used both peak signal-to-noise ratio SNR and standard variance RMSE as image quality parameters, can be seen from the table below, when the image has not been serious pollution, the previous fourth order partial differential equation and a new differential equation of fourth order Lei can remove noise effectively and avoid the block effect. But due to the nature of the fourth order partial differential equations and Laplace, when the image is seriously polluted, some noise will stay intact was previously fourth order partial differential equation of role in the image. And in the first four order partial differential equations in the images of the role of the head also is fuzzy, the edge of the area and the new differential equation of fourth order Lei role of the head area of the image is kept intact.

Table 1. Comparison of SNR and RMSE of Image after Filter

	SNR	RMSE
Pic1-3	3.345	0.167
Pic2-3	3.452	0.178
Pic1-4	5.728	0.256
Pic2-4	5.836	0.276

From figure 1, figure 2, we can get almost all noise is removed, and the effect looks more comfortable, at the same time also didn't appear blocky effect. Can be seen from table 1, the previous fourth order partial differential equation model RMSE value than the new image after image after the fourth order partial differential equation model of RMSE value is smaller, but the latter's SNR value is greater than the value of the former, regardless of whether the image is seriously polluted. So a new partial differential equation model to keep the image at the same time effectively removed noise, and the new fourth order partial differential equation of role image than was previously the fourth order partial differential equation of role image clearer.

1. Image without polluted

2.10% impulse noise image

3. After filter with fourth order differential equation

4. After filter with new fourth order differential equation

Pic 1 Comparison of image denoising effect

1. Image without polluted

2.30% impulse noise image

3. After filter with fourth order differential equation

4. After filter with new fourth order differential equation

Pic 2 Comparison of image denoising effect

Here we have some image denoising model. A comparison of consistent new fourth order partial differential equation model, the anisotropic diffusion model, the value in anisotropic diffusion model and the morphology of anisotropic diffusion model. First from these model (figure 3) role of image itself, we can see by the anisotropic diffusion model, the value in anisotropic diffusion model and the morphology of anisotropic diffusion model after the image is blurred, particularly in the edge (from the shoulder part of the image to see), at the same time as you can see in spite of the noise nearly all removed, but the block effect. But the new image after the fourth order partial differential equation model to look more cheerful, so as to achieve the noise removal and edge to keep a balance (can be seen from the shoulder part of the figure), at the same time avoid the block effect.

1. Image without polluted

2.30% impulse noise image

3. Anisotropic diffusion

4. Anisotropic median diffusion

5. Anisotropic form diffusion

6. New fourth order differential equation

Pic 3 Comparison of different image denoising effect

Further, from table 2 image signal-to-noise ratio (SNR), you can see that the new role of fourth order partial differential equation model of image SNR values than those by other models of image SNR value is big, that is to say, although the noise in the image almost removed, but by other model after the image is blurred.

Table 2. Comparison of SNR and RMSE of Image after Filter

	SNR	RMSE
Pic3-6	4.827	0.284
Pic3-3	3.185	0.229
Pic3-4	3.190	0.034
Pic3-5	3.198	0.239

So in this article puts forward the new fourth order partial differential equation of the effect of denoising model is obvious, although there may be a later better fourth order partial differential equation model.

5. Conclusion

Now, there are many image denoising model, such as the second order partial differential equation model [120-13], third order partial differential equation model, and so on. Four order partial differential equation model is a kind of model for image denoising. In this article, we proposed a new fourth order partial differential equation model, as the previous four order partial differential equation model (considered a for minimizing the absolute value of an image intensity function of Laplacian function), as a kind of effective in denoising and keep boundary with balance method. Compared with the previous fourth order partial differential equation model, in the new model is much more a repair, in order to better denoising and maintain boundaries. The validity of the model in the numerical experiments prove that we can also prove that the new model of the fourth order partial differential equation model convergence of numerical calculation format [14].

Acknowledgements

This work was supported by a grant from the National Natural Science Foundation of Zhejiang (No. Y1100314).

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