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A Weighted Nuclear Norm Method for Tensor Completion

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Abstract

In recent years, tensor completion problem has received a significant amount of attention in computer vision, data mining and neuroscience. It is the higher order generalization of matrix completion. And these can be solved by the convex relaxation which minimizes the tensor nuclear norm instead of the n-rank of the tensor. In this paper, we introduce the weighted nuclear norm for tensor and develop majorization-minimization weighted soft thresholding algorithm to solve it. Focusing on the tensors generated randomly and image inpainting problems, our proposed algorithm experimentally shows a significant improvement with respect to the accuracy in comparison with the existing algorithm HaLRTC.

Keywords: *Tensor completion, weighted nuclear norm, majorization-minimization method, image inpainting*

1. Introduction

Tensor completion, has become a new research focus area and received considerable attention in recent years. It can be treated as a natural generalization of matrix completion. Tensor completion is a procedure for filling in missing entries of a partially known tensor under a low-rank constraint. This problem emerge naturally in a variety of domains, such as neuroscience, computer vision, retail data analysis and information sciences.

Unlike the matrix completion, the main challenge of tensor is the NP-hardness of computing most tensor decompositions, and this challenge pushes researchers to study alternative structure-inducing norms in lieu of the nuclear norm [1, 2]. Recently, several researchers [1, 3-5] extended the framework of nuclear norm regularization for the tensor completion, which leads to a convex optimization problem. In [3], Liu *et al.*, lay the theoretical foundation of low n-rank tensor completion and propose the first definition of the trace norm for tensors. In addition, they propose a solution for the low rank completion of tensors, namely HaLRTC. After that, Gandy *et al.*, [1] use the n-rank of a tensor as a sparsity measure and consider the low-n-rank tensor recovery problem. They introduce a tractable convex relaxation of the n-rank and propose efficient algorithms to solve the low-n-rank tensor recovery problem numerically. Their algorithms are based on the Douglas–Rachford splitting technique and its dual variant, the alternating direction method of multipliers. Several other efficient algorithms can be found in [2, 6, 7].

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In this paper, we also use the n-rank of a tensor as a sparsity measure and consider the low-n-rank tensor completion problem. Motivated by the work in [8], we use the weighted nuclear norm as an approximation of rank function, and develop a majorization-minimization weighted soft thresholding algorithm for solving the relaxation model of the low n–rank tensor completion problem. From the computational results on synthetic data and real world data given in this paper, we can see that the proposed algorithm can yield more accuracy solution.

The remainder of this paper is organized as follows. Section 2 contains a brief introduction to tensor completion. In Section 3 we present our majorization-minimization weighted soft thresholding algorithm for tensor completion (TC-MWST). In Section 4 we compare the TC-MWST algorithm with HaLRTC [3] on randomly generated tensors and then we display the experiments of low-rank image inpainting. Finally we give some concluding remarks in Section 5.

2. Notations and Preliminaries on Tensor Completion

2.1. Preliminaries on Tensor

In this paper, we adopt the nomenclature and the notations in [9] to define the tensor. We denote scalars by lower-case letters, *e.g.*, $a, b, c, \cdots$; vectors as bold lower-case letters, *e.g.*, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \cdots$; and matrices as uppercase letters, *e.g.*, $A, B, C, \cdots$. Tensors are written as calligraphic letters, *e.g.*, $\mathcal{A}, \mathcal{B}, \mathcal{C}, \cdots$. An N-mode tensor is denoted as $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, whose elements are denoted as $x_{i_1 \cdots i_k \cdots i_n}$, where $1 \le i_k \le I_K$, $1 \le K \le N$. The "*unfold*" operation along the n-th mode on a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is defined as $unfold(\mathcal{X}, n) = X_{(n)}$. Specially, the tensor element $(i_1, i_2, \cdots, i_N)$ is mapped to the matrix element (i_n, j), where

$$j = 1 + \sum_{k=1, k \ne n}^{N} (i_k - 1) J_k \text{, with } J_k = \prod_{m=1, m \ne n}^{k-1} I_m .$$

That is, $X_{(n)} \in \mathbb{R}^{I_n \times J}$ and $J = \prod_{k=1, k \ne n}^{N} I_k$. The opposite operation "*fold*" is defined as $fold(X_{(n)}, n) = \mathcal{X}$.

Another important concept is n-rank of a tensor, which is the straightforward generalization of the column (row) rank for matrices. We use r_n to denote the n-rank of an N-mode tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$. It is the rank of the mode-n unfolding matrix $X_{(n)}$.

$$r_n = rank_n(\mathcal{X}) = rank(X_{(n)}) .$$

A tensor of which the n-ranks are equal to r_n is called a rank-$(r_1, r_2, \cdots, r_N)$ tensor.

The inner product of two same-size tensors $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is the sum of the products of their entries, *i.e.*,

$$\langle \mathcal{X}, \mathcal{Y} \rangle = \sum_{i_1=1}^{I_1} \sum_{i_2=1}^{I_2} \cdots \sum_{i_N=1}^{I_N} x_{i_1 \cdots i_k \cdots i_n} y_{i_1 \cdots i_k \cdots i_n} .$$

The corresponding Frobenius norm is $\|\mathcal{X}\|_F = \sqrt{\langle \mathcal{X}, \mathcal{X} \rangle}$. For any tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$, $vec(\mathcal{X})$ denotes the vectorization of $\mathcal{X}$.

2.1. Low n-rank Tensor Completion Problem

In this subsection, we first introduce the case of $N = 2$, that is matrix completion problem,

$$\begin{aligned} &\min \quad rank(X) \\ &\text{s. t.} \quad \mathrm{P}_\Omega(X) = \mathrm{P}_\Omega(M), \end{aligned} \tag{1}$$

where $M \in \mathrm{R}^{n_1 \times n_2}$ is a matrix with n_1 rows and n_2 columns, Ω is the set of indices of samples and P_Ω is the orthogonal projector onto the span of matrices vanishing outside of Ω. Candès *et al.*, [10] had claimed that, under suitable conditions, the following problem is formally equivalent to (1),

$$\begin{aligned} &\min \quad \|X\|_* \\ &\text{s. t.} \quad \mathrm{P}_\Omega(X) = \mathrm{P}_\Omega(M). \end{aligned} \tag{2}$$

Several efficient algorithms have been developed to address the above problem (2) in recent years, such as [11-15]. Recently, J. Xu [16], S. Gaïffas *et al.*, [17] and J. Geng et al. [8] proposed to solve the following weighted version of (2) at each iteration:

$$\begin{aligned} &\min \quad \sum_{i=1}^{n} w_i \sigma_i \left(X \right) \\ &\text{s. t.} \quad \mathrm{P}_\Omega(X) = \mathrm{P}_\Omega(M). \end{aligned} \tag{3}$$

In [1], the author generalized the completion algorithm for the matrix case to higher-order tensors by solving the following optimization problem:

$$\begin{aligned} &\min \quad \|\mathrm{X}\|_* \\ &\text{s. t.} \quad \mathrm{X}_\Omega = \mathrm{T}_\Omega, \end{aligned} \tag{4}$$

where X, T are n-mode tensors with identical size in each mode. In addition, the author also proposed the definition for the tensor trace norm:

$$\|\mathrm{X}\|_* := \sum_{i=1}^{N} \alpha_i \left\| X_{(i)} \right\|_*, \tag{5}$$

where α_i's are constants satisfying $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$.

Note that the problem (4) is a special case of affinely constrained minimization problem

$$\begin{aligned} &\min \quad \|\mathrm{X}\|_* \\ &\text{s. t.} \quad \mathsf{A}(\mathrm{X}) = b, \end{aligned} \tag{6}$$

where $\mathsf{A} : \mathrm{R}^{I_1 \times I_2 \times \cdots \times I_N} \to \mathrm{R}^p$ is a linear transformation and $b \in \mathrm{R}^p$. In the context of the tensor completion problem, the linear operator A is a sampling (or projection/restriction) operator and it is in the form of $\mathsf{A}(\mathrm{X}) = Avec(\mathrm{X})$, where $A \in \mathrm{R}^{p \times I_2 \cdots I_N}$. For a linear operator A we can always write its matrix representation as $\mathsf{A}(\mathrm{X}) = Avec(\mathrm{X})$, where $A \in \mathrm{R}^{p \times I_2 \cdots I_N}$.

3. Proposed tensor Completion Algorithm

In this section, we will introduce the extension of matrix weighted nuclear norm minimization to tensor case, and present the details of the proposed TC-MWST method.

Hunter *et al.*, [18] introduced majoriziation-minimization (MM) algorithm framework, and Geng *et al.*, [8] recently proposed the majoriziation-minimization algorithm for matrix rank minimization. Motivated by these works, we will develop a majoriziation-minimization algorithm for tensor completion.

In this paper, we are the first to introduce the weighted nuclear norm minimization to tensor completion, and the main problem is to solve the following problem:

$$\min \ \|\mathcal{X}\|_{w,*} \quad \text{s. t.} \ \mathcal{A}(\mathcal{X}) = b, \tag{7}$$

where $\|\mathcal{X}\|_{w,*} := \sum_{i=1}^{N} \alpha_i \|X_{(i)}\|_{w,*}$, and $\|X_{(i)}\|_{w,*} = \sum_{j=1}^{r_i} w_{(i)j}\sigma_j(X_{(i)})$ with $\sigma_j(X_{(i)})$ denotes the j-th largest singular value of matrix $X_{(i)}$, r_i is the rank of matrix $X_{(i)}$ and $w_{(i)}$'s are the weight vectors. In the following, we consider an unconstrained problem:

$$\sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*} + \|\mathcal{A}(\mathcal{X}) - b\|_2^2, \tag{8}$$

where $\lambda > 0$ is a penalty parameter.

In the following, we write $\mathcal{X}$ in vector form, and then we get the following equivalent formation of (8):

$$\min \ \sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*} + \|b - Ax\|_2^2, \tag{9}$$

where $A \in \mathbb{R}^{p \times I_2 \cdots I_N}$ is the matrix version of linear operator $\mathcal{A}$, i.e., $\mathcal{A}(\mathcal{X}) = Avec(\mathcal{X})$.
Let

$$L(x) = \sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*} + \|b - Ax\|_2^2. \tag{10}$$

According to the idea of MM method, we should find a function $Q_k(x)$ that coincides with $L(x)$ at x_k but otherwise upper-bounds $L(x)$. Similar to [8], we choose $Q_k(x)$ to be

$$Q_k(x) = \|b - Ax\|_2^2 + (x - x_k)^T(\beta I - A^T A)(x - x_k) + \sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*}, \tag{11}$$

where $\beta = 1.1 \max eig A^T A$.

Expanding $Q_k(x)$ in (11), it holds that

$$Q_k(x) = b^T b + x_k^T(\beta I - A^T A)x_k - 2\beta(x_k + \beta^{-1}A^T(b - Ax_k))x + \beta x^T x + \sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*}. \tag{12}$$

Plugging $z_k = x_k + \beta^{-1}A^T(b - Ax_k)$ into (12), we can rewrite (12) as

$$Q_k(x) = \beta\|x - z_k\|_2^2 + \sum_{i=1}^{N} \lambda\alpha_i \|X_{(i)}\|_{w,*} + b^T b + x_k^T(\beta I - A^T A)x_k - \beta z_k^T z_k. \tag{13}$$

Note that the terms $b^T b + x_k^T(\beta I - A^T A)x_k - \beta z_k^T z_k$ are independent of x, therefore we minimize the following simplified part:

$$\hat{Q}_k(x) = \beta \left\| x - z_k \right\|_2^2 + \Sigma_{i=1}^N \lambda \alpha_i \left\| X_{(i)} \right\|_{w,*}. \tag{14}$$

Now we write x and z_k in tensor form, and rewrite (14) as

$$\hat{Q}_k(x) = \beta \left\| \mathrm{X} - \mathrm{Z}_k \right\|_F^2 + \Sigma_{i=1}^N \lambda \alpha_i \left\| X_{(i)} \right\|_{w,*}. \tag{15}$$

Under the definition of mode-n unfolding , the problem of minimizing (15) can be written as

$$\min \ \Sigma_{i=1}^N \lambda \alpha_i \left\| X_{(i)} \right\|_{w,*} + \frac{\beta}{N} \Sigma_{i=1}^N \left\| X_{(i)} - (Z_k)_{(i)} \right\|_F^2 . \tag{16}$$

The optimization problem (16) is difficult to solve stem from the interdependent weighted nuclear norms. Similar to [3] we introduce additional matrices $M_1, \cdots, M_N$ to split these interdependent terms such that they can be solved independently. Then we obtain the following equivalent formulation:

$$\min_{\mathrm{X}, M_i} \Sigma_{i=1}^N \lambda \alpha_i \left\| M_i \right\|_{w,*} + \frac{\beta}{N} \Sigma_{i=1}^N \left\| M_i - (Z_k)_{(i)} \right\|_F^2$$
$$\text{s. t. } \ M_i = X_{(i)}, \ \forall i \in \{1, 2, \cdots, N\}. \tag{17}$$

Next we introduce a penalty parameter $\gamma > 0$, and the optimization problem (17) can be relaxed as

$$\min_{\mathrm{X}, M_i} \Sigma_{i=1}^N \lambda \alpha_i \left\| M_i \right\|_{w,*} + \frac{\beta}{N} \Sigma_{i=1}^N \left\| M_i - (Z_k)_{(i)} \right\|_F^2 + \frac{1}{2\gamma} \Sigma_{i=1}^N \left\| M_i - X_{(i)} \right\|_F^2 . \tag{18}$$

As we known, an optimal solution of (18) approaches an optimal solution of (17) as $\gamma \to 0$. For convenience, in problem (18) we let $\beta = \gamma$, and obtain the following problem:

$$\min_{\mathrm{X}, M_i} \Sigma_{i=1}^N \lambda \alpha_i \left\| M_i \right\|_{w,*} + \frac{\beta}{N} \Sigma_{i=1}^N \left\| M_i - (Z_k)_{(i)} \right\|_F^2 + \frac{1}{2\beta} \Sigma_{i=1}^N \left\| M_i - X_{(i)} \right\|_F^2 . \tag{19}$$

Computing M_i: We fixed all variables except $M_i, i = 1, 2, \cdots, N$, then M_i is the optimal solution of the following problem.

$$\min_{M_i} \lambda \alpha_i \left\| M_i \right\|_{w,*} + \frac{\beta}{N} \left\| M_i - (Z_k)_{(i)} \right\|_F^2 + \frac{1}{2\beta} \left\| M_i - X_{(i)} \right\|_F^2 . \tag{20}$$

In the following theorem, we will give the optimal solution of (20).

Theorem 1: Given $\lambda > 0$ and $\beta = 1.1 \max eig A^T A$. Let $X = UDiag(\sigma_X)V^T$ be the singular value decomposition for X . $D_\tau(X)$ is the "shrinkage" operator defined as [11]:

$$D_\tau(X) = UDiag(\bar{\sigma})V^T$$

$$\bar{\sigma} = s_\tau(\sigma_X) := \begin{cases} (\sigma_X)_i - \tau_i, \text{ if } (\sigma_X)_i - \tau_i > 0 \\ 0, \quad o.w. \end{cases}.$$

Then for any given $i \in \{1, 2, \cdots, N\}$, $w_{(i)} \in \mathrm{R}^{I_i}$ is weight vector and M_i^* is an optimal solution of (20) if and only if

$$M_i^* = D_{\frac{\alpha_i \lambda w_{(i)}}{\frac{2\beta}{N}+\frac{1}{\beta}}}\left(\frac{\frac{2\beta}{N}\sigma_{(Z_k)_{(i)}} + \frac{1}{\beta}\sigma_{X_{(i)}}}{\frac{2\beta}{N} + \frac{1}{\beta}} \right),$$

where $\sigma_{(Z_k)_{(i)}}$ and $\sigma_{X_{(i)}}$ are the singular values of the matrix $(Z_k)_{(i)}$ and $X_{(i)}$, respectively.

Proof: In general, we have the following property of singular value decomposition. Assume that the matrix X has r positive singular values of $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, then we have

$$\left\| X_{n\times n} \right\|_F^2 = \sum_{i=1}^{r} \sigma_i{}^2 = \left\| \sigma \right\|_2^2, \tag{21}$$

where σ is a vector formed by σ_i, $i = 1, 2, \cdots r$.

Let $M_i = UDiag(\sigma_{M_i})V^T$, $Z_{k(i)} = UDiag(\sigma_{Z_{k(i)}})V^T$ and $X_{(i)} = UDiag(\sigma_{X_{(i)}})V^T$ be the singular value decomposition for M_i, $Z_{k(i)}$ and $X_{(i)}$, respectively. Using the property (21), we can solve the following optimization problem instead of (20):

$$\min_{\sigma_{M_i}} \lambda \alpha_i \left\langle w, \sigma_{M_i} \right\rangle + \frac{\beta}{N} \left\| \sigma_{M_i} - \sigma_{Z_{k(i)}} \right\|_2^2 + \frac{1}{2\beta} \left\| \sigma_{M_i} - \sigma_{X_{(i)}} \right\|_2^2. \tag{22}$$

Another form of (22) is

$$\min_{\sigma_{M_i}} \lambda \alpha_i \textstyle\sum_{j=1}^{I_i} (w_{(i)})_j (\sigma_{M_i})_j + \frac{\beta}{N} \sum_{j=1}^{I_i} ((\sigma_{M_i})_j - (\sigma_{Z_{k(i)}})_j)^2 + \frac{1}{2\beta} \sum_{j=1}^{I_i} ((\sigma_{M_i})_j - (\sigma_{X_{(i)}})_j)^2. \tag{23}$$

Therefore we can solve (23) easily by differentiating it term-wise and using the first-order optimality conditions. Then we obtain the solution to (23):

$$(\sigma_{M_i})_j = \frac{\frac{2\beta}{N}(\sigma_{(Z_k)_{(i)}})_j + \frac{1}{\beta}(\sigma_{X_{(i)}})_j}{\frac{2\beta}{N} + \frac{1}{\beta}} - \frac{\lambda (w_{(i)})_j}{\frac{2\beta}{N} + \frac{1}{\beta}}, \; j = 1, 2, \cdots, I_i.$$

Now, we can get easily the optimal solution of (20) is

$$M_i^* = D_{\frac{\alpha_i \lambda w_{(i)}}{\frac{2\beta}{N}+\frac{1}{\beta}}}\left(\frac{\frac{2\beta}{N}\sigma_{(Z_k)_{(i)}} + \frac{1}{\beta}\sigma_{X_{(i)}}}{\frac{2\beta}{N}+\frac{1}{\beta}}\right)$$. This completes the proof. □

Computing X : We fixed all other variables except $X_i, i = 1, 2, \cdots, N$. Then we can get the optimal X by solving the following problem:

$$\min_{\mathrm{X}} \frac{1}{2\beta}\sum_{i=1}^{N}\left\|M_i - X_{(i)}\right\|_F^2 . \tag{24}$$

It is easy to check that the solution to (24) is given by

$$\mathrm{X}^* = \frac{1}{N}\sum_{i=1}^{N} fold(M_i) . \tag{25}$$

We call our algorithm "TC-MWST", which stands for Majorization-minimization Weighted Soft Thresholding Algorithm for Tensor Completion. Based on the above discussions, the complete TC-MWST algorithm is given in Algorithm 1 below.

Algorithm 1 TC-MWST: Majorization-minimization Weighted Soft Thresholding Algorithm for Tensor Completion

Input: $A, b, \beta, defac, \alpha_i, i = 1, 2, \cdots, N$ and K

Output: X

Initialization: $w_i^0, \lambda^0, \mathrm{X}^0$

for $k = 1$ to K **do**

while not converged, do

$$\mathrm{Z}^k = \mathrm{X}^k + \frac{1}{\beta}\mathsf{A}^*(b - \mathsf{A}\mathrm{X}^k)$$

for $i = 1 : N$

$$M_i^k = D_{\frac{\alpha_i \lambda w_{(i)}^k}{\frac{2\beta}{N}+\frac{1}{\beta}}}\left(\frac{\frac{2\beta}{N}\sigma_{(Z_k)_{(i)}} + \frac{1}{\beta}\sigma_{X^k_{(i)}}}{\frac{2\beta}{N}+\frac{1}{\beta}}\right)$$

end

$$\mathrm{X}^{k+1} = \frac{1}{N}\sum_{i=1}^{N} fold(M_i^k)$$

end while

$$\lambda^{k+1} = defac * \lambda^k,\ \ (w_i)_j^{(k+1)} = 1/(\sigma_j(X_i^{(k+1)}) + \delta),\ j = 1, 2, \cdots, I_i$$

end for

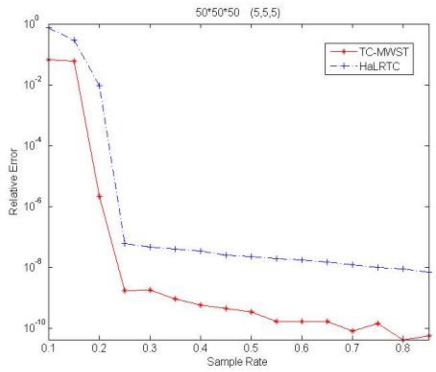
4. Numerical Experiments

In this section, we evaluate the empirical performance of the proposed TC-MWST algorithm both on synthetic and real-world data with the missing data and compare the results with HaLRTC (High Accuracy Low Rank Tensor Completion) [3]. We separate this section into two subsections. In subsection 4.1 the proposed algorithm is tested on the randomly generated low n-rank tensors. Subsection 4.2 tests the algorithm on natural images. All numerical experiments are performed with Matlab (version 2012b) on a desktop computer with a 3.20GHz CPU and 4 GB of memory.

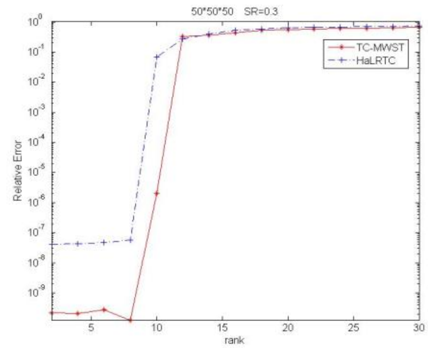
4.1. Synthetic Data

We first conduct the TC-MWST algorithm on several synthetic data sets for the tensor completion tasks. Each test tensor $\mathrm{T} \in \mathrm{R}^{I_1 \times I_2 \times \cdots \times I_N}$ with rank $(r, r, \cdots, r)$ is created by the Tucker decomposition [9]. We first generate a core tensor $\mathrm{A} \in \mathrm{R}^{r \times \cdots \times r}$ with each entry being sampled independently from a standard Gaussian distribution $\mathrm{N}(0,1)$. Then we generate matrices $U^{(1)}, \cdots, U^{(N)}$, with $U^{(i)} \in \mathrm{R}^{I_i \times r}$ by randomly choosing each entry from $\mathrm{N}(0,1)$. Then let $\mathrm{T} := \mathrm{A} \times_1 U^{(1)} \times_2 \cdots \times_N U^{(N)}$. We randomly sample a few entries from tensor T and recover the whole tensor with various sampling ratio (SR) by TC-MWST algorithm and HaLRTC [3]. In the implementation of TC-MWST, we set $\lambda^0 = 0.9\|b\|_\infty$ and $\alpha_i = 1/N, i = 1, 2, \cdots, N$. The parameters of HaLRTC are set to their default values. The relative error (RelErr) of the recovered tensor X is defined by $\mathrm{RelErr} = \|\mathrm{X} - \mathrm{T}\|_F / \|\mathrm{T}\|_F$.

Figure 1 shows the performance of TC-MWST and HaLRTC for random tensor completion problem. Figure 1 (a) shows the relative error for $50 \times 50 \times 50$ tensor with the n-rank is fixed at (r, r, r)=(5, 5, 5) and the sample rate changes form 10% to 85%. On the other hand, in Figure 1.(b), we also report the relative error for the same size tensor with the sample rate is fixed at SR=30% while n-rank r changes from 2 to 30. As we known, when sample rate is fixed, the complexity of the problem increases with the increase of the rank. On the other hand, when the rank is fixed, the complexity increases with the decrease of the SR. Observing the figures, it is clear that TC-MWST outperforms HaLRTC in all cases. Especially in Figure 1 (a), we can see that when SR=10%, TC-MWST can recover the tensor with the relative error 6.79e-2 while HaLRTC with 7.39e-1. Similar results appear in Figure 1 (b), we can see that our proposed algorithm is always able to obtain more accurate solutions.



(a)



(b)

Figure 1. The Performance Comparison for $50 \times 50 \times 50$ Tensors. (a) The Relative Error on Tensors with Sample Rate (SR) between 1% to 85% and Fixed n-rank (r, r, r)=(5, 5, 5) . (b) The Relative Error on Tensors with n-rank (r, r, r) between 2 and 30 and Fixed Sample Rate (SR=30%). All Results are Averages of 10 Independent Trials

In the next experiment we conduct the TC-MWST and HaLRTC algorithms on several cases in which the test tensors have different sizes, n-ranks and sample rates. The average results of 10 independent runs are shown in Table 1. The order of the test tensors varies from three to five, and we also change the n-rank and the sample rate. We can easily observe from Table 1 that our TC-MWST method can always yield more accurate solutions. The relative error obtained by TC-MWST is about 10^2 times better than HaLRTC.

Table 1. Comparison Results of TC-MWST and HaLRTC for Random Problems

Tensor	rank	SR	TC-MWST		HaLRTC	
			RelErr	Time	RelErr	Time
$20\times20\times20$	(2, 2, 2)	0.2	1.73e-08	8	3.12e-01	4
$50\times50\times50$	(5, 5, 5)	0.3	9.47e-10	94	5.16e-08	36
$100\times100\times100$	(5, 5, 5)	0.2	8.22e-10	592	6.58e-08	291
$20\times20\times20\times20$	(2, 2, 2, 2)	0.4	1.01e-09	109	9.16e-08	120
$50\times50\times50\times50$	(4, 4, 4, 4)	0.6	1.59e-10	1099	3.42e-08	989
$20\times20\times20\times20\times20$	(2, 2, 2, 2, 2)	0.3	1.47e-09	935	8.97e-08	912

4.2. Image Simulation

In order to illustrate the performance of our proposed method, we apply TC-MWST to image inpainting [19]. We know that color images can be expressed as third order tensors, and if the images are of low rank we can solve this problem as low n-rank tensor completion problem. In this experiment, we assume the image be well structured, and apply our proposed method on image inpainting of the façade image, which is also used in [3]. The recovery experiment using TC-MWST is present in Figure 2. We randomly remove 50% entries from the original color image. Then we apply our method to the image. From the figure, we can vividly observe that our algorithm is able to recover the natural image with missing information.

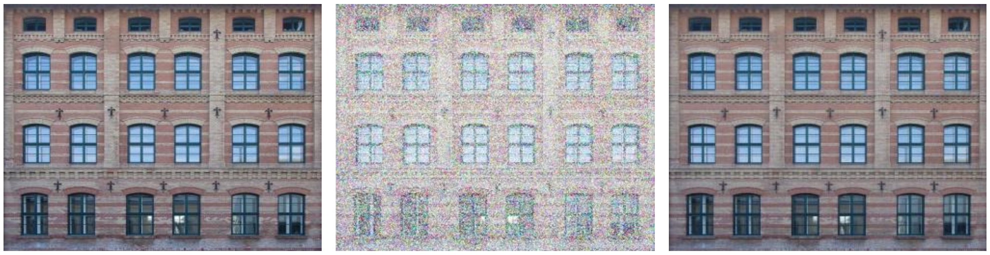


Figure 2. Comparisons in Terms of the Natural Image. From Left to Right: Original Image, 50% Corrupted Image, Recovered Image from 50% Corrupted Noise by TC-MWST (RelErr=7.15e-2)

Next we show some example slices of the MRI data in Figure 3. In this experiment, we test four slice images. The original images are given in the first line of Figure 3. The observations are given in the second line. We randomly remove some entries from the original image, and from left to right is 50%, 60%, 70%, 80%, respectively. The third line shows the reconstruction using TC-MWST. On all four images, the recovery effect is satisfactory. From left to right, the relative error is 2.72e-3, 3.03e-3, 5.95e-3 and 4.38e-2, respectively. From images in the second line, we can hardly get any detail information. However, our MMST algorithm can effectively recover the details of the low-rank image.

5. Conclusion

In this paper, we focus on low n-rank tensor completion problem. We extend the weighted nuclear norm of matrix to that of tensor and apply the majoriziation-minimization (MM) algorithm to solve the tensor n-rank minimization problem. Furthermore, we develop an efficient method called majorization-minimization weighted soft thresholding algorithm for tensor completion (TC-MWST). The experiment results on synthetic data and image processing including natural image and MRI images show the effectiveness of the proposed algorithm.

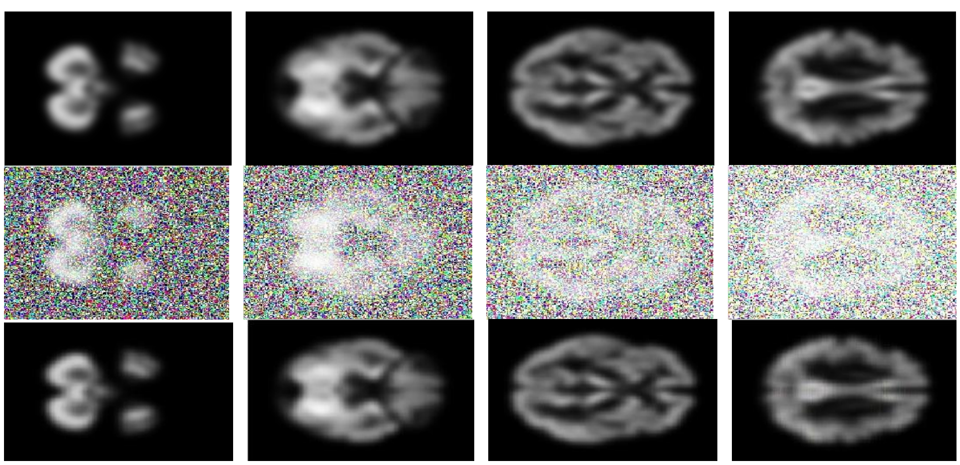


Figure 3. Comparisons in Terms of the MRI Data. The First Line: Original Images. The Second Line: Randomly Remove some Entries from the Original Image (from Left to Right: 50%, 60%, 70%, 80%). The Third Line: Recovered Image by TC-MWST (the Relative Error is 2.72e-3, 3.03e-3, 5.95e-3 and 4.38e-2, respectively)

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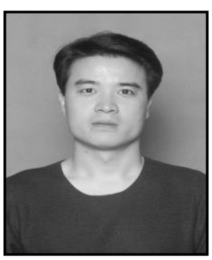

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