International Journal of Signal Processing, Image Processing and Pattern Recognition
Vol.6, No.6 (2013), pp.331-340
http://dx.doi.org/10.14257/ijsip.2013.6.6.30

An Efficient Method for Calculating the Intersection Curve between a Ruled Surface and a Plane

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Abstract

Based on the unique geometric features of the ruled surface, a new efficient algorithm for ruled surface/plane intersection is proposed. The ruled surface is firstly dispersed into a set of line segments, and the ruled surface/plane intersection is transformed to the intersection of a group line segments with a plane. Then a set of ordered intersection points can be obtained by the proposed line/plane intersection algorithm. According to the serial number of every intersection point, all the intersection points are grouped and reorganized. Each group point corresponds to an intersection curve. All the intersection curves can be reconstructed by curve interpolation. Compared with the traditional tracing method, the proposed algorithm can avoid complex calculation including initial points searching and intersection points sorting, which is more efficient and stable.

Keywords: *Ruled surface, Intersection curve, Rectilinear generator, Boundary curve*

1. Introduction

The ruled surface is a basic geometrical element and is widely applied in product design. Calculating the intersection curve between a ruled surface and a plane is an inevitable problem in the technical fields of CAD/CAM, for instance, the surface trimming in geometric modeling, surface slicing in rapid prototyping manufacturing and direction-paralleled tool path planning in NC machining, etc. Since the ruled surface has the geometric features of the sculptured surface, in most existing literature, the intersection of a ruled surface with a plane is usually treated as the intersect of a free-form surface with a plane, in which tracing method is mainly used to calculate the intersection curve [1-3]. However, it is usually difficult to determine the proper initial points for calculating all the intersection curves without omit in tracing method. Unlike common free-form surface, a ruled surface is composed of one-parameter family of straight lines [4]. And the intersection of a line with a plane is relatively simpler to carry out. Given that this property is fully used, the efficiency and accuracy of the intersection calculation will be improved effectively.

In this paper, an efficient method for calculating the intersection curve between a ruled surface and a plane is researched based on the geometric features of ruled surface. A ruled surface is firstly dispersed into a set of straight generatrices. Then the intersection of a ruled surface with a plane can be transformed into a relatively simple problem in which line segment intersects with a plane. By the proposed algorithm, the disadvantages of low efficiency and stability in the traditional tracing method can be effectively overcome.

ISSN: 2005-4254 IJSIP

2. Intersection of a Line Segment with a Surface

As a special kind of surface, Plane has many unique characteristics different from other general freeform surfaces. To simplify the computation and improve the calculation efficiency, the intersection between a line segment and a plane is analyzed independently.

2.1. Intersection of a line segment with a plane

2.1.1. Intersection test of a line segment with a plane: As shown in Figure 1, $\mathbf{P}_0$ is a point on a plane $S(u,v)$, and $\mathbf{n}$ denotes the normal vector of plane S. There is also a line segment L with two endpoints $\mathbf{P}_1$ and $\mathbf{P}_2$. Vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ are respectively constructed with the same starting point $\mathbf{P}_0$, and two different terminal points $\mathbf{P}_1$ and $\mathbf{P}_2$. Assuming that α and β are respectively the angles between $\mathbf{t}_1$ and $\mathbf{t}$, $\mathbf{t}_2$ and $\mathbf{t}$, according to the geometrical theory, α and β can be calculated as shown in Eq. (1).

$$\left.\begin{aligned} \cos\alpha &= \frac{\mathbf{t}_1\mathbf{n}}{|\mathbf{t}_1||\mathbf{n}|} \\ \cos\beta &= \frac{\mathbf{t}_2\mathbf{n}}{|\mathbf{t}_2||\mathbf{n}|} \end{aligned}\right\} \tag{1}$$

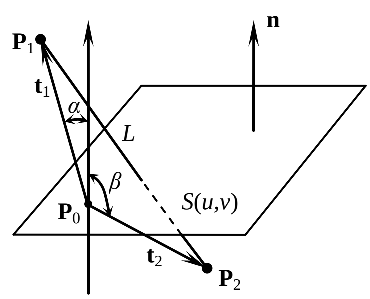


Figure 1. Relative position of a line segment and a plane

As illustrated in Figure 1, when the point $\mathbf{P}_1$ is located on the side in which the normal vector $\mathbf{n}$ points, the corresponding angle α is an acute angle with a negative cosine ($\cos\alpha<0$). While $\mathbf{P}_2$ is located on the side away from the normal vector $\mathbf{n}$, the corresponding angle β is an obtuse angle with a positive cosine ($\cos\beta>0$).

This analysis result shows that when a certain spatial point is located on the side in which the normal vector of a plane points, the included angle between the normal vector and the vector constructed by this point and any point on this plane is an acute angle. On the contrary, when the location of the spatial point is on the side away from the normal vector, the included angle is an obtuse angle. And when the spatial point is on the plane, the included angle is a right angle.

Therefore, based on the above rules, the intersection of a line segment L and a plane $S(u,v)$ can be efficiently tested:

(1) If $\cos\alpha \cdot \cos\beta > 0$, the endpoints $\mathbf{P}_1$ and $\mathbf{P}_2$ of line segment L locate on the same side of plane S, and L does not intersect with $S(u,v)$;

(2) If $\cos\alpha \cdot \cos\beta < 0$, the endpoints $\mathbf{P}_1$ and $\mathbf{P}_2$ of line segment L respectively locate on the different sides of plane S, and L intersects with $S(u,v)$;

(3) If $\cos\alpha = 0, \cos\beta \neq 0$, the endpoint $\mathbf{P}_1$ of line segment L lies on the plane S while another endpoint P_2 is outside of plane $S(u,v)$;

(4) If $\cos\alpha \neq 0, \cos\beta = 0$, the endpoint $\mathbf{P}_2$ of line segment L lies on the plane S while $\mathbf{P}_1$ is outside of plane $S(u,v)$;

(5) If $\cos\alpha = 0, \cos\beta = 0$, the line segment L lies on the plane $S(u,v)$.

2.1.2. Intersection point of a line segment with a plane by vector method: As shown in Figure 2, $\mathbf{n}$ is the normal vector of plane $S(u,v)$. The vector equation of line segment L can be expressed in Eq. (2).

$$\boldsymbol{\rho} = \mathbf{P}_1 + \mathbf{v}t(0 \leq t \leq 1) \tag{2}$$

Where $\boldsymbol{\rho}$ stands for a vector point on L, and $\mathbf{P}_1$ denotes the start point of L. $\mathbf{v}$ is the direction vector of L and $\mathbf{t}$ represents the corresponding parametric variable of point $\boldsymbol{\rho}$.

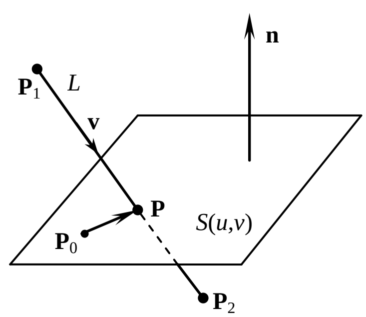


Figure 2. Intersection of a line segment with a plane

If the line segment L intersects with plane $S(u,v)$, the intersection point $\mathbf{P}$ should satisfy the following equation in Eq. (3).

$$\left(\mathbf{P} - \mathbf{P}_0\right) \cdot \mathbf{n} = 0 \tag{3}$$

As the intersection point lies on the line segment L, Eq. (2) should also be satisfied. Couple Eq. (2) and Eq. (3), Eq. (4) can be obtained as following.

$$\left(\mathbf{v}t + \mathbf{P}_1 - \mathbf{P}_0\right) \cdot \mathbf{n} = 0 \tag{4}$$

Based on Eq. (4), the parametric variable t of intersection point $\mathbf{P}$ can be calculated and then the coordinates of point $\mathbf{P}$ can be worked out. If $S(u,v)$ is a parametric plane, the method for calculating the project point from a point onto a plane introduced in literature [5] can be used to reversely calculate the parameters u and v of point $\mathbf{P}$ on plane $S(u,v)$.

2.2. Intersection of a line segment with a freeform surface

As shown in Figure 3, when the two endpoints $\mathbf{P}_1$ and $\mathbf{P}_2$ of a line segment L are on different sides of the surface $R(s,t)$, there will be an intersection point of L and $R(s,t)$.

According to the method in literature [6], the orthogonal projections $\mathbf{P}'_1$ and $\mathbf{P}'_2$ from $\mathbf{P}_1$ and $\mathbf{P}_2$ onto surface $R(s,t)$ can be respectively calculated in Eq. (5).

$$\begin{cases} \mathbf{P}_1'\mathbf{P}_1 \cdot \mathbf{P}_1\mathbf{P}_2 < 0 \\ \mathbf{P}_2'\mathbf{P}_2 \cdot \mathbf{P}_1\mathbf{P}_2 > 0 \end{cases} \tag{5}$$

Where $\mathbf{P}'_1\mathbf{P}_1$ and $\mathbf{P}'_2\mathbf{P}_2$ are separately the vectors constructed from the projective points $\mathbf{P}'_1$ and $\mathbf{P}'_2$ to the corresponding points $\mathbf{P}_1$ and $\mathbf{P}_2$ on L. $\mathbf{P}_1\mathbf{P}_2$ is a vector constructed by two endpoints of L.

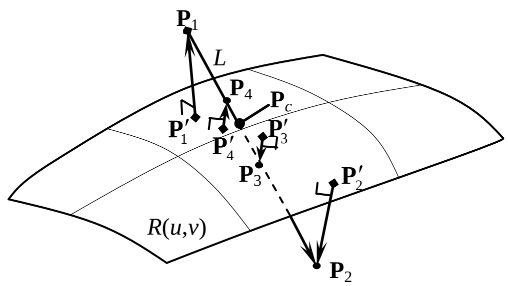


Figure 3. Intersection of a line segment with a surface

The midpoint $\mathbf{P}_3$ of line segment L is extracted and its orthogonal projection $\mathbf{P}'_3$ on surface $R(s,t)$ is calculated by the same method. A vector $\mathbf{P}'_3\mathbf{P}_3$ is constructed to calculate the dot product with $\mathbf{P}_1\mathbf{P}_2$. Then the dichotomy is used to search the intersection point of the line segment with the surface.

(1) If $\mathbf{P}_3'\mathbf{P}_3 \cdot \mathbf{P}_1\mathbf{P}_2 > 0$, the points $\mathbf{P}_3$ and $\mathbf{P}_2$ are located on the same sides of the surface $R(s,t)$. And $\mathbf{P}_3$ is used to replace $\mathbf{P}_2$ in order to construct a new line segment $\mathbf{P}_1\mathbf{P}_3$. Then the midpoint $\mathbf{P}_4$ of this new line segment is utilized to calculate its orthogonal projection $\mathbf{P}'_4$ and to construct a vector $\mathbf{P}'_4\mathbf{P}_4$. If $\mathbf{P}_4'\mathbf{P}_4 \cdot \mathbf{P}_1\mathbf{P}_3 < 0$, the points $\mathbf{P}_4$ and $\mathbf{P}_1$ are on the same sides of surface $R(s,t)$, and $\mathbf{P}_1$ is replaced by $\mathbf{P}_4$ to construct a new line segment $\mathbf{P}_4\mathbf{P}_3$. By the above method, the iterative search process is repeated and new line segment is constructed. The iterative process will terminate until the strength of the line segment is less than the given threshold or the distance from the iterative closet point to the surface equals zero, and the current iterative closet point is the intersection point of line segment L with surface $R(s,t)$.

(2) If $\mathbf{P}_3'\mathbf{P}_3 \cdot \mathbf{P}_1\mathbf{P}_2 < 0$, the points P_3 and P_1 are on the same side of the surface. Point $\mathbf{P}_1$ should be replaced by $\mathbf{P}_3$ to construct a new line segment $\mathbf{P}_3\mathbf{P}_2$, and the intersection point of the line segment L with surface $R(s,t)$ can be obtained by the dichotomy as described in (1).

3. Intersection of a Ruled Surface with a Plane

Figure 4 shows a parametric plane $S(u,v)$ and a ruled surface $R(s,t)$. The equation of the ruled surface $R(s,t)$ is shown in Eq. (6).

$$\mathbf{R}(s,t) = \mathbf{B}(t) + s\mathbf{V}(t) \tag{6}$$

Where s and t are the surface parameters of the ruled surface. $\mathbf{R}(s,t)$ stands for the vector point which corresponds with the parameter (s,t) on the ruled surface. $\mathbf{B}(t)$ and $\mathbf{V}(t)$ are respectively the directrix and the generatrix of the ruled surface $R(s,t)$.

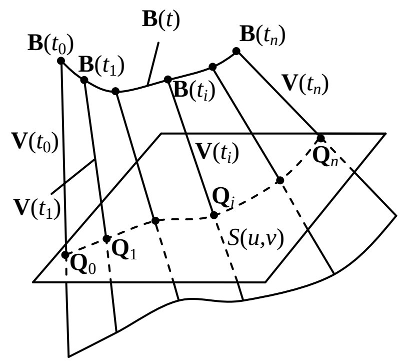


Figure 4. Intersection of a ruled surface with a plane

The intersection curve between a ruled surface and a plane cannot be calculated by algebraic or analytical method, and the numerical method is supposed to use. In this paper, the discrete method is utilized to obtain the intersection curve of a ruled surface with a plane. Firstly, the ruled surface is converted into a group of line segments. By the method described in the above section, the successive intersection points of the line segments with a plane can be calculated accurately. Then all the intersection points are grouped and reorganized. Finally, the intersection curve is obtained by the interpolation method.

3.1. Discretization of a ruled surface

The directrix parameter field of a ruled surface is divided into n sections equally and a sequence of parameters $\{ t_i \mid 0 \le i \le n \}$ are obtained. Parameter t_i can be calculated by Eq. (7).

$$t_i = t_0 + \frac{t_n - t_0}{n} i \tag{7}$$

Where t_0 and t_n are respectively the parameters of the start point and terminal point of the directrix.

Based on these parameters, a corresponding point sequence $\{ \mathbf{B}(t_i) \mid 0 \le i \le n \}$ can be obtained on the directrix as shown in Figure 4. According to the definition of a ruled surface, any point $\mathbf{B}(t_i)$ $(0 \le i \le n)$ on the directrix corresponds with a rectilinear generator as shown in Eq. (8).

$$\mathbf{L}_i = \mathbf{B}(t_i) + s\mathbf{V}(t_i) (0 \le i \le n) \tag{8}$$

By the above method described in section 2.1.1, the intersection tests of all discrete rectilinear generators with a plane are carried out successively. If an intersection occurs, the intersection point can be calculated accurately according to the algorithm described in Section 2.1.2.

3.2. Reconstruction of the intersection curve

The intersection curves may be one or several open curves or closed curves, depending on the structure of the ruled surface and its relative position with the plane. Therefore, intersection points sorting and reconstruction should be performed after a sequence of intersection points are obtained according to the above method. In order to simplify the reconstruction process, the subscript of every intersection point is asked to be the same with that of its corresponding discrete line segment. If some line segment does not intersect with a plane, there is no intersection point with the same subscript in the sequence.

Once all the intersection points are obtained, a traversing process for the sequence of the points is performed in order to reconstruct all the points by the following method:

(1) If the sequence is empty, the ruled surface $R(s,t)$ does not intersect with the plane $S(u,v)$;

(2) If the serial numbers of the subscripts are continuous and the start point is not coincided with the terminal point, the intersection curve is an open curve, as shown in Figure 4;

(3) If the serial numbers of the intersection points are continuous and the start point $\mathbf{Q}_0$ is coincided with the terminal point $\mathbf{Q}_n$, the intersection curve is a close curve, as shown in Figure 5;

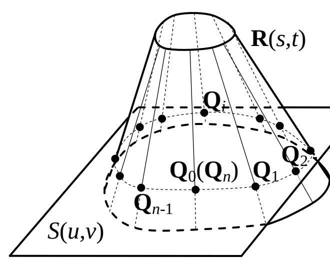


Figure 5. A closed sequence of intersection points

(4) If the serial numbers of the intersection points are not continuous, then several intersection curves are produced and each intersection curve is formed by a group of the successive points, as shown in Figure 6;

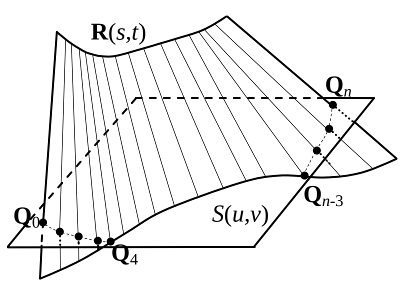


Figure 6. A sequence of discontinuous intersection points

(5) For a boundary plane, when its boundary curves are intersected with a ruled surface, the intersection points are the endpoints of the intersection curve. It's difficult

to obtain the intersection points based on the above discrete method. Therefore, a special process should be performed. As can be seen in Figure 7, the four boundary curves of a boundary surface $S(u,v)(u_0 \leq u \leq u_1, v_0 \leq v \leq v_1)$ are respectively $S(u_0,v)$、$S(u_1,v)$、$S(u,v_0)$、$S(u,v_1)$, of which the curves $S(u_0,v)$ and $S(u,v_1)$ are intersected with the ruled surface and the intersection points are not in the sequence obtained by the discrete method. To calculate the intersection curves accurately, the intersection algorithm of line segments with a ruled surface described in section 2 is essential to find the intersection points $\mathbf{Q}_{u0}$ and $\mathbf{Q}_{v1}$ of the two boundary lines with the ruled surface. Then, points $\mathbf{Q}_{u0}$ and $\mathbf{Q}_{v1}$ are respectively inserted into the start and terminal position of the sequence to reconstruct a new sequence of points;

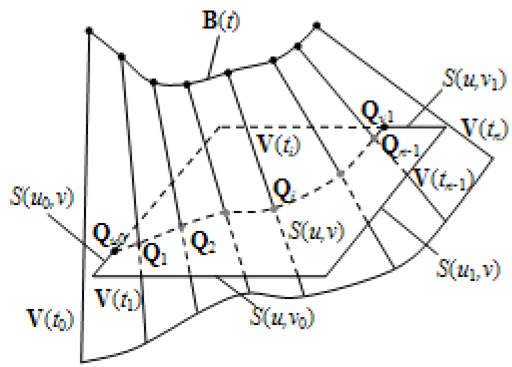


Figure 7. Intersection points on the plane boundary

(6) The two boundary lines of a ruled surface are respectively the start and terminal rectilinear generators, and the other two are boundary curves. If any boundary curve intersects with the plane and the intersection point is not in the intersection sequence, the corresponding intersection point can be calculated by the method introduced in literature [7], then the point should be inserted into the sequence as the above method described. As shown in Figure 8, $\mathbf{Q}_{t1}$ and $\mathbf{Q}_{t2}$ are respectively the intersection points of the two boundary curves with a plane. Meanwhile they are also the endpoints of the two intersection curves.

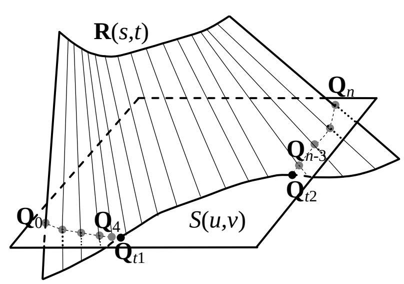


Figure 8. Intersection points on the plane boundary

After all the intersection points are obtained and grouped by the above method, the intersection curves can be reconstructed according the curve interpolation algorithm in literature [8]. If there is more than one curve, the process should be performed one by one.

4. Case study

By the proposed algorithm, an intersection program of a ruled surface with a plane is developed based on the IGES standard in Visual Studio 2005. The intersection curves of a ruled surface with a plane can be calculated by this program and outputted to IGES files, which can be displayed and edited in any CAD software.

Figure 9 shows a sketch when a positive spiral surface intersects with a plane. A positive spiral surface is a principal normal plane of a positive spiral curve. It's a ruled surface with equation $\mathbf{R}(s,t)=(s\cos t, s\sin t, 3.0t)$, where $0<s<20$, $0<t<3\pi$. Plane $S(u,v)$ is a boundary plane which is paralleled to plane YOZ, with four corner points $\mathbf{P}_{00}(10, 35, 25)$, $\mathbf{P}_{01}(10, 35, -25)$, $\mathbf{P}_{11}(10, -8, 25)$, $\mathbf{P}_{11}(10, -8, -25)$. If the plane is defined with its normal vector and a known point on the plane, its equation can be expressed as $\mathbf{n}(\mathbf{S}(u,v)-\mathbf{P}_{00})=0$, where $\mathbf{n}$ is the unit normal vector of plane $S(u,v)$, and its direction vector is (1, 0, 0).

A group of 120 uniform distribution points are determined on the directrix $\mathbf{B}(t)=(20\cos t, 20\sin t, 3.0t)(0<t<3\pi)$ of the positive spiral surface. By extracting the corresponding rectilinear generators one by one, the surface can be converted into a family of straight line segments as shown in Figure 10. According to the proposed algorithm in section 2.1.2, the intersection points of the rectilinear generators with the plane can be obtained, and the corresponding intersection sequence can also be formed as shown in Figure 11. According to the algorithm of a curve with a plane, three boundary intersection points $\mathbf{Q}_{t1}(10, 17.32051, 3.14159)$, $\mathbf{Q}_{t2}(10, -17.32051, 15.70796)$ and $\mathbf{Q}_{t3}(10, 17.32051, 21.99115)$ are calculated as shown in Figure 12. Based on the principle of the minimum distance, the three points are inserted into both the ends of the intersection sequence and two new intersection sequences are obtained. Finally, based on the Cubic B-spline interpolation algorithm, two intersection curves can be obtained as shown in Figure 13.

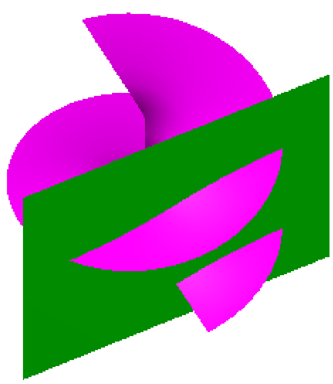


Figure 9. A spiral surface and a plane

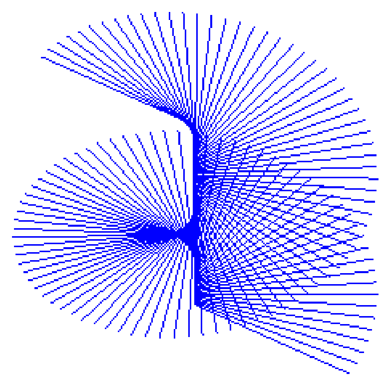
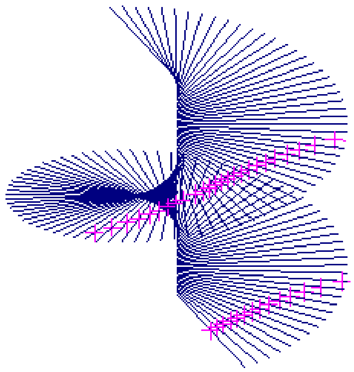


Figure 10. Discrete rectilinear generators



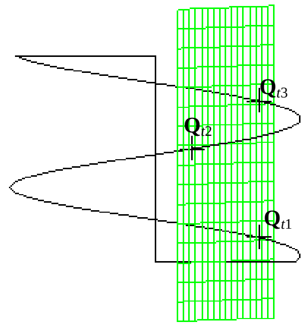


Figure 11. Intersection of lines with a plane **Figure 12. Intersection points on boundary**

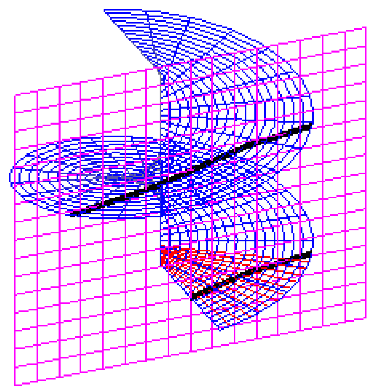


Figure 13. Intersection curves of a positive spiral surface with a plane

5. Conclusions

In this paper, the intersection of a ruled surface with a plane has been converted into the intersection of a plane with line segments or curves. The problems of tracing the initial point, intersection sorting and tracing multi-intersection curves in the traditional tracing method have been overcome, which can reduce the complexity and computation, and improve the algorithm's stability. By the proposed algorithm, all the intersection curves of any ruled surface with a plane can be calculated without omit.

Acknowledgements

This work was supported in part by grants from National Natural Science Foundation of China (Grant No. 51375282), Shandong Provincial Natural Science Foundation of China (Grant No. ZR2010EM013) and Innovation Foundation for Graduate Students of Shandong University of Science & Technology (Grant No. YC130106).

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