

## Performance Analysis of Basis Functions in TVAR Model

G. Ravi Shankar Reddy<sup>1</sup> and Dr. Rameshwar Rao<sup>2</sup>

<sup>1</sup>Dept.of ECE, CVR College of Engineering, Hyderabad, India

<sup>2</sup>Vice - Chancellor, JNT University, Hyderabad, India

<sup>1</sup>ravigosula\_ece39@yahoo.co.in, <sup>2</sup>rameshwar\_rao@hotmail.com

### Abstract

*In this paper Time-varying Auto regressive model (TVAR) based approach for instantaneous frequency (IF) estimation of the nonstationary signal is presented. Time-varying parameters are expressed as a linear combination of constants multiplied by basis functions. Then, the time-varying frequencies are extracted from the time-varying parameters by calculating the angles of the estimation error filter polynomial roots. Since there were many existing basis functions that could be used as basis for the TVAR parameter expansion, one might be interested in knowing how to choose them and what difference they may cause. The performance of different basis functions in TVAR modeling approach is tested with synthetic signals. Our objective is to find an efficient basis for all testing signals in the sense that, for a small number of basis (or) expansion dimension, the basis yields the least error in frequency. In this paper, the optimal basis function of TVAR Model for the instantaneous frequency (IF) estimation of the test signals was obtained by comparing IF estimation precise and anti-noise performance of several types basis functions through simulation.*

**Keywords:** Instantaneous frequency estimation, basis functions, Time-varying autoregressive model, nonstationary signal

### 1. Introduction

Nonstationary signal modeling is a research topic of practical interest, because most temporal signals encountered in real applications, such as speech, biomedical, seismic and radar signals have time-varying statistics[1, 2].The problem of time dependency was usually circumvented by assuming local stationary over a relatively short time interval, in which stationary system identification and analysis techniques are applied. However, this assumption is not always suitable, and methods for nonstationary processes are needed.

Nonstationary signal analysis methods can be categorized into nonparametric and parametric [3]. The nonparametric approaches are based on time-dependent spectral representations, and include the short-time Fourier transform, the time frequency distribution and the evolutionary spectrum due to the uncertainty principle, one cannot get both high time and frequency resolutions using these nonparametric methods[4]

The parametric approaches are based on the linear time-varying (TV) model, in which a nonstationary process is represented using an AR, MA or ARMA model with parameters changing with time. The TV spectrum can be estimated from the TV model parameters, and the instantaneous frequency of the nonstationary signal can be extracted. In contrast with nonparametric approaches, good accuracy in signal representation and high frequency resolution in spectral estimation can be obtained by using parametric approaches even for short data sequences [5]

A time-varying autoregressive (TVAR) approach is used for modeling non stationary signals, and frequency information is then extracted from the TVAR parameters. Two methods are used for estimating the TVAR parameters [6]: the adaptive recursive estimation method and deterministic basis function expansion method. The adaptive recursive estimation methods are stochastic approach, where the coefficients of the associated models are treated as random processes with some stochastic model structure; the most popular methods to deal with this class of models are the least mean square (LMS) and the recursive least square (RLS), and Kalman filtering algorithms. The basis function expansion method is a deterministic parametric modeling approach, where the associated time-varying coefficients are expanded as a finite sequence of pre-determined basis functions; generally, these coefficients are expressed using a linear (or) nonlinear combination of a finite number of such basis functions. The problem then becomes time invariant, and the unknown new adjustable model parameters are those involved in the expansions. Hence, the initial time-varying modeling problem is reduced to deterministic regression selection and parameter estimation.

Adaptive algorithms, such as the least mean square (LMS) and the recursive least square (RLS) use a dynamic model for adapting the TVAR parameters and are capable of tracing signals with weak (or) medium non-stationary dynamics. Adaptive algorithms are sensitive to the noise. They also failed to track the time-varying frequency of the signal, if the frequency changed very fast. However, they were efficient in tracking the frequency jump [7].

The basis function method is capable of tracing both the fast and the slow time-varying frequencies. A key advantage of using basis functions is that a considerable reduction in the number of parameters needed to track each TV coefficients can be obtained [7]. Hence, this model is focused in the present study. However, the selection of the expansion dimension and the basis function is questionable since there is no fundamental theorem on how to choose them [7]. It is ideally expected that when the expansion dimension is infinite, the result of the frequencies estimation from any basis function is the same, which will exactly equal to the true frequency [7]. But this is impractical, since the computation may require infinite memory, and infinite computational time consumption.

Numerous solutions have been projected, in the literature such as time basis functions, Legendre polynomial, Chebyshev polynomial, Discrete prolate spheroidal sequence (DPSS), Fourier basis, Discrete cosine basis, Walsh basis, none of these solutions seems to be perfect, since the selection of  $u_{m,n}$  desires some priori information upon the time variations present in  $x_n$ .

It is generally mentioned in [12] that there are possibly two ways for selecting the basis function. If some prior knowledge about the physical process of time variation is available, the basis functions should be chosen such that the prominent trends in parameter change is retained. If a priori information is unavailable, which might be the case for complicated physical systems, Niedzwiecki, [12] suggests that the selection of the basis function should rely on general approximation, such as the Taylor and Fourier series approximation, and the function most commonly used are the time polynomial and the Fourier, or the cosine functions, since their realization is easy, and they can fairly accurate a broad range of variations.

The paper is presented as follows. It explains the Time-varying Autoregressive modeling in Section 2. In Section 3 it explains the selection of basis and order determination. In Section 4 it gives the steps to estimate IF, based on TVAR model. The investigational results of estimating IF in noisy environment using different basis functions are presented in Section 5. Concluding remarks are given in Section 6.

## 2. TVAR Modeling

The non stationary discrete-time stochastic process  $x_n$  is represented by  $p^{\text{th}}$  order TVAR model as

$$x_n = - \sum_{k=1}^p a_{k,n} x_{n-k} + v_n \quad (1)$$

Here  $a_{k,n}$  are time-varying coefficients and  $v_n$  is a stationary white noise process and whose mean is zero and variance is  $\sigma_v^2$ . According to the time-varying coefficients evolution, TVAR is likely to be categorized in to two group's i.e. adaptive method and basis function approach.

TVAR model based on the basis function technique is able to trace a strong non-stationary signal, that's why this model is focused in the present study. In this technique, each of its time-varying coefficients are modeled as linear combination of a set of basis functions [6].

The purpose of the basis is to permit fast and smooth time variation of the coefficients. If we denote  $u_{m,n}$  as the basis function and consider a set of  $(q + 1)$  function for a given model, we can state the TVAR coefficients in general as

$$a_{k,n} = \sum_{m=0}^q a_{km} u_{m,n} \quad (2)$$

From (2) we examine that, we have to calculate the set of parameters  $a_{km}$  for  $\{k=1,2,\dots,p; m=0,1,2,\dots,q; a_{0m}=1\}$  in order to compute the TVAR coefficients  $a_{k,n}$  and the TVAR model is absolutely specified by this set. The TVAR coefficients are designed as follows, we consider single realization of the process  $x_n$ . For a given realization of  $x_n$  we can analyze (1) as a time-varying linear prediction error filter and consider  $v_n$  to be the prediction error

$$v_n = x_n - \hat{x}_n \quad (3)$$

$$\text{where } \hat{x}_n \triangleq - \sum_{k=1}^p a_{k,n} x_{n-k} \quad (4)$$

The total squared prediction error, which is as well as the error in modeling  $x_n$ , is now specified by

$$\epsilon_p = \sum_{\tau} |v_n|^2$$

Substitute (2) in (4) and the prediction error  $v_n$  can be written as

$$v_n = x_n + \sum_{k=1}^p \sum_{m=0}^q a_{km} u_{m,n} x_{n-k} \quad (5)$$

The total squared prediction error can be formulated as

$$\epsilon_p = \sum_{\tau} \left| x_n + \sum_{k=1}^p \sum_{m=0}^q a_{km} u_{m,n} x_{n-k} \right|^2 \quad (6)$$

For modeling the non stationary stochastic process  $x_n$ , using covariance technique, we make no assumptions on the data outside  $[0, N-1]$ . In equation (6)  $\tau$  is the interval over which the summation is performed and set  $\tau = [p, N - 1]$ . By minimizing the mean squared prediction error in (6) we can estimate the time-varying parameters  $a_{km}$  [6]. We can minimize the mean squared prediction error in (6) by means of setting the gradient of  $\epsilon_p$  with respect to  $a_{lg}^*$  zero

$$\frac{\partial \epsilon_p}{\partial a_{lg}^*} = \sum_{\tau} \frac{\partial v_n v_n^*}{\partial a_{lg}^*} = \sum_{\tau} v_n \frac{\partial v_n^*}{\partial a_{lg}^*} = 0 \quad (7)$$

$$\{l = 1, 2, \dots, p; g = 0, 1, \dots, q\}$$

Where

$$v_n^* = x_n^* + \sum_{l=1}^p \sum_{g=0}^q a_{lg}^* u_{g,n}^* x_{n-l}^*$$

And the derivative of  $v_n^*$  with respect to  $a_{lg}^*$

$$\frac{\partial v_n^*}{\partial a_{lg}^*} = u_{g,n}^* x_{n-l}^*$$

Consequently (7) becomes,

$$\sum_{\tau} v_n u_{g,n}^* x_{n-l}^* = 0 \quad (8)$$

The above mentioned condition is similar to the orthogonality law encountered in stationary signal modeling. Substitute (5) in (8) we have

$$\sum_{\tau} \left( x_n + \sum_{k=1}^p \sum_{m=0}^q a_{km} u_{m,n} x_{n-k} \right) u_{g,n}^* x_{n-l}^* = 0 \quad (9)$$

Now we define a function  $c_{mg}(l, k)$  as shown below,

$$c_{mg}(l, k) \triangleq \sum_{\tau} u_{m,n} x_{n-k} u_{g,n}^* x_{n-l}^* \quad (10)$$

Using the above definition in (9) we have,

$$\sum_{k=1}^p \sum_{m=0}^q a_{km} c_{mg}(l, k) = -c_{0g}(l, 0) \quad (11)$$

The above equation represents a system of  $p(q+1)$  linear equations. The above system of linear equations can be efficiently represented in matrix form as follows.

Define a column vector  $a_m$  as follows

$$a_m = [a_{1m} \ a_{2m} \ \dots \ a_{pm}]^T, \quad \text{where } m = 0, 1, \dots, q \quad (12)$$

We can use the function (10) to find the following matrix for  $0 \leq (m, g) \leq q$

$$C_{mg} = \begin{bmatrix} c_{mg}(1,1) & c_{mg}(1,2) & \cdots & c_{mg}(1,p) \\ c_{mg}(2,1) & c_{mg}(2,2) & \cdots & c_{mg}(2,p) \\ \vdots & \vdots & \ddots & \vdots \\ c_{mg}(p,1) & c_{mg}(p,2) & \cdots & c_{mg}(p,p) \end{bmatrix} \quad (13)$$

The above matrix is of size  $p \times p$  and all the different values for  $m$  and  $g$  resulting in  $(q+1) \times (q+1)$  such matrices, by means of these matrices, we can now describe a block matrix as shown below,

$$C = \begin{bmatrix} C_{00} & C_{01} & \cdots & C_{0q} \\ C_{10} & C_{11} & \ddots & C_{1q} \\ \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot \\ C_{q0} & C_{q1} & \cdots & C_{qq} \end{bmatrix} \quad (14)$$

The above Block matrix  $C$  has  $(q+1) \times (q+1)$  elements and each element is a matrix of size  $p \times p$ , which implies the Block matrix  $C$  of size  $p(q+1) \times p(q+1)$ .

Now we describe a column vector  $d_m$  as shown below

$$d_m = [c_{0m}(1,0) \quad c_{0m}(2,0) \quad \cdots \quad c_{0m}(p,0)]^T \quad (15)$$

where  $m = 0, 1, \dots, q$  By using the definitions from (12)-(15) we can represent the system of linear equations in (11) in a compact matrix form as follows

$$\underbrace{\begin{bmatrix} C_{00} & \cdots & C_{0q} \\ \vdots & \ddots & \vdots \\ C_{q0} & \cdots & C_{qq} \end{bmatrix}}_c \underbrace{\begin{bmatrix} a_0 \\ \vdots \\ a_q \end{bmatrix}}_a = - \underbrace{\begin{bmatrix} d_0 \\ \vdots \\ d_q \end{bmatrix}}_d \quad (16)$$

$$Ca = -d$$

The above equations reduce to the Yule-walker equations (YWE) for a stationary AR model, as soon as  $q=0$ . By solving the above matrix equation, we can obtain the set of TVAR parameters  $a_{km}$  (elements of  $a$ ), the predictor coefficients  $a_{k,n}$  can be calculated using (2)

### 3. Parameter Selection

The TVAR parameter selection is basically depends on three degrees of freedom, such as the TVAR order  $p$ , the basis function dimension  $q$ , and the set of basis functions  $u_{m,n}$ .

#### 3.1. Choice of the Basis Functions

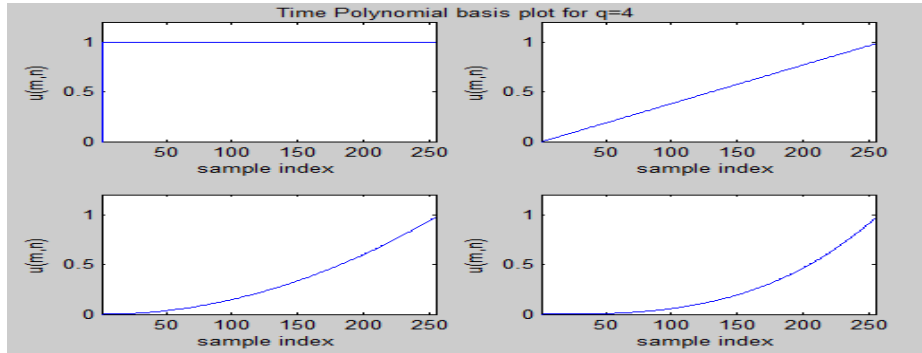
The basis functions  $u_{m,n}$  must be independent and non-zero for  $n=0, 1, \dots, N-1$ , and  $u_{m,n}=1$ , for  $n=0$ . If a priori information about the signal variation is known, the basis functions should be chosen such that the trends in parameter change is retained. In case, when a priori information is unavailable selection of basis is trial and error. According to equation (2), no particular constraint is imposed on the basis  $u_{m,n}$ , consequently one will be able to track only variations which are approximable by this set of functions. Numerous solutions have been projected, in the literature such as time basis functions, Legendre polynomial, Chebyshev polynomial, Discrete prolate spheroidal (DPSS) sequence, Fourier basis, Discrete

cosine basis, Walsh basis, none of these solutions seems to be perfect, since the selection of  $u_{m,n}$  desires some priori information upon the time variations present in  $x_n$ . In this article we compare 7 different basis functions.

**1. Time Polynomial Basis Function**

$$u_{m,n} = \left(\frac{n-p}{N}\right)^m \tag{17}$$

$m = 0, 1, \dots, \dots, q; n = p, p + 1, \dots, N - 1.$   
 N denotes the length of the data record that is being modeled.



**Figure 1. Time Polynomial Basis Function for q=4**

**2. Legendre Polynomial Basis Function**

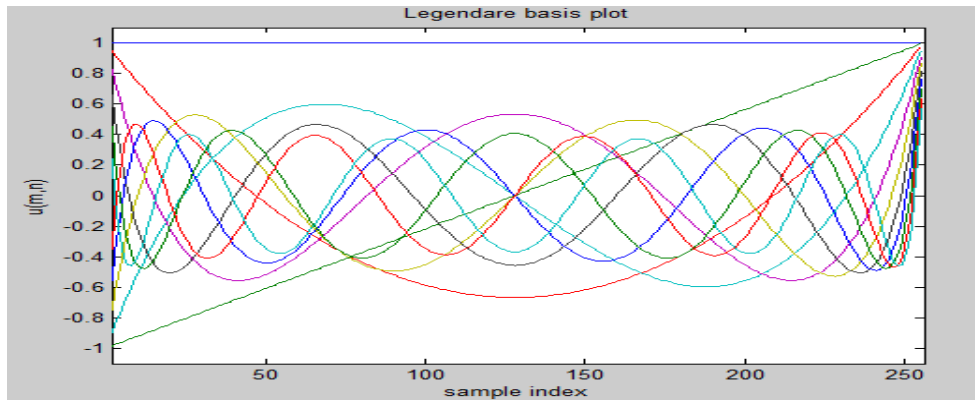
$$u_{0,n} = 1,$$

$$u_{1,n} = \frac{2(n-1)}{N-2} = y, \tag{18}$$

$$u_{2,n} = \frac{3y^2-1}{2}$$

$$u_{m+1,n} = \frac{(2m+1)yu_{m,n} - mu_{m-1,n}}{(m+1)}$$

Where  $m=0, 1, 2, \dots, q, n=1, 2, \dots, N$



**Figure 2. Legendre Polynomial Basis Function for q=9**

### 3. Chebyshev Basis Function

$$u_{m,n} = \cos[m \cos^{-1}(k - 1)] \quad (19)$$

Where  $k = \frac{2(n-1)}{(N-1)}$  and  $1 \leq n \leq N$

$m = 0, 1, 2, \dots, q$

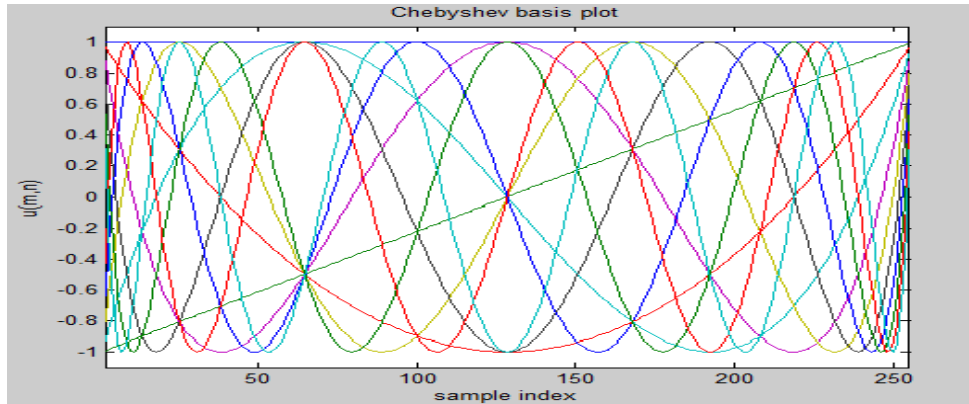


Figure 3. Chebyshev Polynomial Basis Function for  $q=9$

### 4. Fourier Basis Function

$$u_{m,n} = \begin{cases} \cos(\omega mn) & \text{for even values of } m \\ \sin(\omega mn) & \text{for odd values of } m \end{cases} \quad (20)$$

Where  $\omega = \frac{1}{N}$

$m = 0, 1, 2, \dots, q, n = 1, 2, \dots, N$

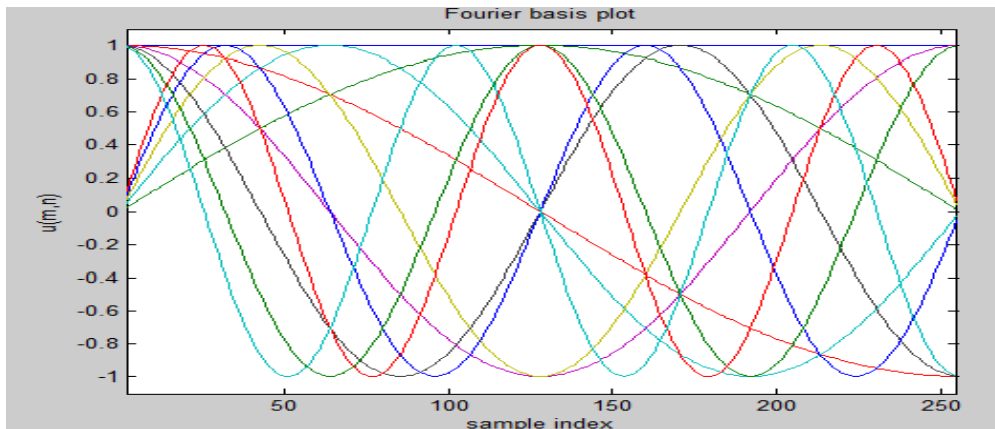


Figure 4. Fourier Basis Function for  $q=9$

### 5. Discrete Cosine Basis Function

$$u_{m,n} = \alpha(m) \cos\left(\frac{\pi m(2n+1)}{2N}\right)$$

Where

$$\alpha(m) = \begin{cases} \sqrt{\frac{1}{N}} & m = 0 \\ \sqrt{\frac{2}{N}} & m = 0,1,2 \dots \dots q \end{cases} \quad (21)$$

$n=1,2,\dots,N$

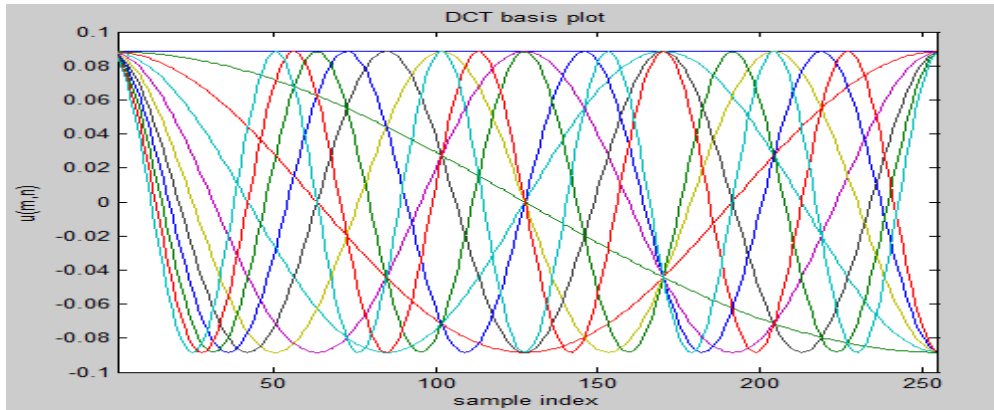


Figure 5. Discrete Cosine Basis Function for  $q=9$

### 6. Discrete Prolate Spheroidal Sequence (DPSS):

$u_{m,n}$  =  $m^{th}$  sequences most concentrate in the frequency band  $|w| \leq 2\pi W$ , where  $W$  is half bandwidth.

Where  $n = 0,1,2,\dots,N$   
 $m = 0,1,2,\dots,q$

$N$  is the total number of samples

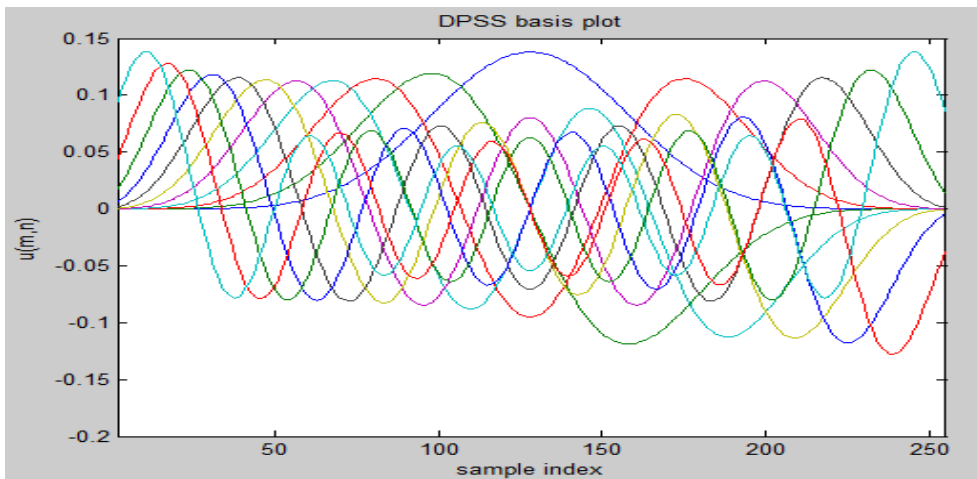


Figure 6. Discrete Prolate Spheroidal Sequence for  $q=9$



### 7. Walsh Basis Function

Walsh functions consisting of a number of fixed-amplitude square pulses interposed with zeros. Walsh functions can be generated a number of ways. One is to define Walsh functions as the following

$$u_{m,n} = \prod_{r=0}^{R-1} \text{sgn}(\cos m_r 2^r \pi n) \tag{22}$$

Where  $n=0, 1, 2, \dots, N$ ;  $m=0, 1, 2, \dots, q$ ;  $R=2^N$

$$m = \sum_{r=0}^{R-1} m_r 2^r$$

Here  $m_r = 0$  (or)  $1$

For example,  $u_{5,n}$  has the following form:

$$m=5=1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0,$$

$$m_2=1, m_1=0, m_0=1,$$

$$u_{5,n} = \text{sgn}(\cos m_2 2^2 \pi n) \text{sgn}(\cos m_1 2^1 \pi n) \text{sgn}(\cos m_0 2^0 \pi n)$$

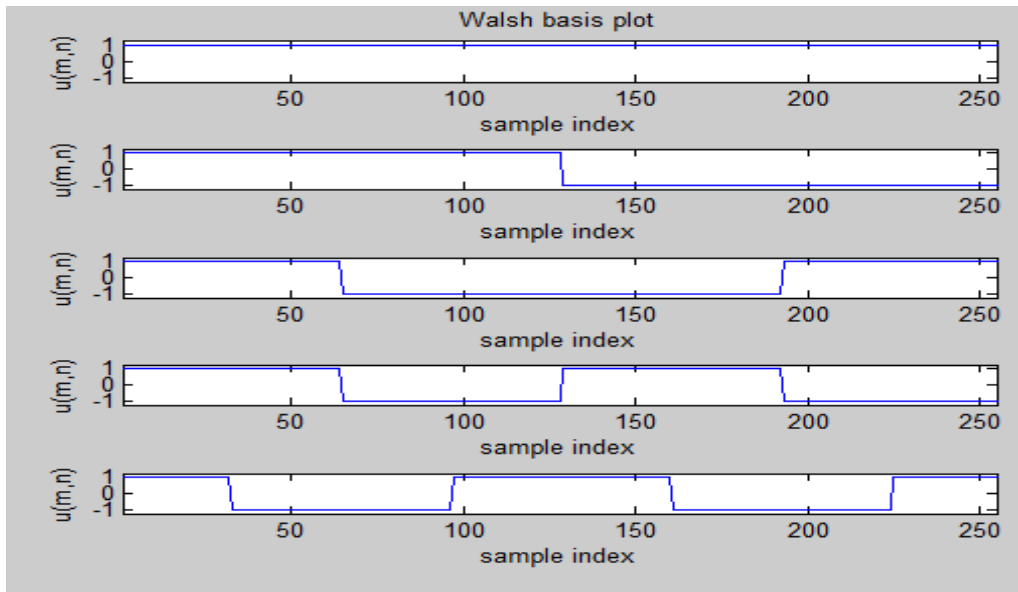


Figure 7. Walsh Basis Function for q=4

### 3.2. Order Selection

In the presence of noise The TVAR model can distinguish several time-varying spectral peaks well. However it is sensitive to model order change. False spectral peaks may be produced by the TVAR modeling approach, when an erroneous model order is chosen. Thus, the determination of right model order in TVAR modeling is a significant issue. There are few techniques in choice of TVAR model order. For instance, Bayesian technique [15] and Akaike information criterion (AIC) [16] are used for the determination of model orders in

TVAR models. In this article, we consider the choice of model order as a Maximum-likelihood (ML) estimation [17] technique. In this technique, by maximizing the likelihood function we can determine the model order

### Maximum Likelihood Estimation (MLE)

The TVAR Model for the non stationary discrete-time stochastic process  $x_n$  is

$$x_n = - \sum_{k=1}^p \sum_{m=0}^q a_{km} u_{m,n} x_{n-k} + v_n$$

The above can be represented in compact form as

$$x_n = - Z^T[n]a + v_n \tag{23}$$

Where  $Z[n]$  is

$$Z[n] = \Phi[n] \otimes u[n] \tag{24}$$

Here,  $\otimes$  denote Kronecker multiplication.

$$\Phi[n] = [x_{n-1}, x_{n-2} \dots \dots x_{n-p}]^T \tag{25}$$

$$u[n] = [u_{0n}, u_{1n} \dots \dots u_{qn}]^T \tag{26}$$

Moreover

$$a = [a_1^T, a_2^T \dots \dots a_p^T] \tag{27}$$

Here

$$a_k^T = [a_{k0}, a_{k1} \dots \dots a_{kq}] \tag{28}$$

Step-1: compute

$$Z[n] = \Phi[n] \otimes u[n]$$

Step-2: calculate

$$C = - \left( \sum_{n=p}^N Z[n] Z^T[n] \right)^{-1} \left( \sum_{n=p}^N Z[n] x[n] \right) \tag{29}$$

Step-3: Estimate

$$\hat{\beta} = \frac{1}{N} \sum_{n=0}^N [x_n - C^T Z[n]]^2 \tag{30}$$

Step-4: Obtain the cost function

$$J(p, q) = \frac{p(q + 1) + 2 - N}{2} \log(2\pi\hat{\beta}) - \frac{1}{2} \log \left| \sum_{n=0}^N Z[n]Z^T[n] \right| \quad (31)$$

Step-5: Maximize the above cost function to select the expansion dimension  $q=q_{opt}$  and the model order  $p=p_{opt}$ , where

$$p_{opt} \in \{1, 2, 3, 4, \dots, p_{max}\}$$

$$q_{opt} \in \{0, 1, 2, 3, 4, \dots, q_{max}\}$$

#### 4. Instantaneous Frequency Estimation using TVAR Model

- 1) Compute TVAR model order  $p$  and  $q$  using MLE Algorithm choose the basis function  $u_{m,n}$   $m=1, 2, \dots, q, n=1, 2, \dots, N$
- 2) For covariance technique of signal modeling set  $\tau = [p, N - 1]$  and compute  $c_{m,g}(l, k)$  by means of equation (10) to find the matrix  $C_{mg}$  in (13), subsequently, set up the matrix  $C$  in (14), as well, use  $c_{m,g}(l, k)$  to calculate  $d_g$  in (15)
- 3) Calculate the TVAR parameters  $a_{km}$  by solving  $Ca = -d$ , in (16) and form the coefficients  $a_{k,n}$  using (2)
- 4) Solve the roots of the time-varying autoregressive polynomial formed by TVAR linear prediction filter.  $A(z; n) = 1 + \sum_{k=1}^p a_{k,n}z^{-k}$  at each instant  $n$  to find the time-varying poles:  $P_{i,n}, i=1, 2, \dots, p$
- 5) The Instantaneous frequency (IF) of the non stationary signal, for each sample instant  $n$  can be estimated from the instantaneous angles of the poles using the formula

$$f_{i,n} = \frac{\arg[P_{i,n}]}{2\pi} \text{ for } |P_{i,n}| \simeq 1$$

#### 5. Simulation Results

In this section, seven basis functions (time polynomial, Legendre polynomial, Chebyshev polynomial, DPSS, Fourier, Discrete cosine, Walsh) are compared based on the accuracy in estimating the time-varying frequency of testing signals. Our experience shows that different basis functions show their own unique tractability and accuracy. The time-polynomial and the Legendre functions yield the same result. This is because both functions are linearly related and capable of spanning exactly the same subspace.

For our comparison in the time-varying frequency estimation, all the seven basis functions are tested with synthetic signals. Our objective is to find an efficient basis for all testing signals in the sense that, for a small number of basis (or) expansion dimension, the basis yields the least error in frequency estimation.

Order  $p$  and dimension  $q$  (obtained from the MLE Algorithm) was considered suitable in that it is the smallest, but yields approximately the least error in frequency estimation (*i.e.*, increasing  $q$  higher than this number would not yield much difference in estimation error).

Basis functions were tested in a noise environment with SNR=20dB, with eight synthetic signals. These eight signals were generated such that their time-varying frequencies were exactly known. We concluded that the basis function is best suitable for a given test signal in the sense that, the TVAR Model order  $p$  and Basis function dimension  $q$  is low to get the

least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension is high

**1) Signal test 1** was a linear chirp that has a linearly-varying frequency from  $0.01F_s$  to  $0.45F_s$  over 256 samples. This signal was generated by using equation  $x_1(n)=\cos(\pi\mu n^2+2\pi f_1 n)$  where  $\mu=1.6e5, f_0=100\text{Hz}$ , and sampling rate  $F_s=10000\text{Hz}$ .

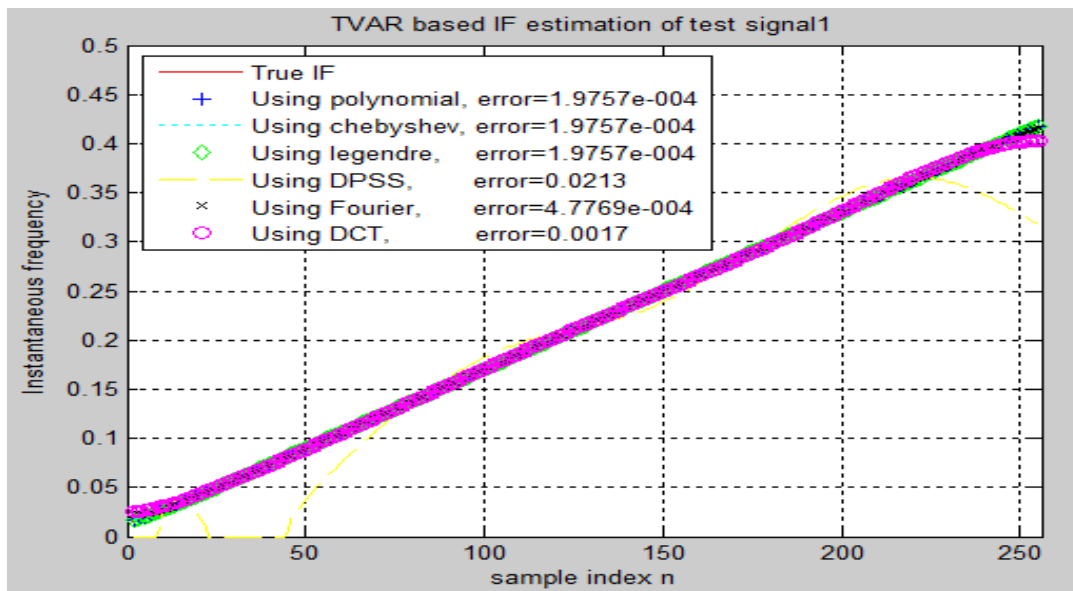
$$\text{IF law } f_i(n)=f_0 + \mu n, \quad 1 \leq n \leq 256 \quad (32)$$

TVAR model order for different basis are computed using Maximum likelihood estimation (MLE) Algorithm, the results are tabulated below in Table 1

**Table 1. TVAR Model Order Estimation of Signal Test1 using MLE Algorithm**

Basis function	TVAR Model Order	Basis function dimension
	P	q
Time polynomial	2	4
Legendre polynomial	2	4
Chebyshev polynomial	2	6
DPSS	2	4
Fourier	2	4
Discrete Cosine	2	10

Except the cosine function, all other basis functions have small errors in estimating the time-varying frequency of the signal test1. For this signal Best basis function is any basis function except cosine basis function since cosine basis function requires high basis function dimension (q) to get least error in frequency estimation.



**Figure 8. Instantaneous Frequency Estimate of Signal Test 1 using Different Basis**

2) **Signal Test 2** was a chirp signal with a normalized frequency variation in a parabolic shape from 0.02 to over 0.46 over 256 samples. It was created by using the equation

$$x_2(n) = \cos\left(\frac{2}{3}\pi \mu n^3 + 2\pi f_0 n\right),$$

Where  $\mu = 6.4e6$ ,  $f_0 = 200\text{Hz}$ ,  $F_s = 10000\text{Hz}$ .

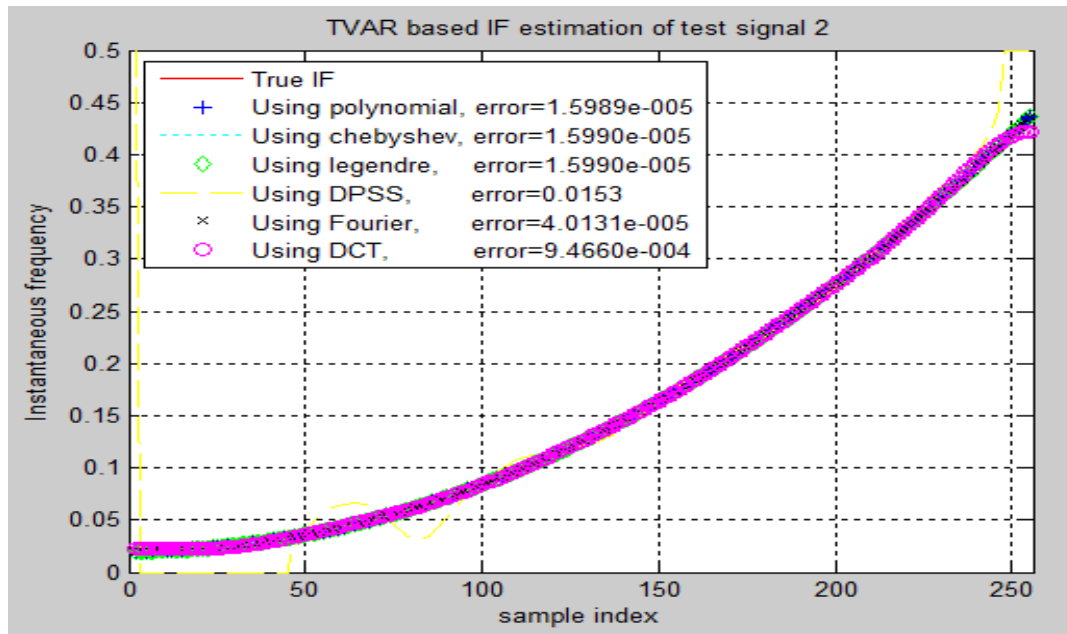
$$\text{IF law } f_i(n) = f_0 + \mu n^2, \quad 1 \leq n \leq 256 \quad (33)$$

TVAR model order for different basis are computed using Maximum likelihood estimation (MLE) Algorithm, the results are tabulated below in Table 2

**Table 2. TVAR Model Order Estimation of Signal Test2 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	10
Legendre polynomial	2	10
Chebyshev polynomial	2	10
DPSS	2	10
Fourier	2	8
Discrete Cosine	2	8

For this signal any basis is suitable since in terms of TVAR Model order p and Basis function dimension q is same for all basis and also estimation error is also similar for all basis.



**Figure 9. Instantaneous Frequency Estimate of Signal Test 2 using Different Basis**

3) **Signal test 3** was a sinusoid with a normalized frequency change in parabolic shape from 0.05 to a maximum at 0.3 and then back to 0.05 over 256 samples. This signal may be thought of as an example of an ideal Doppler signal. It was given by  $x_3(n) = \cos(-\frac{2}{3}\pi\mu n^3 + 2\pi\mu t_d n^2 - 2\pi\mu t_d^2 n + 2\pi f_0 n)$

Where  $\mu = 1.06e8$ ,  $f_0 = 5500\text{Hz}$ , and  $t_d =$  time delay which is of 130 samples.

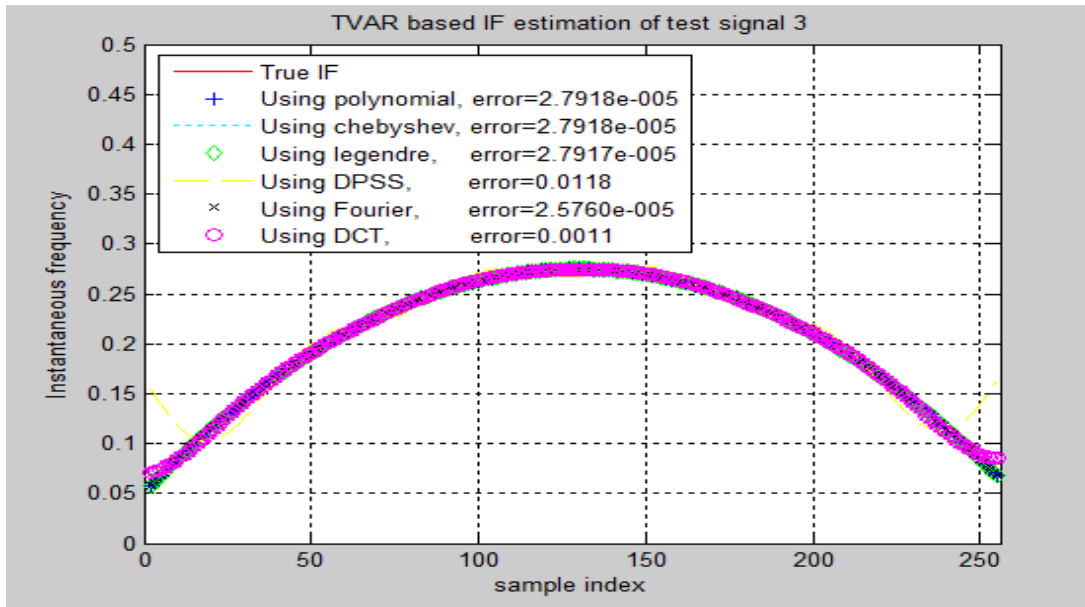
IF law

$$f_i(n) = f_0 + 2\mu t_d n - \mu t_d^2 - \mu n^2, \quad 1 \leq n \leq 256 \quad (34)$$

**Table 3. TVAR Model Order Estimation of Signal Test3 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	16
Legendre polynomial	2	16
Chebyshev polynomial	2	14
DPSS	2	16
Fourier	2	8
Discrete Cosine	2	8

For this signal Fourier basis and Discrete Cosine basis are best suitable basis since the TVAR Model order p and Basis function dimension q is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension is high as shown in Table 3.



**Figure 10. Instantaneous Frequency Estimate of Signal Test 3 using Different Basis**

4) **Signal test 4** a sinusoid with a normalized frequency piecewise *linearly* varying over  $N = 256$  samples, increasing from  $f_0 = 0.1$  to  $f_{max} = 0.4$  over the first  $N_1 = 128$  samples, then decreasing back to  $f_0 = 0.1$  over the next 128 samples

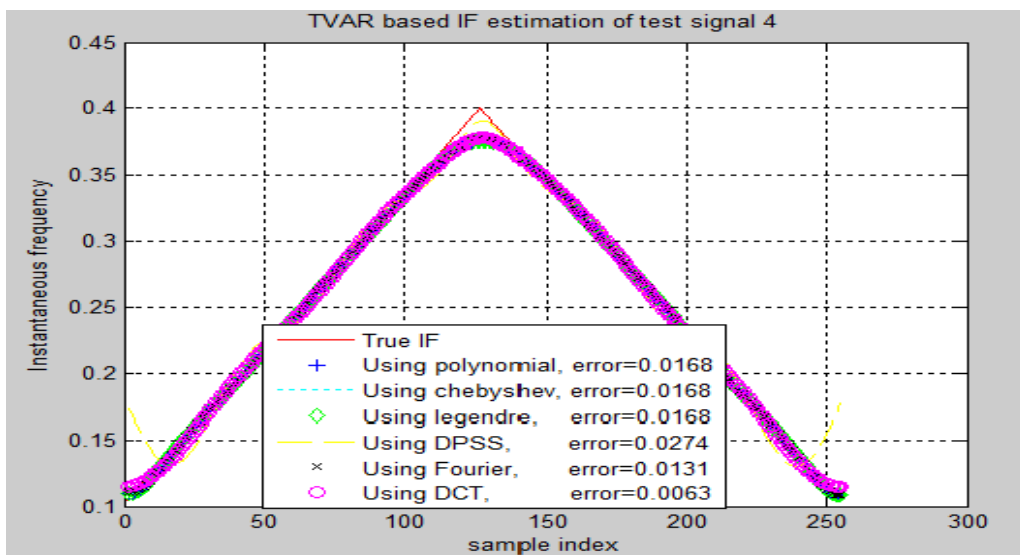
$$x_4(n) = \begin{cases} \left[ \cos\left(2\pi \left(f_0 + \frac{\mu}{2} n\right) n\right) \right], & 1 \leq n \leq 128 \\ \left[ \cos\left(2\pi \left(f_{max} - \mu \left(\frac{n}{2} - N_1\right)\right) n\right) \right], & 129 \leq n \leq 256 \end{cases} \quad (35)$$

Where  $\mu = \left| \frac{f_{max} - f_0}{N_1} \right|$  and  $f_i(n) = \begin{cases} f_0 + \mu n, & 1 \leq n \leq 128, \\ f_{max} - \mu(n - N_1), & 129 \leq n \leq 256, \end{cases} \quad (36)$

**Table 4. TVAR Model Order Estimation of Signal Test4 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	24
Legendre polynomial	2	24
Chebyshev polynomial	2	20
DPSS	2	24
Fourier	2	14
Discrete Cosine	2	10

For this signal Discrete Cosine basis is best suitable basis since the TVAR Model order p and Basis function dimension q is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension is high as shown in Table 4.



**Figure 11. Instantaneous Frequency Estimate of Signal Test 4 using Different Basis**

5) **Signal test 5** a sinusoid with a normalized frequency nonlinearly varying over  $N = 256$  samples in a *quadratic* manner, decreasing from  $f_0 = 0.4$  to  $f_{min} = 0.1$  over the first  $N_1 = 128$  samples then increasing back to  $f_0 = 0.4$  over the next 128 samples.

$$x_5(n) = \begin{cases} \cos \left\{ 2\pi \left[ f_{\min} + \mu \left( \frac{n^2}{3} - nN_1 + N_1^2 \right) \right] n \right\} \\ \cos \left\{ 2\pi \left[ f_0 - \mu \left( \frac{n^2}{3} - nN + N^2 \right) \right] n \right\} \end{cases} \quad (37)$$

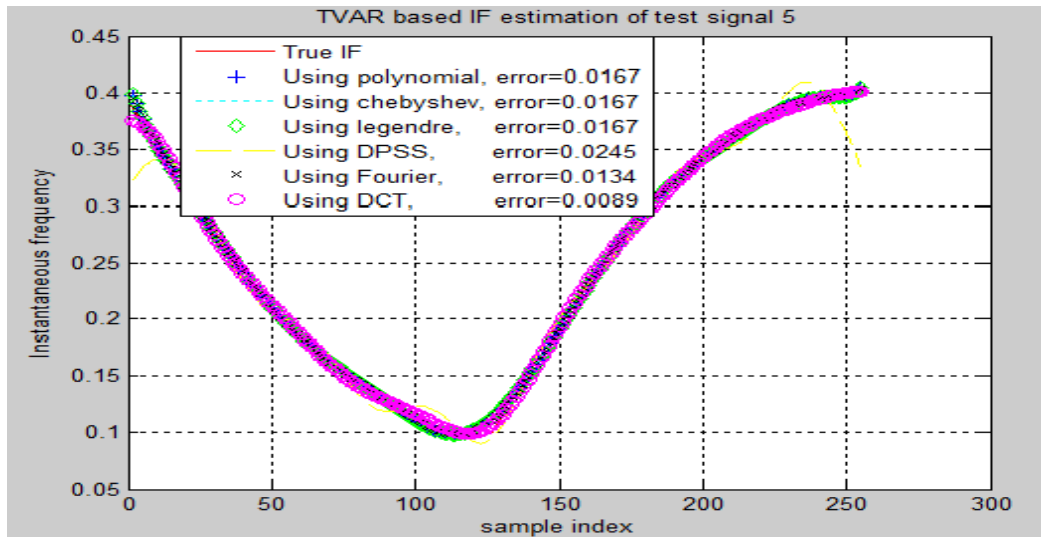
Where  $\mu = \left| \frac{f_0 - f_{\min}}{N_1^2} \right|$ , and

$$f_i(n) = \begin{cases} f_{\min} + \mu (n - N_1)^2 & 1 \leq n \leq 128 \\ f_0 - \mu (n - N)^2 & 129 \leq n \leq 256 \end{cases} \quad (38)$$

**Table 5. TVAR Model Order Estimation of Signal Test 5 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	24
Legendre polynomial	2	24
Chebyshev polynomial	2	18
DPSS	2	24
Fourier	2	10
Discrete Cosine	2	10

For this signal Fourier basis and Discrete Cosine basis are best suitable basis since the TVAR Model order  $p$  and Basis function dimension  $q$  is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension high as shown in Table 5.



**Figure 12. Instantaneous Frequency Estimate of Signal Test 5 using Different Basis**



6) **Signal test 6** a sinusoid with a normalized frequency nonlinearly varying in a *periodic* manner over  $N = 256$  samples, starting from  $f_0 = 0.25$  and oscillating between

$$f_{\max} = 0.4 \text{ and } f_{\min} = 0.1, \text{ with a sweeping rate of } \mu_f = \frac{3.2}{N}.$$

$$x_6(n) = \cos\left\{2\pi\left[f_0 n - \frac{\mu}{2\pi\mu_f} \cos(2\pi\mu_f n)\right]\right\}, \quad 1 \leq n \leq 256, \quad (39)$$

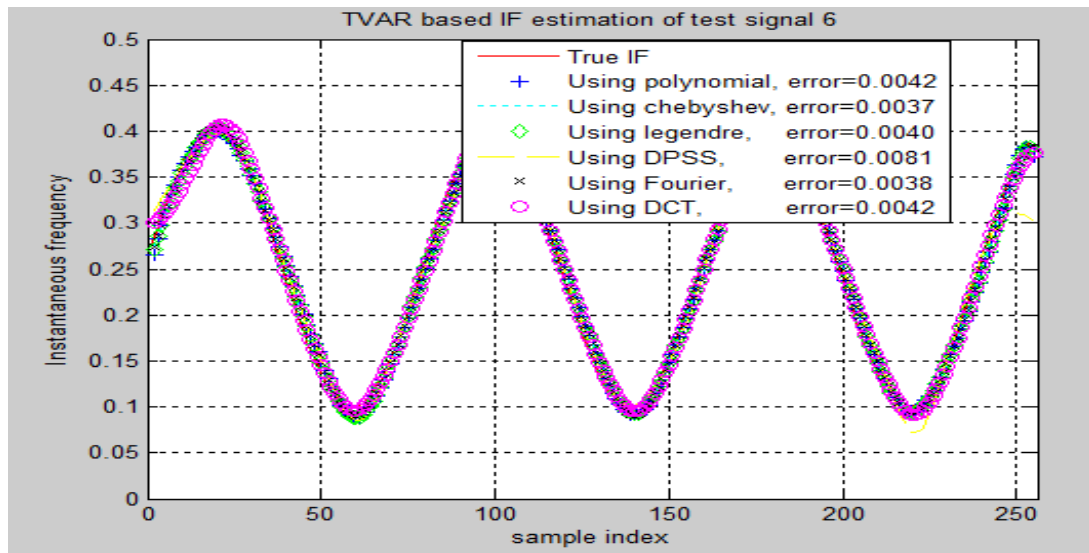
Where  $\mu = \left|\frac{f_{\max} - f_{\min}}{2}\right|$  and

$$f_i(n) = f_0 + \mu \sin(2\pi\mu_f n) \quad 1 \leq n \leq 256 \quad (40)$$

**Table 6. TVAR Model Order Estimation of Signal Test 6 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	26
Legendre polynomial	2	26
Chebyshev polynomial	2	18
DPSS	2	26
Fourier	2	14
Discrete Cosine	2	10

For this signal Discrete Cosine basis is best suitable since the TVAR Model order  $p$  and Basis function dimension  $q$  is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension high as shown in Table 6.



**Figure 13. Instantaneous Frequency Estimate of Signal Test 6 using Different Basis**

7) **Signal Test 7** a sinusoid with a frequency *jump*. The frequency remains constant at to  $f_0=0.1$  for the first 127 samples and then it jumps to  $f_N=0.4$  at the 128<sup>th</sup> sample and remains constant over the next 128 samples

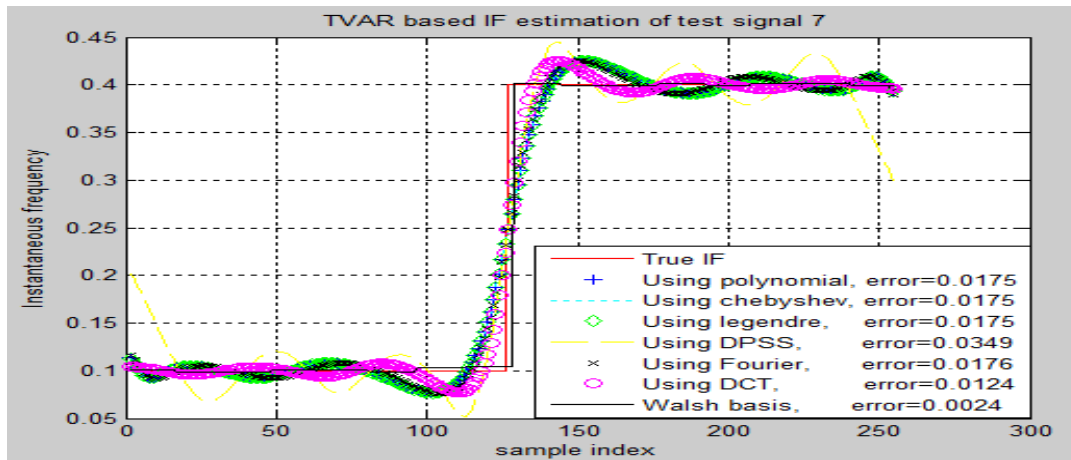
$$x_7(n)=\begin{cases} \cos(2\pi f_0 n) & 1 \leq n \leq 127 \\ \cos[2\pi(f_0 + \Delta f)n], & 128 \leq n \leq 256 \end{cases} \quad (41)$$

$$\text{Where } \Delta f= f_N - f_0 \text{ and } f_i(n)=\begin{cases} f_0 & 1 \leq n \leq 127, \\ f_0 + \Delta f, & 128 \leq n \leq 256, \end{cases} \quad (42)$$

**Table 7. TVAR Model Order Estimation of Signal Test7 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	28
Legendre polynomial	2	28
Chebyshev polynomial	2	22
DPSS	2	28
Fourier	2	18
Discrete Cosine	2	18
Walsh	2	10

For this signal Walsh basis is best suitable basis since the TVAR Model order p and Basis function dimension q is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension is high as shown in Table 7.



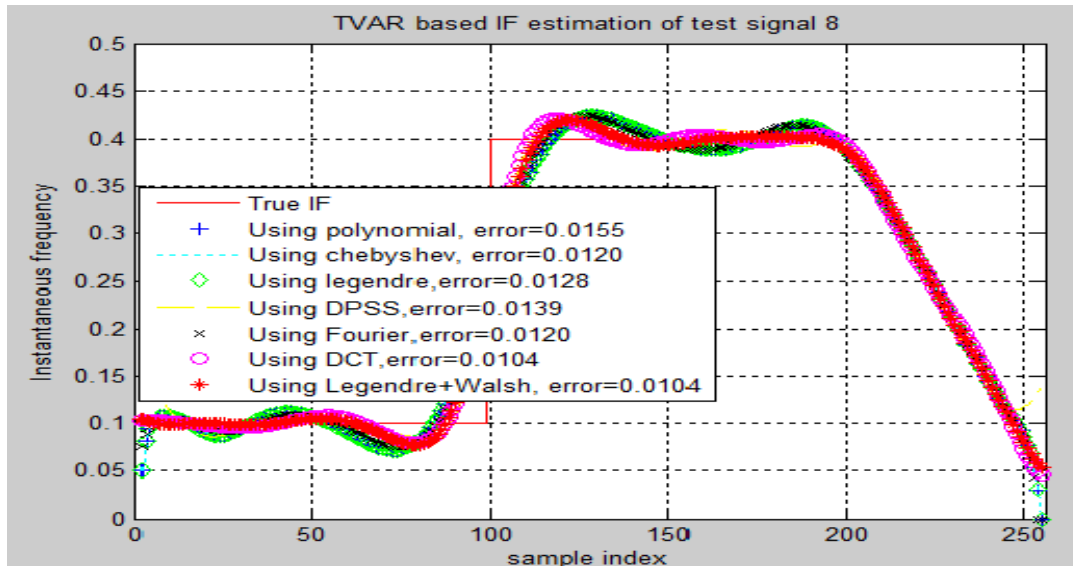
**Figure 14. Instantaneous Frequency Estimate of Signal Test 7 using Different Basis**

8) **Signal Test 8** was a highly nonstationary sinusoid that has a normalized frequency, jumping from 0.1 to 0.4 and then linearly decreasing from 0.4 to 0.1. The length of this signal was 256 samples.

**Table 8. TVAR Model Order Estimation of Signal Test8 using MLE Algorithm**

Basis function	TVAR Model order P	Basis function dimension q
Time polynomial	2	32
Legendre polynomial	2	30
Chebyshev polynomial	2	26
DPSS	2	30
Fourier	2	18
Discrete Cosine	2	18
Legendre +Walsh	2	10

This signal exhibit both fast and slowly varying dynamics at different stages of the signal, for this signal the combinations of both Legendre and Walsh basis function are best suitable basis since the TVAR Model order p and Basis function dimension q is low to get the least estimation error as compared to other basis, for remaining basis to get the similar error we required basis function dimension is high as shown in Table 8.



**Figure 15. Instantaneous Frequency Estimate of Signal Test 8 using Different Basis**

**Table 9. Summary of the Best Basis Functions that Yield the Least Error for each Signal Test**

Signal	Best Basis function
Signal test 1	Any, except cosine function
Signal test 2	Any
Signal test 3	Discrete Cosine, Fourier
Signal test 4	Fourier

Signal test 5	Discrete Cosine and Fourier
Signal test 6	Discrete Cosine, Fourier
Signal test 7	Walsh
Signal test 8	Walsh + Legendre

Table 9 is a summary of the results from our tests. It is obvious that we cannot decisively select the best single basis function that is suitable for all signal tests. It is more(or)less dependent on the characteristics of signals.

1) If the signal is a chirp (or) has a frequency that varies linearly, then the, Time polynomial basis, Legendre polynomial basis, DPSS basis are suitable as the basis for the parameter expansion.

2) If the signal has a frequency that varies periodically, the Fourier basis and Discrete Cosine basis are suitable as the basis for the parameter expansion.

3) Walsh basis functions should be used when dynamics are expected to exhibit fast transients and burst-like dynamics, whereas Legendre polynomials are more appropriate for smoothly changing dynamics.

4) If the signal exhibit both fast and slowly varying dynamics at different stages of the signal, then the combinations of both Legendre and Walsh basis functions are suitable as the basis for the parameter expansion.

5)If the characteristics of the signals were not known, we recommend the Fourier basis function as the basis expansion of the TVAR parameters, since the Fourier basis function yielded reasonable accuracy in the frequency estimation for all eight nonstationary signals.

## 6. Conclusions

Several basis functions were compared in estimating the time-varying frequency of the non stationary signals. It is obvious that we cannot decisively select the best single basis function that is suitable for all signal tests. It is more (or) less dependent on the characteristics of signals. In conclusion, the polynomial basis is superior to the other basis for IF estimation of the signal that has a frequency that varies linearly. If the signal has a frequency that varies periodically, the Fourier basis and Discrete Cosine basis are suitable as the basis for the parameter expansion, and we also concluded Walsh basis functions should be used when the dynamics are expected to exhibit fast transients and burst-like dynamics, where as Legendre polynomials are more appropriate for smoothly changing dynamics. If the signal exhibit both fast and slowly varying dynamics at different stages of the signal ,then the combinations of both Legendre and Walsh basis functions are suitable as the basis for the parameter expansion.Ingeneral,if the characteristics of the signals were not known, we recommend the Fourier basis function as the basis expansion of the TVAR parameters,

## References

- [1] T. S. Rao, "The fitting of non-stationary time-series models with time-dependent parameters", *Journal of the Royal Statistical Society Series B (Methodological)*, vol. 32, no. 2, pp. 312-322.
- [2] Y. Grenier, "Time-dependent ARMA modeling of non stationary signals", *IEEE Transactions on Acoustics, Speech, and signal Processing*, vol. 31, no. 4, (1983) August, pp. 899-911.
- [3] K. C. Sharman and B. Friedlander, "Time-varying auto regressive modeling of a class of non stationary signals", *IEEE Conference on Acoustics, Speech and Signal Processing*, vol. 9, (1984) March, pp. 227-230.
- [4] L. Cohen, "Time-Frequency Analysis", Prentice Hall, Chapter 3, Englewood Cliffs, NJ, (1995).
- [5] P. Shan and A. A. (Louis) Beex, "High-resolution instantaneous frequency estimation based on time-varying AR modeling", *IEEE 4<sup>th</sup> International Symposium on Time-Frequency and time-Scale Analysis (TFTS'98)*, pittsburgh, pa, (1998) October.
- [6] R. K. Pally, "Implementation of instantaneous frequency estimation based on time-varying AR modeling", M. S. thesis, Virginia Tech, (2009) April.
- [7] C. Sodsri, "Time-varying autoregressive models for non-stationary acoustic signal and its frequency analysis", PhD dissertation submitted to Pennsylvania State University, Pennsylvania, (2003).
- [8] X. Luo, "Time-varying Autoregressive modeling of non stationary signals", M. S. thesis, The University of Tennessee, Knoxville, (2005) May.
- [9] A. A. (Louis) Beex and P. Shan, "A time-varying Prony method for instantaneous frequency estimation at low SNR", *IEEE International Symposium on Circuits and Systems*, vol. 3, (1999) May, pp. 3-8.
- [10] S. Mukhopadhyay and P. Sircar, "Parametric modeling of nonstationary signals: A unified approach", *Signal Processing*, vol. 60, (1997), pp. 135-152.
- [11] S. M. Kay, "Modern Spectral Estimation: Theory and Application", Prentice-Hall, Englewood Cliffs, NJ, (1988).
- [12] M. Neidzwiecki, "Identification of Time-varying Processes", John Wiley & Sons, Chicester, England, (2000).
- [13] H. W. Ruizou and K. H.Chon, "A Robust Time-Varying Identification Algorithm Using Basis Functions", *Annals of Biomedical Engineering*, vol. 31, (2003), pp. 840-853.
- [14] R. Chabonnier, M. Barlaud, G. Alengrin and J. Menez, "Results on AR Modeling of Nonstationary Signals", *Signal Processing*, vol. 12, no. 2, (1987), pp. 143-151.
- [15] F. Kozin and F. Nakajima, "The order determination problem for Linear Time-Varying AR Models", *IEEE Transactions on Automatic Control*, vol. AC-25, no. 2, (1980), pp. 250-257.
- [16] J. J. Rajan and P. J. W. Rayner, "Generalized Feature Extraction for Time-Varying Autoregressive Model", *IEEE Transactions on Signal Processing*, vol. 44, no. 10, (1996), pp. 2498-2507.
- [17] K. B. Eom, "Analysis of Acoustic Signatures from Moving Vehicles Using Time-Varying Autoregressive Models", *Multisignal Systems and Signal Processing*, vol. 10, (1999), pp. 357-378.
- [18] M. B. Priestley, "Non-linear and Non-stationary Time Series Analysis", Academic Press Inc., San Diego, CA, (1988).
- [19] B. Boashash, "Estimating and interpreting the instantaneous frequency of a signal-part1: Fundamentals", *IEE Proceedings*, vol. 80, no. 4, (1992) April, pp. 520-538.

## Authors

**G. Ravi Shankar Reddy** is presently working as an Associate Professor, CVR College of Engineering ,Hyderabad,India .He received the B.Tech degree in Electronics and Communications Engineering from Jawaharlal Nehru Technological University,Hyderabad,India,in 2001.He received the M.Tech degree in Systems and Signal Processing from Osmania University,Hyderabad,India in 2004. Presently he is pursuing his Ph.D. from Jawaharlal Nehru Technological University,Ananatapur,India.Under the guidance of Dr.Rameshwar Rao.His research area is Nonstationary Signalprocessing.

**Dr.Rameshwar Rao** is currently Vice Chancellor of Jawaharlal Nehru Technological University, Hyderabad, India. He guided several PhD students in the area of VLSI and Signal processing. He has more than 50 publications in various National and International Journals .His research interests are in VLSI and Signalprocessing.

