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Image Restoration Based on L1 + L1 Model

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Abstract

In this paper, we firstly propose a new image restoration model including non-smooth ℓ^1-norm data-fidelity term and non-smooth ℓ^1-norm regularization term based on the bilateral total variation regularization. Secondly, we prove the existence of minimal solutions of our proposed energy functional model. Thirdly, we consider the convergence of the discrete numerical algorithm, and obtain that the limit point of the solution sequence is the minimal point of our proposed energy functional. Finally, we give some experimental simulation results in the case of the single noisy image without blurring, multiple different noisy images without blurring, single noisy image with blurring, and multiple different noisy images with different blurring, respectively. The restoration results show our model works effectively.

Keywords: *image restoration, bilateral total variation, minimal solution, mathematical induction method*

1. Introduction

Image denoising and deblurring are two fundamental problems in the field of image processing. Image denoising is to enhance images by reducing some degradation. Image deblurring is to recover the original, sharp image by using a mathematical model of the blurring process. The key issue is that some information is "hidden" and can only be recovered if we know the details of the blurring process.

The observed image is the convolution of a shift invariant blurring function with the true image plus some additive noise. Let $u(x)$ be an original image, $z(x)$ be an observed image, and $h(x)$ be a point spread function (PSF). A degraded image model can be described as $z(x)=\int_\Omega h(s)u(x-s)ds+n(x).$ Here $n(x)$ is an additive noise, $x=(x_1,x_2)$ is a vector. Let u, z and n be discrete original image, degraded image and PSF, respectively. Then, a discrete image formation process can be put into the matrix-vector form $z=Hu+n,$ where H is a Toeplitz matrix which is built according to the discrete PSF h. Assuming that the discrete image u has $m\times n$ pixels, notation $N=m\times n$, then u, z and n are $N\times 1$ vectors arranged by row, and H is a matrix of $N\times N$.

Blurred image restoration has been referred in many works when the PSF is known [1, 3-5]. One of the most successful regularization approaches is the TV regularization method [20], so the mathematical model can be stated as follows [23],

$$\min_u E(u)=\frac{1}{2}\|Hu-z\|_{\ell^2}^2+\alpha\int_\Omega|\nabla u|dx,$$

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where α is a positive constant. However, the disadvantage of the model leads to over-smooth and produce block-effect.

Recently, many researchers have discussed the local minimization problem, or the global minimum [6-7, 10-11, 15-16, 18]. The minimal problem is

$$\min_u E(u) = \frac{1}{2}\|Hu - z\|_{\ell^2}^2 + \lambda\Phi(u),$$

where $\|\cdot\|$ is the Euclidean norm, $\Phi : \mathbf{R}^N \mapsto \mathbf{R}$ is a piecewise C^q smooth regularization term, $\Phi(u) := \sum_{i=1}^{r} \phi(G_i u)$, where $G_i : \mathbf{R}^N \mapsto \mathbf{R}^N$ is a difference operator, $\phi : \mathbf{R} \mapsto \mathbf{R}$ is an increasing function[1,3-4,13-15,19-21]. Especially, Geman, *et al.*, [13] studied the cost function involving convex ℓ^2-norm data-fidelity term and ℓ^α-norm regularization term, that is, $\phi(t) = |t|^\alpha$, $1 \le \alpha \le 2$. We call the model as $\ell^2 + \ell^\alpha$ model, where $1 \le \alpha \le 2$.

In 2004, Nikolova[17] offered another image restoration model, and proved the existence of solutions of the minimal functional model. The functional model is

$$\min_u E(u) = \|Hu - z\|_{\ell^1} + \beta\Psi(u),$$

where $\Psi(u)$ is a regular function, *i.e.*, $\Psi(u) := \sum_{i=1}^{r} \varphi(G_i u)$, G_i is a difference operator for each $i = 1, 2, \cdots, r$, and $\varphi : \mathbf{R} \mapsto \mathbf{R}$ is a potential function. The following potential functions are some smooth and convex edge-preserving [2, 4, 9, 17]. Especially, Bouman, *et al.*, [4] thought the cost function involving non-convex ℓ^1-norm data-fidelity term and convex ℓ^α-norm regularization term, *i.e.*, $\varphi(t) = |t|^\alpha$, where $1 < \alpha \le 2$. We call the model as $\ell^1 + \ell^\alpha$ model, where $1 < \alpha \le 2$. Afterwards, Li et al.[17] analyzed the case $\varphi(t) = \sqrt{\alpha + t^2}$, where $\alpha > 0$.

In 1998, Tomasi, *et al.*, [22] first put forward the concept of bilateral filtering, the advantages of the bilateral filtering are not only to consider space distance between two pixels, but also to consider illumination distance between two pixels. In 2002, Based on the spirit of TV criterion, and a related the bilateral filter technique, Fariu, *et al.*, [12] proposed the bilateral total variation(BTV) regularization, which is computationally cheap to implement, and preserves edges. The regularizing function looks like

$$\Gamma_{\text{BTV}} = \sum_{k=-p_1}^{p_1} \sum_{l=-p_2}^{p_2} \alpha^{|k|+|l|} \left\|u - S_x^k S_y^l u\right\|_{\ell^1},$$

where matrices operators S_x^k and S_y^l shift X by k, and Y by l pixels in horizontal and vertical directions respectively, presenting several scales of derivatives. The scalar weight α, $0 < \alpha < 1$, is applied to give a spatially decaying effect to the sum of the regularization terms.

After the above analysis, we will consider a cost function including non-smooth ℓ^1-norm data-fidelity term and non- smooth ℓ^1-norm regularization term based on the bilateral total variation in this paper. We call the model as $\ell^1 + \ell^1$ model.

The rest of the paper is organized as follows. In section 2, we will construct a novel image restoration model based on the bilateral total variation regularization term. In section 3, we will prove the existence of minimal solutions of our proposed model. In section 4, we will consider the convergence of discrete numerical algorithm by the mathematical induction method. In section 5, we show some experimental results. In section 6, we give some discussions and conclusions.

2. Bilateral Total Variation-Based Image Restoration Model

Given R degraded images, we describe as $z_r = H_r u + n_r$, $r = 1, 2, \cdots, R$, where n_r is an additional noise, $r = 1, 2, \cdots, R$.

Using R degraded images, and the advantages of BTV regularization preserving edges, we consider the following minimization functional model

$$\min_u E(u) = \sum_{r=1}^{R} \left\| H_r u - z_r \right\|_{\ell^1} + \lambda \sum_{k=-p_1}^{p_1} \sum_{l=-p_2}^{p_2} \alpha^{|k|+|l|} \left\| u - S_x^k S_y^l u \right\|_{\ell^1}, \tag{2.1}$$

where λ, α are positive constant, $0 < \alpha < 1$, p_1, p_2 are two positive integers. The first term is called data-fidelity term, *i.e.*, the residual error term. We adopted ℓ^1-norm, instead of ℓ^2-norm, since when the error is slightly large, the growth rate of ℓ^1-norm is slower than that of ℓ^2-norm. Hence, the term is also robust. The second term is called the BTV regularization term, S_x^k and S_y^l can see the section 1. We call the model (2.1) as $\ell^1 + \ell^1$ model.

In order to simplify in the following section, we only consider the case $R = 1$,

$$\min_{u \in \mathbf{R}^N} E(u) = \sum_{i=1}^{N} \left| b_i^T u - z_i \right| + \lambda Q(u), \tag{2.2}$$

where $Q(u) = \sum_{s \in S} \alpha^{|s|} \left\| G_s u \right\|_{\ell^1}$, $S = \left\{ km + l \,\middle|\, \forall k = -p_1, \cdots, p_1,\ l = -p_2, \cdots, p_2 \right\}$, $|s| = |k| + |l|$, $\forall k = -p_1, \cdots, p_1,\ l = -p_2, \cdots, p_2,\ s = km + l,\ \forall k = -p_1, \cdots, p_1,\ l = -p_2, \cdots, p_2,\ \ G_s \in \mathbf{R}^{N \times N}$, $H = \left(b_i^T \right)$, $G_s u$ is a $N \times 1$ vector by rows of $\left(I - S_x^k S_y^l \right) u$.

Suppose H^{-1} exists, put $y = Hu - z$, so $u = H^{-1}(z + y)$, the functional (2.2) becomes

$$\min_{y \in \mathbf{R}^N} E_z(u) = \sum_{i=1}^{N} |y_i| + \lambda Q_z(y), \tag{2.3}$$

where $Q_z(y) = Q\left(H^{-1}(z + y) \right)$, $\mathrm{Q}(u) = \sum_{s \in S} \alpha^{|s|} \left\| G_s u \right\|_{\ell^1} = \sum_{s \in S} \alpha^{|s|} \sum_{i=1}^{N} \varphi\left(g_{s,i}^T u \right)$, $\varphi : \mathrm{R} \to \mathrm{R}$, $\varphi(t) = |t|$, $g_{s,i}^T$ is the *ith* row of G_s, $i = 1, \cdots, N$. For every $y \in \mathbf{R}^N$, $E_z(y)$ is 0-coercive, and $E_z(y) \geq 0$. $Q_z(y)$ is convex and continuous. Clearly, $E(u)$ reaches its minimum at $\hat{u} \in \mathbf{R}^N$, if and only if, $E_z(u)$ reaches its minimum at $\hat{y} = H\hat{u} - z$.

3. Existence of Minimal Solutions of Energy Functional (2.3)

Now, we construct a relaxation of the minimal energy functional

$$\min_{y \in R^N} E_z^{\varepsilon}(y) = \sum_{i=1}^{N} |y_i| + \lambda Q_z^{\varepsilon}(y), \tag{3.1}$$

where $Q_z^{\varepsilon}(y) = Q^{\varepsilon}\left(H^{-1}(z + y) \right) = \sum_{s \in S} \alpha^{|s|} \sum_{i=1}^{N} \varphi_{\varepsilon}\left(g_{s,i}^T H^{-1}(z + y) \right)$, $\varphi_{\varepsilon}(t) = \sqrt{\varepsilon + t^2}$, $\varepsilon > 0$. Some notations can be referred in the section 2.

According to equations (2.3) and (3.1), we can get the following propositions

Proposition 1 $E_z^{\varepsilon}(y)$ is uniformly convergent to $E_z(y)$ while $\varepsilon \to 0^+$.

Proposition 2 Suppose H^{-1} exists, and given $\varepsilon > 0$, for every $z \in R^N$, the functional $E_z^{\varepsilon}(y)$ in (3.1) is 0-coercive, *i.e.*, $E_z^{\varepsilon}(y) \to \infty$, if $\|y\| \to \infty$.

Proposition 3 The function $Q_z^\varepsilon(y)$ is convex and C^1-continuous about y, and that, for every $z \in R^N$, $\varepsilon > 0$, $\forall y \in \bar{B}(0,\varepsilon)$, $\forall t \in [-\varepsilon,\varepsilon]$, $i = 1,\cdots,N$, there is $\eta > 0$ such that for every $Q_z^\varepsilon(y+te_i) - Q_z^\varepsilon(y) \ge tD_iQ_z^\varepsilon(y) + \eta t^2$, $i = 1,\cdots,N$, where $D_iQ_z^\varepsilon(y)$ is the *ith* partial derivative of $Q_z^\varepsilon(y)$.

Proof: Since $\varphi_\varepsilon''(t) > 0$, then $Q_z^\varepsilon(y)$ satisfies the case in Remark 1 [18]. ▌

Proposition 4 Fixed z, y, the functional $E_z^\varepsilon(y)$ is an increasing function, *i.e.*, for every $\varepsilon_1 < \varepsilon_2$, there is $E_z^{\varepsilon_1}(y) \le E_z^{\varepsilon_2}(y)$, and $E_z^\varepsilon(y) \ge 0$.

Lemma 1 [14, 17] Suppose H^{-1} exists, and given $\varepsilon > 0$, then $E_z^\varepsilon(y)$ reaches its minimum at $\hat{y}_\varepsilon \in R^N$, i.e. $E_z^\varepsilon(\hat{y}_\varepsilon) \le E_z^\varepsilon(y), \quad \forall y \in R^N$.

Proof: The function $E_z^\varepsilon(y)$ does admit a minimum for every $z \in R^N$. This minimum is both local and global [**14, 17**]. We call the minimal point $\hat{y}_\varepsilon$, and the minimum $E_z^\varepsilon(\hat{y}_\varepsilon)$. ▌

Theorem 1 Suppose H^{-1} exists, and for every $z \in R^N$, then there is a minimal point $\hat{y}_0$, such that $E_z(\hat{y}_0) \le E_z(y), \quad \forall y \in R^N$.

Proof: Taking a strict monotonically decreasing positive sequence $\{\varepsilon_n\}$, and $\varepsilon_n \to 0$ while $n \to \infty$. Taking $\varepsilon_0 = 1$, and given $\varepsilon_n > 0$, there are the following conclusions by Lemma 1,

1° For $\varepsilon = \varepsilon_0$, there are a minimal point $\hat{y}_{\varepsilon_0}$ and $M(>0)$ only relevant with ε_0, such that

$$E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}) \le E_z^{\varepsilon_0}(y), \ \ \forall y \in R^N, \ \ 0 \le E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}) \le M. \tag{3.2}$$

2° For $\varepsilon = \varepsilon_1$, there are a minimal point $\hat{y}_{\varepsilon_1}$, such that

$$E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_1}) \le E_z^{\varepsilon_1}(y), \ \ \forall y \in R^N. \tag{3.3}$$

and since $\varepsilon_1 < \varepsilon_0$, according to Proposition 4, we have

$$E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_0}) \le E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}), \tag{3.4}$$

$$E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_1}) \le E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_0}), \tag{3.5}$$

From the inequalities (3.2), (3.4) and (3.5), we can see

$$0 \le E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_1}) \le E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_0}) \le E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}) \le M,$$

then, $0 \le E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_1}) \le E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}) \le M$.

In a similar way, we can get

$$0 \le \cdots \le E_z^{\varepsilon_n}(\hat{y}_{\varepsilon_n}) \le \cdots \le E_z^{\varepsilon_1}(\hat{y}_{\varepsilon_1}) \le E_z^{\varepsilon_0}(\hat{y}_{\varepsilon_0}) \le M. \tag{3.6}$$

Therefore, we can conclude that the energy sequence $\left\{E_z^{\varepsilon_n}(\hat{y}_{\varepsilon_n})\right\}$ is uniformly bounded, i.e. $\sum_{i=1}^{N}\left|\hat{y}_{\varepsilon_n,i}\right| \le E_z^{\varepsilon_n}(\hat{y}_{\varepsilon_n}) = \sum_{i=1}^{N}\left|\hat{y}_{\varepsilon_n,i}\right| + \lambda Q_z^{\varepsilon_n}(\hat{y}_{\varepsilon_n}) \le M$, where M is a generalized positive constant. Then, $\left\|\hat{y}_{\varepsilon_n}\right\| \le M$. As we know, there are a subsequence $\left\{\hat{y}_{\varepsilon_{n_k}}\right\}$ and a point $\hat{y}_0$, such that $\lim_{k\to\infty} \hat{y}_{\varepsilon_{n_k}} = \hat{y}_0$. According to Lemma 1, there is $E_z^{\varepsilon_n}(\hat{y}_{\varepsilon_n}) \le E_z^{\varepsilon_n}(y)$ for each $y \in R^N$. Then $E_z^{\varepsilon_n}(\hat{y}_0) = \lim_{k\to\infty} E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_{n_k}}\right) \le E_z^{\varepsilon_n}(y)$, $\forall y \in R^N$.

By Proposition 1, we have $E_z(\hat{y}_0)=\lim_{n\to\infty}E_z^{\varepsilon_n}(\hat{y}_0)\le\lim_{n\to\infty}E_z^{\varepsilon_n}(y)=E_z(y),\ \forall y\in R^N$. ▌

4. Convergence of Discrete Numerical Algorithm

In section 3, we have proved that there is a minimal point $\hat{y}_0$ for the functional (2.3). In what follows, we will give a specific calculation of the minimal point.

4.1 Minimal Solution of Relaxation Energy Functional (3.1)

Given $\varepsilon>0$, we can obtain that the functional $E_z^\varepsilon(y)$ reaches its minimum at $\hat{y}_\varepsilon$ using Lemma 1. Here, we will get the minimal solution $\hat{y}_\varepsilon$ using numerical iterative method.

Lemma 2[17] Suppose H^{-1} exists, and given $\varepsilon>0$, then the functional $E_z^\varepsilon(y)$ reaches its minimum at $\hat{y}_\varepsilon\in R^N$ if and only if

$$-1\le\lambda D_i Q_z^\varepsilon(\hat{y}_\varepsilon)\le 1,\ \text{ if } i\in\hat{h}, \tag{4.1}$$

$$sign(\hat{y}_{\varepsilon,i})+\lambda D_i Q_z^\varepsilon(\hat{y}_\varepsilon)=0,\ \text{ if } i\in\hat{h}^c. \tag{4.2}$$

where $\hat{h}=\{i\in\{1,\cdots,N\}:\hat{y}_{\varepsilon,i}=0\}$, $\hat{h}^c$ is its complement of $\hat{h}$. Moreover, for any $i\in\hat{h}^c$, we have

$$\text{if } \lambda D_i Q_z^\varepsilon(\hat{y}_\varepsilon-\hat{y}_{\varepsilon,i}e_i)<-1,\ \text{then } \hat{y}_{\varepsilon,i}>0, \tag{4.3}$$

$$\text{if } \lambda D_i Q_z^\varepsilon(\hat{y}_\varepsilon-\hat{y}_{\varepsilon,i}e_i)>-1,\ \text{then } \hat{y}_{\varepsilon,i}<0, \tag{4.4}$$

Lemma 3[17] Suppose H^{-1} exists, and given $\varepsilon>0$, there is a constant $\eta>0$ such that for every $k\in N$,

$$E_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k-1)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)-E_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)$$
$$\ge\lambda\eta\left(y_{\varepsilon,i}^{(k-1)}-y_{\varepsilon,i}^{(k)}\right)^2,\ \forall i\in\{1,\cdots,N\}.$$

In the following, we will proceed to calculate the discrete form of the minimal point $\hat{y}_\varepsilon$ of the functional (3.1).

Fixed $\varepsilon>0$, let $y_\varepsilon^{(0)}\in R^N$ be a starting point. At every iteration $k=1,2,\cdots$, the new iterate $y_\varepsilon^{(k)}$ is obtained from $y_\varepsilon^{(k-1)}$ by calculating successively each one of its entries $y_{\varepsilon,i}^{(k)}$ using one-dimensional minimization:

For any $i=1,2,\cdots,N$, find $y_{\varepsilon,i}^{(k)}$ such that

$$E_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)\le E_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},t,y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right),\ \forall t\in R. \tag{4.5}$$

According to the inequality (4.5), the solution obtained at step $i-1$ of iteration k is $\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)$.

Then the function $t\to E_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},t,y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)$ is strictly convex and 0- -coercive. Hence, $y_{\varepsilon,i}^{(k)}$ is well defined and unique [14].

Suppose H^{-1} exists, and given $\varepsilon>0$, the solution at step $i-1$ of iteration k, the entry $y_{\varepsilon,i}^{(k)}$ is determined using Theorem 1:

1° if $i\in\{1,2,\cdots,N\}$, calculate $\xi_{\varepsilon,i}^{(k)}=\lambda D_i Q_z^\varepsilon\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},0,y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)$,

2^{o} if $-1 \le \xi_{\varepsilon,i}^{(k)} \le 1$, then $y_{\varepsilon,i}^{(k)} = 0$. otherwise $y_{\varepsilon,i}^{(k)}$ is the unique solution of

$sign\left(y_{\varepsilon,i}^{(k)}\right) + \lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)}, \cdots, y_{\varepsilon,i-1}^{(k)}, 0, y_{\varepsilon,i+1}^{(k-1)}, \cdots, y_{\varepsilon,N}^{(k-1)}\right) = 0$,

where if $\xi_{\varepsilon,i}^{(k)} < -1$, then $y_{\varepsilon,i}^{(k)} > 0$; if $\xi_{\varepsilon,i}^{(k)} > -1$, then $y_{\varepsilon,i}^{(k)} < 0$.

Theorem 2 Suppose H^{-1} exists, and fixed $\varepsilon > 0$, for $k \to \infty$, the sequence $\left\{y_{\varepsilon}^{(k)}\right\}$ defined by (4.5) converges to a point $\hat{y}_{\varepsilon}$, such that $E_z^{\varepsilon}\left(\hat{y}_{\varepsilon}\right) \le E_z^{\varepsilon}\left(y\right)$, $\forall y \in R^N$. And there is a positive M such that $0 \le E_z^{\varepsilon}\left(\hat{y}_{\varepsilon}\right) \le M$, where M is only relevant with the fixed ε.

Proof: To simplify the notation, let $y_{\varepsilon,[i]}^{(k)}$ denote the intermediate solution at step i of iteration k for any $i = 1, \cdots, N$,

$$y_{\varepsilon,[i]}^{(k)} = \left(y_{\varepsilon,1}^{(k)}, \cdots, y_{\varepsilon,i-1}^{(k)}, y_{\varepsilon,i}^{(k)}, y_{\varepsilon,i+1}^{(k-1)}, \cdots, y_{\varepsilon,N}^{(k-1)}\right). \tag{4.6}$$

For $i = 0$, put $y_{\varepsilon,[0]}^{(k)} = y_{\varepsilon}^{(k-1)}$. Notice that $y_{\varepsilon,[N]}^{(k)} = y_{\varepsilon}^{(k)}$. For every $k \in N$, (4.5) shows that

$E_z^{\varepsilon}\left(y_{\varepsilon,[i]}^{(k)}\right) \le E_z^{\varepsilon}\left(y_{\varepsilon,[i-1]}^{(k)}\right), \ \forall i = 1, \cdots, N.$

Then $E_z^{\varepsilon}\left(y_{\varepsilon}^{(k)}\right) \le E_z^{\varepsilon}\left(y_{\varepsilon}^{(k)}\right), \forall k \in N$. The sequence $E_z^{\varepsilon}\left(y_{\varepsilon}^{(k)}\right)$ is monotonically decreasing and bounded below by $E_z^{\varepsilon}\left(\hat{y}_{\varepsilon}\right)$, i.e. $\lim\limits_{k\to\infty} E_z^{\varepsilon}\left(y_{\varepsilon}^{(k)}\right) = E_z^{\varepsilon}\left(\hat{y}_{\varepsilon}\right)$. Hence, we have

$$E_z^{\varepsilon}\left(y_{\varepsilon}^{(k-1)}\right) - E_z^{\varepsilon}\left(y_{\varepsilon}^{(k)}\right) = \sum_{i=1}^{N}\left(E_z^{\varepsilon}\left(y_{\varepsilon,[i-1]}^{(k-1)}\right) - E_z^{\varepsilon}\left(y_{\varepsilon,[i]}^{(k)}\right)\right) \tag{4.7}$$

$$\ge \lambda\eta \sum_{i=1}^{N}\left(y_{\varepsilon,i}^{(k-1)} - y_{\varepsilon,i}^{(k)}\right)^2. \tag{4.8}$$

for every $k \in N$. The inequality in (4.8) is obtained by applying Lemma 3 to every term on the right side of (4.7). It follows that the sequence $y_{\varepsilon}^{(k)}$ is convergent, *i.e.*,

$\lim\limits_{k\to\infty} y_{\varepsilon}^{(k)} = \hat{y}_{\varepsilon}$, and $\hat{h} = \left\{i \in \{1, \cdots, N\} : \hat{y}_{\varepsilon,i} = 0\right\}$.

And then, we will show that $\hat{y}_{\varepsilon}$ satisfies the conditions given in Lemma 2.

1^{o} if $\hat{h}$ is nonempty, for every $i \in \hat{h}$, the convergence of $y_{\varepsilon,i}^{(k)} \to 0 (k \to \infty)$ can be produced in two different ways.

(I) If existing an integer n_i, for all $k \ge n_i$, we have $y_{\varepsilon,i}^{(k)} = 0, \ \forall k \ge n_i$, then

$-1 \le \lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)}, \cdots, y_{\varepsilon,i-1}^{(k)}, y_{\varepsilon,i}^{(k)}, y_{\varepsilon,i+1}^{(k-1)}, \cdots, y_{\varepsilon,N}^{(k-1)}\right) \le 1.$

And because $D_i Q_z^{\varepsilon}$ is continuous, we get (4.1) when $k \to \infty$.

(II) Otherwise, there is a subsequence, for simplicity denoted $y_{\varepsilon,i}^{(k)}$, such that $y_{\varepsilon,i}^{(k)} \ne 0, \ \forall k \in N$. Then, any such $y_{\varepsilon,i}^{(k)}$ satisfies the equation

$sign\left(y_{\varepsilon,i}^{(k)}\right) + \lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)}, \cdots, y_{\varepsilon,i-1}^{(k)}, 0, y_{\varepsilon,i+1}^{(k-1)}, \cdots, y_{\varepsilon,N}^{(k-1)}\right) = 0, \ \forall k \in N.$

At this time, we will show that there is an integer n_i and a constant $\sigma_{\varepsilon,i} \in \{-1, 1\}$, such that $k \ge n_i \ \Rightarrow \ sign\left(y_{\varepsilon,i}^{(k)}\right) = \sigma_{\varepsilon,i}$.

Suppose the contrary: for each k, there is $j_k > j$ so that $sign\left(z_i^{(j_k)}\right) = -sign\left(z_i^{(k)}\right)$. Then there is a subsequence, denoted $y_{\varepsilon,i}^{(k)}$, such that $sign\left(y_{\varepsilon,i}^{(k)}\right)=(-1)^k$ for all $k\in N$,

k is odd $\Rightarrow$ $\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)\geq 1.$

k is even $\Rightarrow$ $\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)\leq 1.$

This result contradicts the fact that since $Q_z^{\varepsilon}(y)$ is C^1, that is

$\lim_{k\to\infty} D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)=D_i Q_z^{\varepsilon}\left(\hat{y}_{\varepsilon}\right).$

(III) If $\sigma_i=1$, we have $y_{\varepsilon,i}^{(k)}>0,\ \forall k\geq n_i$. For $k\to\infty$, we have

$1+\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)=0.$

(IV) If $\sigma_i=-1$, you can similarly get $-1+\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},y_{\varepsilon,i}^{(k)},y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)=0.$

2^{o} If $\hat{h}^c$ is nonempty, for all $k\in N,\ \forall i\in\hat{h}^c$, we have

$sign\left(y_{\varepsilon,i}^{(k)}\right)+\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon,1}^{(k)},\cdots,y_{\varepsilon,i-1}^{(k)},0,y_{\varepsilon,i+1}^{(k-1)},\cdots,y_{\varepsilon,N}^{(k-1)}\right)=0.$

Put $\varsigma:=\frac{1}{2}\min_{i\in\hat{h}^c}\left|\hat{y}_{\varepsilon,i}\right|$, there is $n\in N$ such that $y_{\varepsilon}^{(k)}\in B\left(\hat{y}_{\varepsilon},\varsigma\right)$ for all $k\geq n.$ Since for every $i\in\hat{h}^c$, the function $y_{\varepsilon}\to\left|y_{\varepsilon,i}\right|+\lambda D_i Q_z^{\varepsilon}\left(y_{\varepsilon}\right)$ is continuous on $B\left(\hat{y}_{\varepsilon},\varsigma\right)$, at the limit $k\to\infty$, we get (4.2). ▌

4.2 Minimal Solution of the Energy Functional (2.3)

In this subsection, we first construct a minimizing sequence by using Lemma 2, and then use the mathematical induction method to prove the convergence of the minimal sequence, and get that the limit point of the minimal sequence is the minimum point of the minimal energy functional (2.3).

Taking a strictly monotone decreasing positive sequence $\varepsilon_n\downarrow 0,$ and $\varepsilon_0=1.$

For $\varepsilon=\varepsilon_0=1$, and given an initial point $y_{\varepsilon_0}^{(0)}=y_1^{(0)}$. At each iteration $k=1,2,\cdots$, the new iterate $y_{\varepsilon_0}^{(k)}$ is obtained from $y_{\varepsilon_0}^{(k-1)}$ by calculating successively each one of its entries $y_{\varepsilon_0,i}^{(k)}$ using one-dimensional minimization.

For all $i=1,2,\cdots,N$, find $y_{\varepsilon_0,i}^{(k)}$ such that

$$E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)\leq E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},t,y_{\varepsilon_0,i+1}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right),\ \forall t\in R. \quad (4.9)$$

According the inequality (4.5), the solution obtained at step $i-1$ of iteration k is

$\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right).$

Then the function $t\to E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},t,y_{\varepsilon_0,i+1}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)$ is strictly convex and 0-coercive. Hence, $y_{\varepsilon_0,i}^{(k)}$ is well defined and unique [14].

According to Theorem 2, we can get the following conclusion after calculating

Proposition 5 Supposing H^{-1} exists, and given $\varepsilon=\varepsilon_0=1,\ y_{\varepsilon_0}^{(0)}=y_1^{(0)}\in R^N$ as a starting point, for $k\to\infty$, the sequence $\left\{y_{\varepsilon_0}^{(k)}\right\}$ defined by (4.5) converges to a point $\hat{y}_{\varepsilon_0}$, such that

$E_z^{\varepsilon_0}\left(\hat{y}_{\varepsilon_0}\right)\leq E_z^{\varepsilon_0}\left(y\right),\ \forall y\in R^N.$

And there is a positive M such that $0 \le E_z^{\varepsilon_0}\left(\hat{y}_{\varepsilon_0}\right) \le M$, where M is only relevant with the fixed ε_0.

In a similar way, we can get some other conclusions

Proposition 6 Supposing H^{-1} exists, and given $\varepsilon = \varepsilon_1 < \varepsilon_0, y_{\varepsilon_1}^{(0)} = y_1^{(0)} \in R^N$ as a staring point, for $k \to \infty$, the sequence $\left\{y_{\varepsilon_1}^{(k)}\right\}$ defined by (4.5) converges to a point $\hat{y}_{\varepsilon_1}$, such that $E_z^{\varepsilon_1}\left(\hat{y}_{\varepsilon_1}\right) \le E_z^{\varepsilon_1}(y),\ \forall y \in R^N$.

Proposition 7 Supposing H^{-1} exists, and given $\varepsilon = \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_0,\ y_{\varepsilon_n}^{(0)} = y_1^{(0)} \in R^N$ as a starting point, for $k \to \infty$, the sequence $\left\{y_{\varepsilon_n}^{(k)}\right\}$ defined by (4.5) converges to a point $\hat{y}_{\varepsilon_n}$, such that $E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_n}\right) \le E_z^{\varepsilon_n}(y),\ \forall y \in R^N$.

In what follows, we will prove the convergence of the constructed sequence $\left\{\hat{y}_{\varepsilon_n}\right\}$ using the mathematical induction method.

Theorem 3 Suppose H^{-1} exists, if the sequence $\left\{\hat{y}_{\varepsilon_n}\right\}$ are the minimal solutions of the corresponding relaxation energy functional sequence $\left\{E_z^{\varepsilon_n}(y)\right\}$, for $n \to \infty,\ \varepsilon \to 0,$ there is a convergent subsequence $\left\{\hat{y}_{\varepsilon_{n_k}}\right\}$ limiting to $\hat{y}_0$, and the energy functional (2.3) reaches to the minimum at the point $\hat{y}_0$, that is $\lim\limits_{k\to\infty} \hat{y}_{\varepsilon_{n_k}} = \hat{y}_0$, and $E_z\left(\hat{y}_0\right) \le E_z(y),\ \forall y \in R^N$.

Proof: Here, we will prove the convergence of the sequence $\left\{\hat{y}_{\varepsilon_n}\right\}$ using the mathematical induction method. We only prove the case $\varepsilon_1 < \varepsilon_0$, so proved other cases in the similar way.

For any $i = 1, \cdots, N$, let $y_{\varepsilon_n, i}^{(k)}$ be the intermediate solution at step i of iteration k. According to the above notation, Notice

$$y_{\varepsilon_n,[i]}^{(k)} = \left(y_{\varepsilon_n,1}^{(k)}, \cdots, y_{\varepsilon_n,i-1}^{(k)}, y_{\varepsilon_n,i}^{(k)}, y_{\varepsilon_n,i+1}^{(k-1)}, \cdots, y_{\varepsilon_n,N}^{(k-1)}\right),$$

$$y_{\varepsilon_0}^{(0)} = y_1^{(0)} = \left(y_{1,1}^{(0)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right),\ y_{\varepsilon_0}^{(0)} = y_{\varepsilon_1}^{(0)} = \cdots = y_{\varepsilon_n}^{(0)}.$$

(a) Use the inductive hypothesis for any i when $k = 1$

1° For $i = 1$, notice $y_{\varepsilon_0,[1]}^{(1)} = \left(y_{\varepsilon_0,1}^{(1)}, y_{\varepsilon_0,2}^{(0)}, \cdots, y_{\varepsilon_0,N}^{(0)}\right),\ y_{\varepsilon_1,[1]}^{(1)} = \left(y_{\varepsilon_1,1}^{(1)}, y_{\varepsilon_1,2}^{(0)}, \cdots, y_{\varepsilon_1,N}^{(0)}\right)$.

By (4.9), we have

$$E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[1]}^{(1)}\right) = E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(1)}, y_{\varepsilon_0,2}^{(0)}, \cdots, y_{\varepsilon_0,N}^{(0)}\right) = E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(1)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right) \le E_z^{\varepsilon_0}\left(t, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right),$$

$$E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[1]}^{(1)}\right) = E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(1)}, y_{\varepsilon_1,2}^{(0)}, \cdots, y_{\varepsilon_1,N}^{(0)}\right) = E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(1)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right) \le E_z^{\varepsilon_1}\left(t, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right).$$

Since $\varepsilon_1 < \varepsilon_0$, and the uniqueness of $y_{\varepsilon_0,1}^{(1)},\ y_{\varepsilon_1,1}^{(1)}$, using Proposition 4, we have

$$E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(1)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right) \le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(1)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right) \le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(1)}, y_{1,2}^{(0)}, \cdots, y_{1,N}^{(0)}\right),$$

that is, $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[1]}^{(1)}\right) \le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[1]}^{(1)}\right)$.

2° Suppose that $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i]}^{(1)}\right) \le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i]}^{(1)}\right)$ holds when $i = i$, and $i < N$, that is

$$E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i]}^{(1)}\right) = E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(1)}, \cdots, y_{\varepsilon_1,i-1}^{(1)}, y_{\varepsilon_1,i}^{(1)}, y_{\varepsilon_1,i+1}^{(0)}, y_{\varepsilon_1,i+2}^{(0)}, \cdots, y_{\varepsilon_1,N}^{(0)}\right)$$

$$\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(0)},y_{\varepsilon_0,i+2}^{(0)},\cdots,y_{\varepsilon_0,N}^{(0)}\right)$$
$$\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(0)},y_{\varepsilon_0,i+2}^{(0)},\cdots,y_{\varepsilon_0,N}^{(0)}\right)=E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i]}^{(1)}\right).$$

3^{o} Now, we will prove that $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i+1]}^{(1)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i+1]}^{(1)}\right)$ holds when $i=i+1$ for every $i=1,\cdots,N-1$. Notice

$y_{\varepsilon_0,[i+1]}^{(1)}=\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(1)},y_{\varepsilon_0,i+2}^{(0)},\cdots,y_{\varepsilon_0,N}^{(0)}\right)$,

$y_{\varepsilon_1,[i+1]}^{(1)}=\left(y_{\varepsilon_1,1}^{(1)},\cdots,y_{\varepsilon_1,i-1}^{(1)},y_{\varepsilon_1,i}^{(1)},y_{\varepsilon_1,i+1}^{(1)},y_{\varepsilon_1,i+2}^{(0)},\cdots,y_{\varepsilon_1,N}^{(0)}\right)$,

According to 2^{o} and Proposition 4, we can get

$$\begin{aligned}E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i+1]}^{(1)}\right)&=E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(1)},\cdots,y_{\varepsilon_1,i-1}^{(1)},y_{\varepsilon_1,i}^{(1)},y_{\varepsilon_1,i+1}^{(1)},y_{\varepsilon_1,i+2}^{(0)},\cdots,y_{\varepsilon_1,N}^{(0)}\right)\\&\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(1)},y_{\varepsilon_1,i+2}^{(0)},\cdots,y_{\varepsilon_1,N}^{(0)}\right)\\&=E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(1)},y_{\varepsilon_0,i+2}^{(0)},\cdots,y_{\varepsilon_0,N}^{(0)}\right)\\&\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(1)},\cdots,y_{\varepsilon_0,i-1}^{(1)},y_{\varepsilon_0,i}^{(1)},y_{\varepsilon_0,i+1}^{(1)},y_{\varepsilon_0,i+2}^{(0)},\cdots,y_{\varepsilon_0,N}^{(0)}\right)=E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i+1]}^{(1)}\right).\end{aligned}$$

(b) Consider i when $k=k$

1^{o} For $i=1$, notice $y_{\varepsilon_0,[1]}^{(k)}=\left(y_{\varepsilon_0,1}^{(k)},y_{\varepsilon_0,2}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)$, $y_{\varepsilon_1,[1]}^{(k)}=\left(y_{\varepsilon_1,1}^{(k)},y_{\varepsilon_1,2}^{(k-1)},\cdots,y_{\varepsilon_1,N}^{(k-1)}\right)$, there is $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[1]}^{(k)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[1]}^{(k)}\right)$.

2^{o} Suppose that there is $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i]}^{(k)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i]}^{(k)}\right)$ holds when $i=i,\ i<N$, that is,

$$\begin{aligned}E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i]}^{(k)}\right)&=E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(k)},\cdots,y_{\varepsilon_1,i-1}^{(k)},y_{\varepsilon_1,i}^{(k)},y_{\varepsilon_1,i+1}^{(k-1)},y_{\varepsilon_1,i+2}^{(k-1)},\cdots,y_{\varepsilon_1,N}^{(k-1)}\right)\\&\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k-1)},y_{\varepsilon_0,i+2}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)=E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i]}^{(k)}\right).\end{aligned}$$

3^{o} Now, we will prove that $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i+1]}^{(k)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i+1]}^{(k)}\right)$ holds when $i=i+1$ for every $i=1,\cdots,N-1$. Notice

$y_{\varepsilon_0,[i+1]}^{(k)}=\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k)},y_{\varepsilon_0,i+2}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)$,

$y_{\varepsilon_1,[i+1]}^{(k)}=\left(y_{\varepsilon_1,1}^{(k)},\cdots,y_{\varepsilon_1,i-1}^{(k)},y_{\varepsilon_1,i}^{(k)},y_{\varepsilon_1,i+1}^{(k)},y_{\varepsilon_1,i+2}^{(k-1)},\cdots,y_{\varepsilon_1,N}^{(k-1)}\right)$.

According to 2^{o} and Proposition 4, we can get

$$\begin{aligned}E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i+1]}^{(k)}\right)&=E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(k)},\cdots,y_{\varepsilon_1,i-1}^{(k)},y_{\varepsilon_1,i}^{(k)},y_{\varepsilon_1,i+1}^{(k)},y_{\varepsilon_1,i+2}^{(k-1)},\cdots,y_{\varepsilon_1,N}^{(k-1)}\right)\\&\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k)},y_{\varepsilon_1,i+2}^{(k-1)},\cdots,y_{\varepsilon_1,N}^{(k-1)}\right)\\&\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k)},y_{\varepsilon_0,i+2}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)\\&\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k)},\cdots,y_{\varepsilon_0,i-1}^{(k)},y_{\varepsilon_0,i}^{(k)},y_{\varepsilon_0,i+1}^{(k)},y_{\varepsilon_0,i+2}^{(k-1)},\cdots,y_{\varepsilon_0,N}^{(k-1)}\right)=E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i+1]}^{(k)}\right).\end{aligned}$$

(c) Consider i when $k=k+1$

1^{o} For $i=1$, notice $y_{\varepsilon_0,[1]}^{(k+1)}=\left(y_{\varepsilon_0,1}^{(k+1)},y_{\varepsilon_0,2}^{(k)},\cdots,y_{\varepsilon_0,N}^{(k)}\right)$, $y_{\varepsilon_1,[1]}^{(k+1)}=\left(y_{\varepsilon_1,1}^{(k+1)},y_{\varepsilon_1,2}^{(k)},\cdots,y_{\varepsilon_1,N}^{(k)}\right)$.

According to Proposition 4 and the above assumptions when $k=k$, we have

$$E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[1]}^{(k+1)}\right)=E_z^{\varepsilon_1}\left(y_{\varepsilon_1,1}^{(k+1)},y_{\varepsilon_1,2}^{(k)},\cdots,y_{\varepsilon_1,N}^{(k)}\right)$$

$$\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(k+1)},y_{\varepsilon_1,2}^{(k)},\cdots,y_{\varepsilon_1,N}^{(k)}\right)\le E_z^{\varepsilon_1}\left(y_{\varepsilon_0,1}^{(k+1)},y_{\varepsilon_0,2}^{(k)},\cdots,y_{\varepsilon_0,N}^{(k)}\right)$$
$$\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,1}^{(k+1)},y_{\varepsilon_1,2}^{(k)},\cdots,y_{\varepsilon_1,N}^{(k)}\right)=E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[1]}^{(k+1)}\right).$$

2^{o} Suppose that there is $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i]}^{(k+1)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i]}^{(k+1)}\right)$ holds when $i=i,\ i<N$. In the similar way, we can get $E_z^{\varepsilon_1}\left(y_{\varepsilon_1,[i+1]}^{(k+1)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0,[i+1]}^{(k+1)}\right)$ holds when $i=i+1,\ i+1\le N$.

Considering the above results after using the mathematics method in the case $0<\varepsilon_1<\varepsilon_0=1$, we can see $E_z^{\varepsilon_1}\left(y_{\varepsilon_1}^{(k+1)}\right)\le E_z^{\varepsilon_0}\left(y_{\varepsilon_0}^{(k+1)}\right),\ \forall k\in N,$ where

$$y_{\varepsilon_0}^{(k+1)}=\left(y_{\varepsilon_0,1}^{(k)},y_{\varepsilon_0,2}^{(k)},\cdots,y_{\varepsilon_0,N}^{(k)}\right),\quad y_{\varepsilon_1}^{(k+1)}=\left(y_{\varepsilon_1,1}^{(k)},y_{\varepsilon_1,2}^{(k)},\cdots,y_{\varepsilon_1,N}^{(k)}\right)$$

According to Theorem 2, we can gain that the two sequences $y_{\varepsilon_1}^{(k+1)}$ and $y_{\varepsilon_0}^{(k+1)}$ both converge when $k\to\infty$. Hence, there is

$$0\le E_z^{\varepsilon_1}\left(\hat{y}_{\varepsilon_1}\right)=\lim_{k\to\infty}E_z^{\varepsilon_1}\left(y_{\varepsilon_1}^{(k+1)}\right)\le\lim_{k\to\infty}E_z^{\varepsilon_0}\left(y_{\varepsilon_0}^{(k+1)}\right)=E_z^{\varepsilon_0}\left(\hat{y}_{\varepsilon_0}\right)\le M,$$

where $M(>0)$ is only relevant with the fixed ε_0.

In the same way, we can prove the following conclusion

$$0\le\cdots\le E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_n}\right)\le\cdots\le E_z^{\varepsilon_1}\left(\hat{y}_{\varepsilon_1}\right)\le E_z^{\varepsilon_0}\left(\hat{y}_{\varepsilon_0}\right)\le M.$$

Then, the energy sequence $E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_n}\right)$ is bound, that is,

$$\sum_{i=1}^{N}\left|\hat{y}_{\varepsilon_n,i}\right|\le E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_n}\right)=\sum_{i=1}^{N}\left|\hat{y}_{\varepsilon_n,i}\right|+\lambda Q_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_n}\right)\le M.$$

So, $\left\|\hat{y}_{\varepsilon_n}\right\|\le M$. There is a subsequence $\hat{y}_{\varepsilon_{n_k}}$ such that $\lim_{k\to\infty}\hat{y}_{\varepsilon_{n_k}}=\hat{y}_0$. Using Lemma 1, we can get $E_z^{\varepsilon_n}\left(\hat{y}_0\right)=\lim_{k\to\infty}E_z^{\varepsilon_n}\left(\hat{y}_{\varepsilon_{n_k}}\right)\le E_z^{\varepsilon_n}\left(y\right),\ \forall y\in R^N$. And since $\varepsilon_n\to 0$ ($n\to\infty$), based on Proposition 1, we have

$$E_z\left(\hat{y}_0\right)=\lim_{n\to\infty}E_z^{\varepsilon_n}\left(\hat{y}_0\right)\le\lim_{n\to\infty}E_z^{\varepsilon_n}\left(y\right)=E_z\left(y\right),\ \forall y\in R^N. \ ▌$$

5. Numerical Experimental and Simulation Results

In this section, we will provide some experimental results, which show that our proposed image restoration model based on the bilateral total variance regularization term works very well. To simplify the numerical calculation, we use the steepest descent method to find the minimal solution of the energy functional (2.1). So, the evolution equation

$$\frac{\partial u}{\partial\tau}=-\sum_{r=1}^{R}H_r^{*}\cdot sign\left(H_r u-z_r\right)-\lambda\sum_{k=-p_1}^{p_1}\sum_{l=-p_2}^{p_2}\alpha^{|k|+|l|}\left(I-S_x^{-k}S_y^{-l}\right)\cdot sign\left(u-S_x^{k}S_y^{l}u\right)$$

where H_r^{*} represents conjugate transpose of the Toeplitz matrix H_r.

In our experiments, we adopt the normal Camera-man image with 256×256 pixels, p_1 and p_2 are chosen as 1 or 2, and add Gaussian noise with zero-mean, σ^2 -variance, or Gaussian blurring. In what follows, signal-to-noise-ratio(SNR) is computing as follows

$$SNR=\frac{\sum_{i,j}\left(u_{ij}-\bar{u}_{ij}\right)^2}{\sum_{i,j}\left(z_{ij}-u_{ij}\right)^2},\quad \bar{u}=\frac{\sum_{i,j}u_{ij}}{m\times n}.$$

where u is an original image, z is a noisy image. All codes are performed in Matlab. (Note that B/N means blurred and noisy.)

(a) (b) (c) (d)

Figure 1. Restoration of Single Noisy Image, but without Blurring (a) Original Image, (b) Noisy Image Noise Variance $\sigma^2 = 0.1$, but no Blurring (SNR=3.03 dB), (c)Result of ROF Model (SNR=4.46 dB), (d) Result of our Model (SNR=5.15 dB)

In Figure 1, we denoise using single noisy image without blurring, and compare the restoration result of our model with that of ROF model. In Figure 1(c), using the ROF model, we can see that the whole image will become constant in the end, but in Figure 1(d), we can get that the local domain will be constant by using our proposed model. Through comparing the SNR in Figure 1(c) with that in Figure 1(d), we obtain that our proposed model works better than the ROF model does while employing only one noisy image.

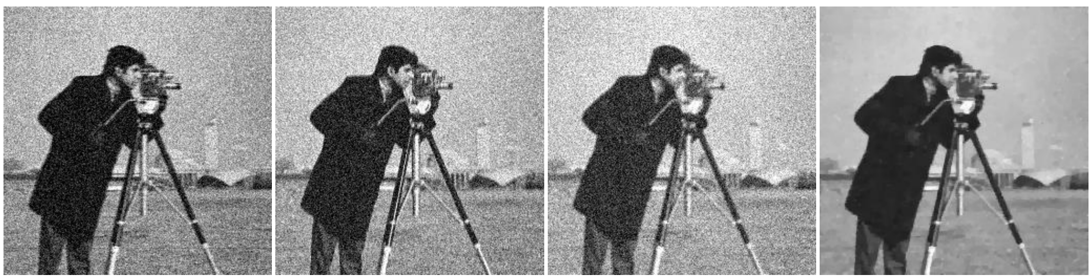
(a) (b) (c) (d)

Figure 2. Restoration of Three Different Noisy Images, but without Blurring (a) The 1st Noisy Image, Noise Variance $\sigma^2 = 0.05$, but no Blurring(SNR=4.95 dB), (b)The 2nd Noisy Image, Noise Variance $\sigma^2 = 0.1$, but no Blurring (SNR=3.05 dB), (c)The 3rd Noisy Image, Noise Variance $\sigma^2 = 0.15$, but no Blurring (SNR=1.89 dB), (d) Result of our Model (SNR=6.50 dB)

In order to show the advantage of our proposed model, which can restore image by using multiple degraded images, we do two experiments in Figure 2 and Figure 4. Through comparing Figure 1(d) with Figure 2 (d), we can receive the fact that the effect of using multiple noisy images to restore is better than that of using only single degraded image.

(a) (b)

Figure 3. Restoration of Single Noisy Image with Blurring (a) B/N Image, Noise Variance $\sigma^2 = 0.1$, with 1×3 Horizontal Motion Blurring (SNR=2.68 dB), (b) Result of our Method (SNR=4.57 dB).



(a) (b) (c) (d)

Figure 4. Restoration of Three Different Noise Images with Three Different Blurring (a) The 1st B/N Image, Noise Variance $\sigma^2 = 0.05$, with 1×3 Horizontal Motion Blurring (SNR=4.32 dB), (b)The 2nd B/N Image, Noise Variance $\sigma^2 = 0.1$, with 3×1 Vertical Motion Blurring (SNR=2.66 dB), (c)The 3rd B/N Image, Noise Variance $\sigma^2 = 0.15$, with 3×3 Gaussian Blurring, Gaussian Kernel Variance $\sigma_0 = 3$ (SNR=1.59 dB), (d) Result of our Model (SNR=5.34 dB)

In Figure 3, we use single noisy image with blurring to denoise and deblur. In Figure.4, we adopt multiple degraded images with different noise and different blurring to wipe off noise and restore deblurring. Comparing Figure 1(d) and Figure 3(b) with Figure 2(d) and Figure 4(d), respectively, we conclude that our proposed model works better by using multiple degraded images than it does by only single degraded image, even if the degraded images include noise and blurring.

6. Discussion and Conclusion

Based on many variational models mentioned in Section 1, we present a unified method for image denoising and deblurring by using single or multiple degraded images. In our model, we use non-convex ℓ^1-norm data-fidelity term and ℓ^1-norm regularization term. Although we have proved the existence of discrete numerical form of the minimal functional (2.3), we don't adopt the discrete format to do our numerical experiments because of the complexity and difficulty of taking a strictly monotone decreasing positive sequence $\{\varepsilon_n\}$, then we employ the steepest descent method to find the minimal solution of energy functional(2.3). However, experiments show that firstly, our model works better that the ROF model when single degraded image is considered; Secondly, the restoration results with multiple degraded images yields better than with single degraded image.

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