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Control of Stochastic Resonance in Overdamped Fractional Langevin Equation

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Abstract

A system based on the model of overdamped fractional Langevin equation is constructed, the relation between fractional order and stochastic resonance is studied, it is found that stochastic resonance phenomenon is not produced or not obvious with given certain weak periodic signal and noise intensity. Applying external periodic signal is proposed to produce or strengthen stochastic resonance. The numerical simulations show that the stochastic resonance phenomenon can be effectively controlled, by changing the external periodic signal's frequency or amplitude. We find the non-monotonic behaviors between the driving parameters and signal-to-noise(SNR). Optimal stochastic resonance can be achieved by adjusting one of the parameters as frequency and amplitude.

***Keywords**: Stochastic resonance, overdamped fractional Langevin equation, control*

1. Introduction

Stochastic resonance (SR) is a nonlinear phenomenon that exploits internal or external noise to enhance a system's response to a monochromatic signal [1]. Originally proposed as a potential mechanism for the occurrence of the terrestrial ice ages, SR has since been demonstrated in diverse fields, involving physical, chemical and biological systems [2-4]. In most of literatures, models that used to demonstrate SR are usually restricted to integer-order equations, but many complicate systems are difficult to be characterized by traditional integer-order equations, such as anomalous diffusion which exists in the amorphous semiconductors [5]. In normal diffusion, the relation of mean-square displacement and time is $\langle \delta x^2(t) \rangle \propto t$. While in the anomalous diffusion, their relation is $\langle \delta x^2(t) \rangle \propto t^{\alpha}$, where $\alpha > 1$ and $0 < \alpha < 1$ indicate superdiffusion and subdiffusion [6]. The fractional Langevin equation (FLE) is mainly used to model such anomalous diffusion, that replacing the usual friction term by a power-law-type memory in Generalized Langevin equation (GLE) [7,8]. In recent decades, SR phenomenon is investigated in some systems described by FLE. For example, SR phenomenon is observed in subdiffusive transport occurring in a viscoelastic environment based on the coupling of Brownian particle to a thermal bath of harmonic oscillators[9], the FLE can also be derived from a standard Hamiltonian model of Brownian motion[10]. Gao Shi-Long et al. explored SR in FLE, discussed the physical meaning and showed the SNR gain in FLE is better than that of the integer-order situation [11].

In previous studies, optimal SR is obtained by adjusting the intensity of the noise, but in

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the actual physical phenomenon, in the case of a given noise and weak input signal, the SR is difficult or not capable to be occurred. In order to regulate and motivate SR, an external periodic signal can be applied to control the SR [12].

In this paper, in studying the SR caused by overdamped fractional Langevin equation in bistable system, we note that adding external signal can enhance or depress the response of a noisy bistable system to monochromatic signals, significantly influence its natural SR. We found the relation between the driving parameters and signal-to-noise (SNR) is non-monotonic. Optimal SR can be achieved by adjusting one of the parameters as frequency and amplitude.

The structure of the paper is as follows. In Section 2 we present the basic model investigated and the relation between fractional order and stochastic resonance. Section 3 presents the method applying an external period signal. In Section 4 we analyze the behavior of the output response, and explore the main results of this paper. Section 5 contains some brief concluding remarks.

2. Models

The fractional calculus operator ${}_aD_t^\alpha$ is defined as

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \mathrm{Re}(\alpha) > 0 \\ 1 & \mathrm{Re}(\alpha) = 0 \\ \int_a^t (d\tau)^{-\alpha} & \mathrm{Re}(\alpha) < 0 \end{cases} \tag{1} $$

where α is real order, a and t are the bounds of the operation.

There are three definitions to describe the fractional calculus, they are Grünwald-Lentnikov (GL) definition, the Riemann-Liouville (RL) and the Caputo definition [13-14]. The GL is given as

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{\left[\frac{t-a}{h}\right]}(-1)^j\binom{\alpha}{j} f(t-jh) \tag{2} $$

where $[\cdot]$ means the integer part, $\binom{\alpha}{j}$ is the binomial coefficients. The RL definitions is given as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \tag{3} $$

for ($n-1<\alpha<n$) and where $\Gamma(\cdot)$ is the Gamma function. The Caputo's definition is written as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \tag{4} $$

for ($n-1<\alpha<n$). Usually, we take a lower limit $a=0$ for the above three definitions.

We consider a GLE for a Brownian particle with mass *m*,

$$m\ddot{x}+\int_0^t \eta(t-t')\dot{x}(t')dt' = -\frac{\partial U(x)}{\partial x}+F(t)+\xi(t) \tag{5}$$

When in overdamped limit $m \to 0$, and in the bistable state, then $U(x) = -\frac{a}{2}x^2+\frac{b}{4}x$, $F(t) = A\cos(2\pi\omega t)$. By taking a power law decaying memory kernel $\eta(t) = |t|^{-\alpha}/\Gamma(1-\alpha)$ with $0<\alpha<1$, $\Gamma(\cdot)$ is standard gamma-function, the reduced equation is:

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-t')^{-\alpha}\dot{x}(t')dt' = ax - bx^3 + A\cos(2\pi\omega t) + \xi(t) \tag{6}$$

According to the Caputo's definition, the left of above equation is defined as ${}_0^C D^{\alpha} x(t)$, thus the overdamped fractional Langevin equation is defined[11]:

$${}_0^C D^{\alpha} x(t) = ax(t) - bx^3(t) + A\cos(2\pi\omega t) + \xi(t) \tag{7}$$

where ${}_0^C D^{\alpha} x(t)$ is the α order fractional order derivative to $x(t)$ by using Caputo's definition, and $0<\alpha<1$, $ax(t)-bx^3(t)$ is the bistable potential function, $A\cos(2\pi\omega t)$ is a low-frequency weak signal, $\xi(t)$ is a zero-mean, Gaussian white noise, whose autocorrelation function is $\langle \xi(t)\xi(0)\rangle = 2D\delta(t)$, *D* is the intensity of noise.

We choose the parameters a=b=1, A=0.3, f=0.02, D=1 for Eq.(7). By changing the fractional order α while the other parameter are fixed, the time domain and frequency spectrum of output response *x*(*t*) are observed and analyzed. As shown in Figure 1, there is no SR when $\alpha = 0.95$, and the peak amplitude of power spectrum is 0.08832. When $\alpha = 0.7$, the SR is significantly occurred, its peak amplitude of power spectrum 0.5033>0.08832, as shown in Figure 2, but when α decreases to 0.55, the SR phenomenon is depressed, its peak amplitude of power spectrum decreased to 0.3311, as shown in Figure 3. It can be seen from Fig.4 that there is one fractional order α that maximize the SNR of output response.

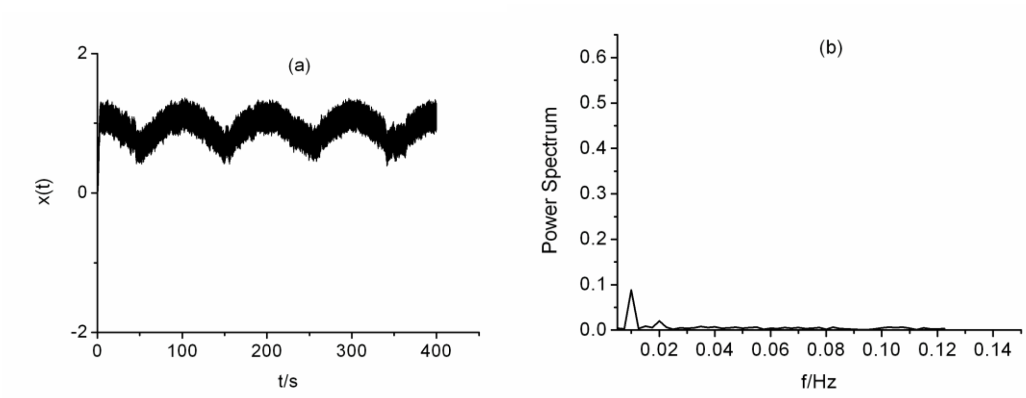

Figure 1. Output response *x*(*t*) when $\alpha = 0.95$: (a) time domain waveforms; (b) power spectrum

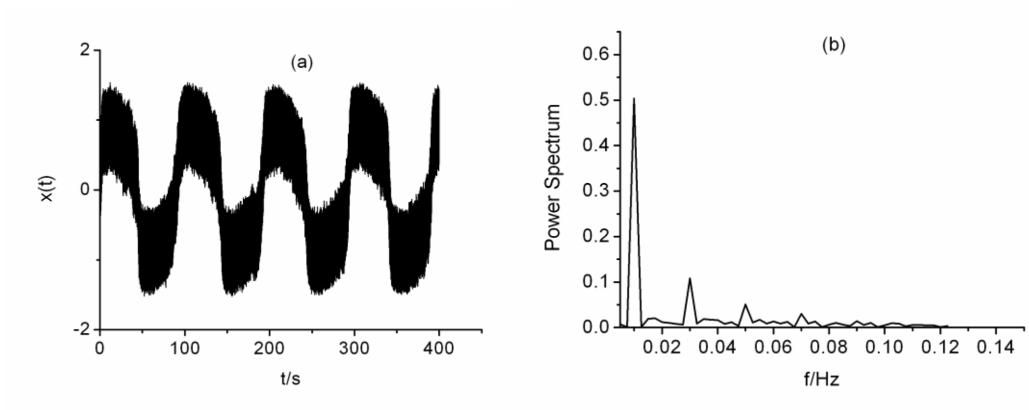

Figure 2. Output response *x*(*t*) when $\alpha = 0.7$: (a) time domain waveforms; (b) power spectrum

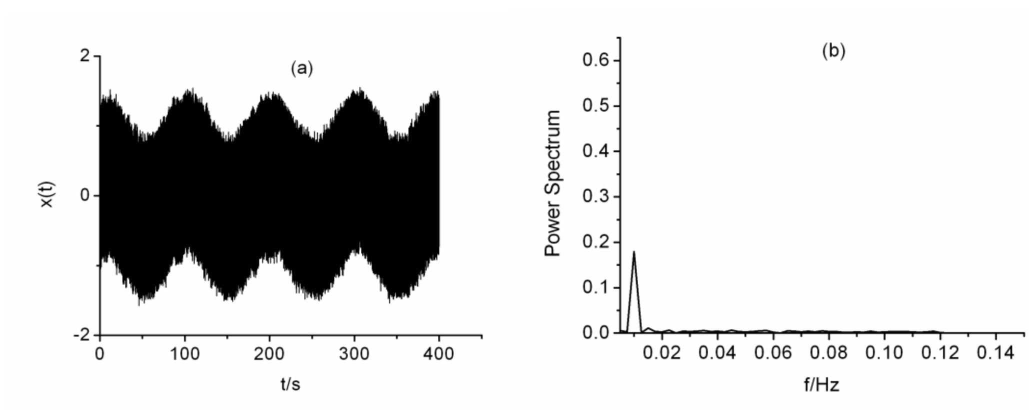

Figure 3. Output response *x*(*t*) when $\alpha = 0.55$: (a) time domain waveforms; (b) power spectrum

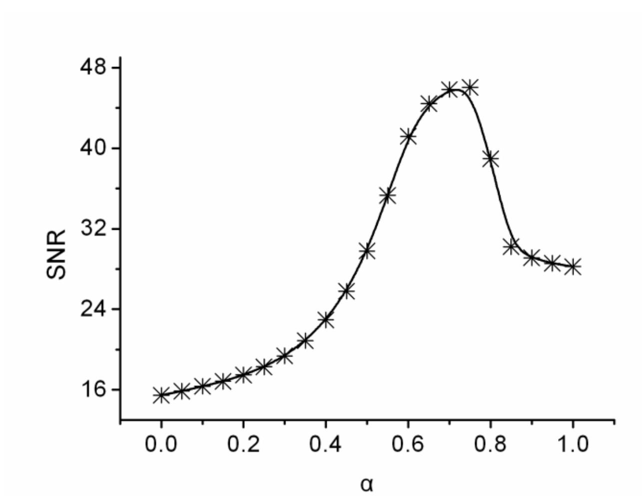

Figure 4. Relation of SNR versus fractional order α

It can be seen that fractional order α relates with the memory of Brown particle movement. When $\alpha \to 1$, the Brown particle movement loses memory, the system turns into traditional integer-order bistable SR. When $\alpha \to 0$, the Brown particle movement has same memory of past each moment. When $0 < \alpha < 1$, the Brown particle movement of overdamped fractional Langevin system is consisted of the intrawell random fluctuations and the transition between two potential wells. Optimal SR is achieved when the period signal, noise and system parameter are chosen of certain value [11]. If the signal, noise and system parameter are not well matched, SR would not appear or negligible, an external period signal can be applied to control the overdamped fractional Langevin system to regulate and motivate SR.

3. Controlling SR with Periodic Signal

As shown in Figure 5, the external periodic signal is introduced into the overdamped fractional Langevin system, that is a control signal applied to the Eq.(7), and the dynamic equation is:

$$ {}_{0}^{C}D^{\alpha} x(t) = ax(t) - bx^{3}(t) + A\cos(2\pi\omega t) + B\cos(2\pi\Omega t) + \xi(t) \qquad (8) $$

Where B is the amplitude and Ω is the frequency of applied external periodic signal. If the parameters of overdamped fractional Langevin system are set, and the weak periodic signal and noise signal are given, the amplitude B and frequency Ω can be taken as control variables to analyze the system output signal $x(t)$ as an observable variable.

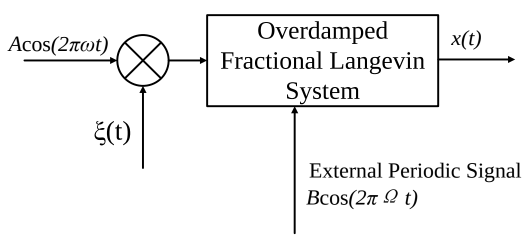

Figure 5. Controlling SR with period signal

4. Numerical Simulation Results and Analysis

Eq.(8) shows that the output response $x(t)$ is related with the parameter a and b, parameters A and ω of the input signal $A\cos(\omega t)$, parameters B and Ω of control signal $B\cos(\Omega t)$, and intensity D of the noise $\xi(t)$. The following analysis is based on the overdamped fractional Langevin system with parameters: a=b=1, A=0.3, f=0.02, D=1 and fractional order $\alpha = 0.75$. The variable external periodic signals and fractional order are applied to the system; the control effects to the output response are realized and analyzed.

4.1 Amplitude changing while frequency fixed

When B = 0, $\Omega = 11$ there is no external periodic signal applied to the system, Figure 6 shows the time-domain and frequency spectrum of output response $x(t)$. As shown in Figure 6(b), it can be seen that the output response $x(t)$ cannot cross the barrier into another potential well, it only fluctuate in the single potential well, thus the phenomenon of SR is not occurred, the corresponding output power spectrum is very small P=0. 08158.

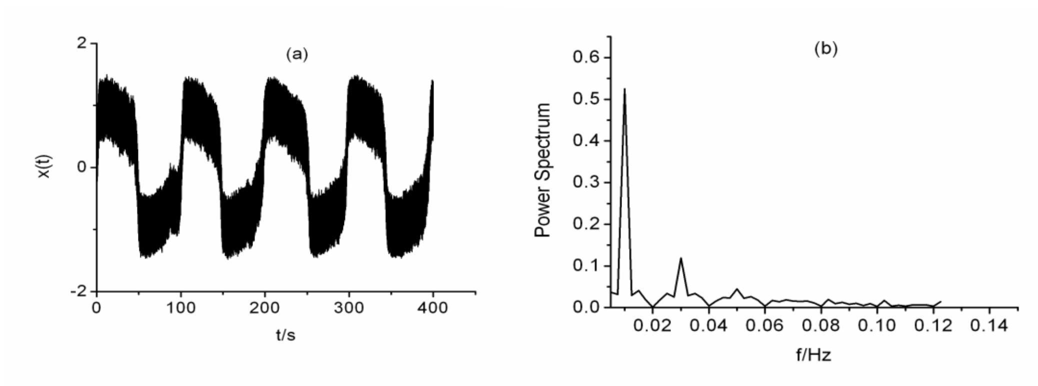

Figure 6. Output response *x*(*t*) when *B*=11:
(a) time domain waveforms; (b) power spectrum

When the amplitude of external periodic signal B is increased to 20, while other parameters remain unchanged, the time domain and power spectrum of output response are shown in Figure 7. It can be seen from Figure 7(b) that the peak amplitude of power spectrum 0.3676 <0.5247, that indicates SR is depressed by increasing the amplitude of external periodic signal.

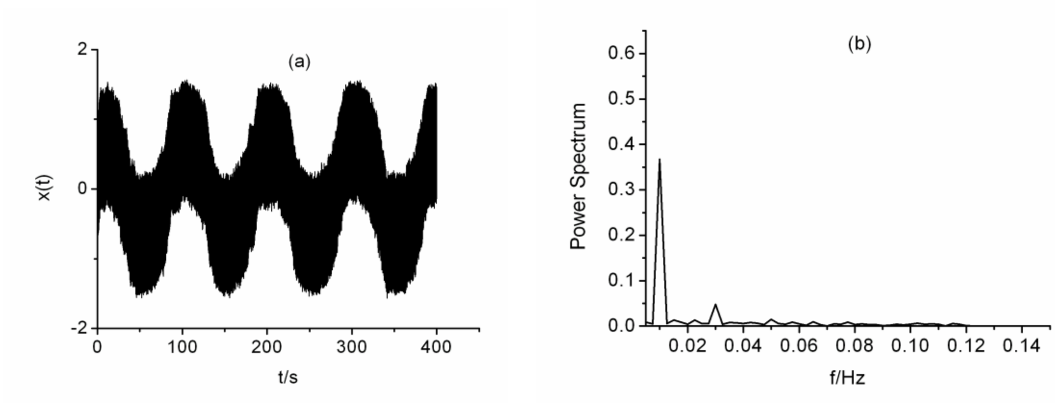

Figure 7. Output response *x*(*t*) when *B*=20:
(a) time domain waveforms; (b) power spectrum

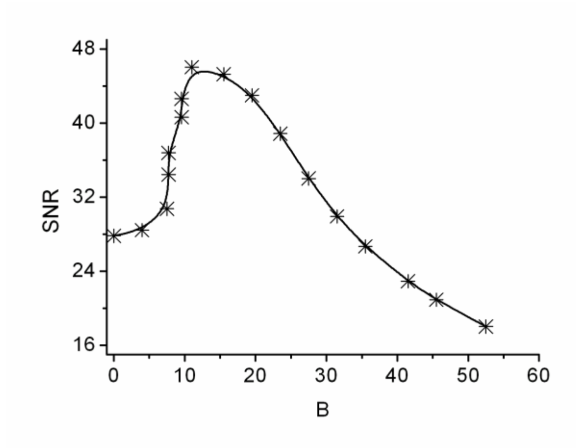

Figure 8. Relation of SNR versus amplitude *B*

It can be concluded from the above analysis that after introducing the external periodic signal (frequency fixed while amplitude variable), SR can be enhanced by increasing the amplitude. But when very large B>>11 it will destroy the SR by rendering the inter-well barrier insignificant and the potential effectively monostable, so SNR decreases monotonically, thus depress the SR. As shown in Figure 8, when the other parameters are fixed while only *B* changes, there exists one value that make the SNR of output reach maximum, then the system have optimal SR.

4.2 Frequency changing while amplitude fixed

As shown in Figure 9, there is no SR when $\Omega = 50$, and the peak amplitude of power spectrum is 0.1113. When $\Omega = 10$, the SR is significantly occurred, its peak amplitude of power spectrum 0.522>0.1113, as shown in Figure 10, but when Ω decreases to 4, the SR phenomenon is depressed, its peak amplitude of power spectrum decreases to 0.3311, as shown in Figure 11. From the Figure 12, it can be seen that there is one frequency that maximize the SNR of output response.

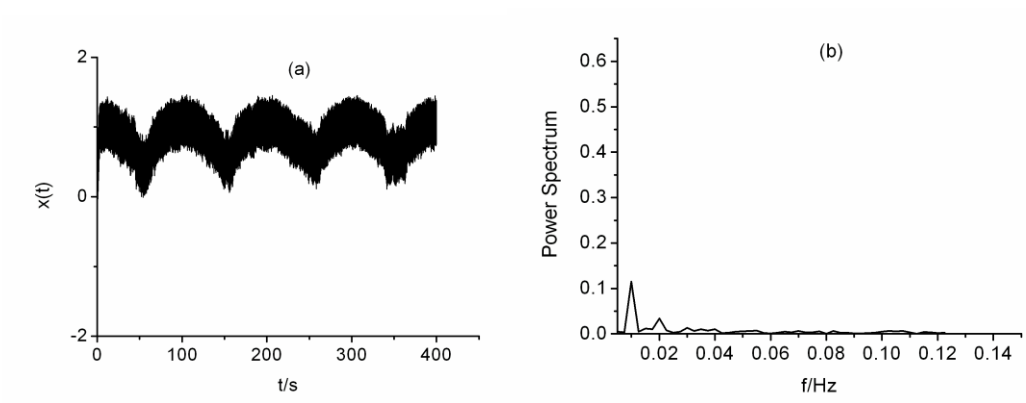

Figure 9. Output response *x*(*t*) when $\Omega = 50$:
(a) time domain waveforms; (b) power spectrum

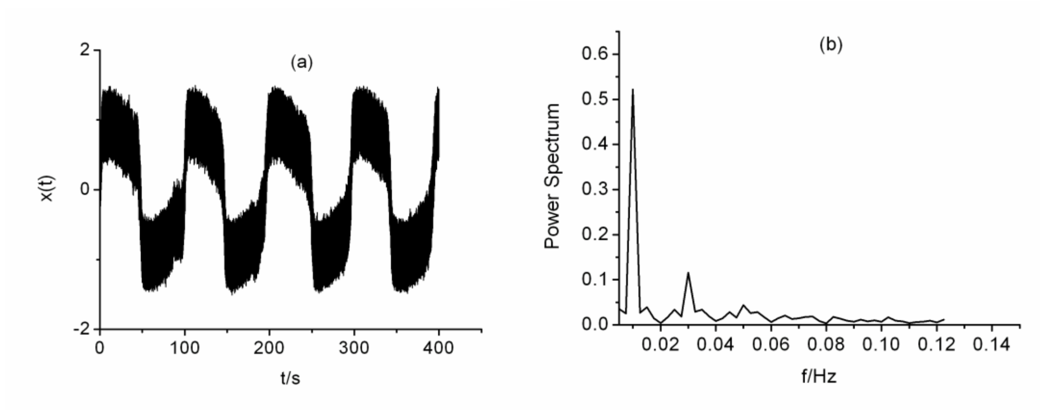

Figure 10. Output response *x*(*t*) when $\Omega = 10$:
(a) time domain waveforms; (b) power spectrum

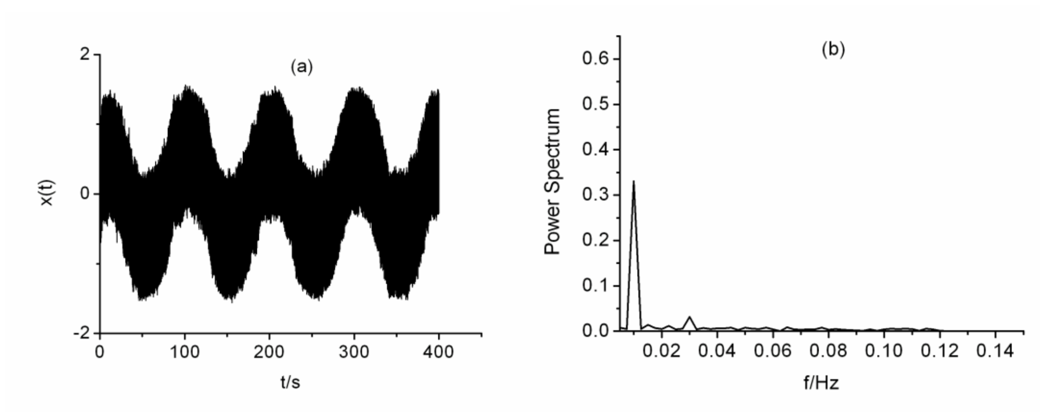

Figure 11. Output response *x*(*t*) when $\Omega = 4$:
(a)time domain waveforms; (b) power spectrum

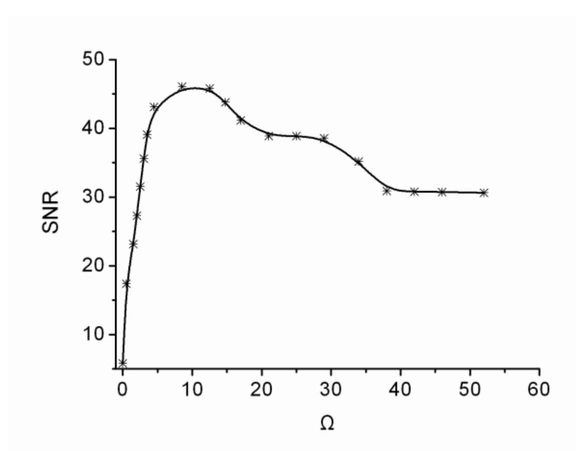

Figure 12. Relation of SNR versus frequency Ω

5. Conclusions

The effect of external periodic signal control for SR of overdamped fractional Langevin equation is analyzed. By changing the frequency, amplitude or the fractional order, the external periodic signal can lead the fluctuations in the single potential well thus transition between two potential wells, SR can be controlled to be enhanced or depressed, the optimal SR state can be achieved by changing each one of the parameters.

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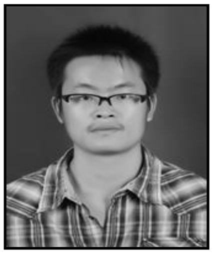

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