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Time-frequency Analysis Based on the S-transform

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Abstract

S-transform is a new time-frequency analysis method, which is deduced from short-time Fourier transform and continue Wavelet transform. It has much better performance than traditional time-frequency method. Therefore, in this paper, the basic principle of is briefly introduced and the relationships between is analyzed by theoretical derivation. According to the simulation experiments, the time-frequency space characteristics of short-time Fourier transform, Wigner-Ville distribution and S-transform are contrasted. As the results shown, the window of S-transform has a progressive frequency dependent resolution. So the S-transform has a great flexibility and utility in the processing of non-stationary signal. Compare with the time-frequency spectrum of three different analysis methods under various noise conditions, it is obvious that S-transform has much better anti-noise performance than that of traditional methods for non-stationary signal processing. Based on the superior time-frequency resolution, the S-transform spectrum can be used to describe the structure of incoming signal effectively.

Keywords: *Time-Frequency Analysis S-transform; Short-time Fourier Transform; Continue Wavelet Transform; Wigner-Ville distribution*

1. Introduction

In non-stationary signal processing field, the time-frequency transform is an important method. The Fourier transform is only map the signal from one-dimensional time domain to one-dimensional frequency domain. After the transformation, the signal has a very good frequency resolution, but the time resolution lost completely. This is not for non-stationary signal processing. Time-frequency analysis method is one that converses the signal from time domain to time-frequency domain and analyzes the non-stationary signals. The short-time Fourier transform, continue Wavelet transform and Wigner-Ville distribution [1] are the most common way to analyze the non-stationary signals.

The S-transform [2] is first proposed by R. G. Stockwell in 1996. It is unique in that it provides frequency-dependent resolution while maintaining a direct relationship with the Fourier spectrum. It is an extension of the ideas of the short-time Fourier transform and is based on a moving and scalable localizing Gaussian window. It is shown to have come desirable characteristics that are absent in continue Wavelet transform. Pinneger [3, 4] *et al.*, used it to process the non-stationary signals added Gaussian white noise. The article explores the S-transform from two perspectives, and described the time-frequency analysis features through simulation experiments. The S-transform has good performance of noise reduction in comparative experiments.

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2. The Introduction of S-Transform

2.1. Deduce S-transform from Short-time Fourier Transform

The Fourier transform lacks the skill to position time and frequency at the same time. It is not available for the time-frequency localization. Gabor first proposed that it could adopt a moving and scalable localizing Gaussian window as the base function. It defined by

$$w_{\tau,\omega}(t) = w(t-\tau)e^{j\omega t} \tag{1}$$

The definition of short-time Fourier transform[5] is

$$STFT(\tau, f) = \int_{-\infty}^{+\infty} x(t)w(t-\tau)e^{-j2\pi ft}dt \tag{2}$$

According to the uncertainty principle, the time-bandwidth product could not get smaller without limits. The Gaussian window is able to combine the time domain and frequency domain. The Gaussian window is defined as

$$\omega(t) = \frac{1}{\delta\sqrt{2\pi}} e^{-\frac{t^2}{2\delta^2}} \tag{3}$$

According to the nature of Gaussian window, δ is the scale factor to change the width of Gaussian window. In order to make the width of Gaussian window have a better self-adaptability to different frequency components. We defined δ as $\delta(f) = \frac{1}{|f|}$, it is a frequency-related function.

The new function is [6]

$$\omega(t, f) = \frac{|f|}{\sqrt{2\pi}} e^{-\frac{t^2 f^2}{2}} \tag{4}$$

The window needs to be 1 mean, as follows:

$$\int_{-\infty}^{+\infty} \omega(t, f)\,\mathrm{d}t = 1 \tag{5}$$

We can get the expression of S-transform:

$$ST(\tau, f) = \int_{-\infty}^{+\infty} x(t)w(t-\tau, f)e^{-j2\pi ft}dt = \int_{-\infty}^{+\infty} x(t)\frac{|f|}{\sqrt{2\pi}} e^{-\frac{f^2(\tau-t)^2}{2}} e^{-i2\pi ft}dt \tag{6}$$

2.2. Deduce S-transform from Continue Wavelet Transform

The definition of continue Wavelet transform is

$$W(\tau, d) = \int_{-\infty}^{+\infty} x(t)\omega(t-\tau, d)dt \tag{7}$$

$\omega(t,d)$ is wavelet mother function. The dilation factor d controls the width of wavelet basis. It also controls the frequency resolution. But the mother wavelet function has to content the admissibility condition on the mother wavelet.

The mother wavelet is defined as

$$\omega(t,f)=\frac{|f|}{\sqrt{2\pi}}e^{-\frac{t^2 f^2}{2}}e^{-j2\pi ft} \tag{8}$$

The S-transform of function $x(t)$ is defined as a continue Wavelet transform with a special mother wavelet multiplied by the phase factor

$$S(\tau,f)=e^{j2\pi f\tau}\int_{-\infty}^{+\infty}x(t)\omega(t-\tau,f)dt \tag{9}$$

The dilation factor d is the inverse of the frequency f. And the wavelet in (7) doesn't satisfy the admissibility condition. So the S-transform is

$$S(\tau,f)=\int_{-\infty}^{+\infty}x(t)\frac{|f|}{\sqrt{2\pi}}e^{-\frac{(\tau-t)^2 f^2}{2}}e^{-j2\pi ft}dt \tag{10}$$

2.3. The inverse S-transform

The S-transform is a representation of local spectrum. It could expect a simple operation of averaging the local spectra over time to give the Fourier spectrum.

$$X(f)=\int_{-\infty}^{+\infty}S(\tau,f)d\tau \tag{11}$$

The Fourier transform of $x(t)$ is

$$X(f)=\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt \tag{12}$$

The relationship between Fourier transform and S-transform is

$$X(f)=\int_{-\infty}^{+\infty}x(t)e^{-j2\pi ft}dt=\int_{-\infty}^{+\infty}S(\tau,f)d\tau \tag{13}$$

The signal $x(t)$ could exactly recoverable from S-transform[7].

$$x(t)=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}S(\tau,f)d\tau\right\}e^{j2\pi ft}df \tag{14}$$

3. The Discrete S-transform

3.1. Deduce S-transform from Short-time Fourier Transform

Because the close relationship between Fourier transform and S-transform, the S-transform can be written as operations on the Fourier spectrum $X(f)$ of $x(t)$

$$ST(\tau,f)=\int_{-\infty}^{+\infty}\left[X(f+\alpha)e^{-\frac{2\pi^2\alpha^2}{f^2}}\right]e^{-i2\pi\alpha\tau}d\alpha \tag{15}$$

Where $(f \neq 0)$.

The Fourier transform of $x(t)$ is $X(f)$. α is a frequency variable.

When it has a frequency of 0 Hz, we can get

$$ST(\tau,0)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}x(t)\,\mathrm{d}t \tag{16}$$

Let $x[kT], k=0,1,2,...,N-1$ denote a discrete time series corresponding to $x(t)$ with a time sampling interval of T. The discrete Fourier transform is given by[1]

$$X\left[\frac{n}{NT}\right]=\frac{1}{N}\sum_{k=0}^{N-1}x[kT]e^{-\frac{i2\pi nk}{N}} \tag{17}$$

where $n=0,1,2,...,N-1$.

Using (10), Let $f_s=\frac{1}{T}$ be the sampling frequency , f_0 be the frequency step, $M=\frac{f_s}{f_0}$ and $m=-\frac{M}{2},...,\frac{M}{2}-1$. $p=0,...,N-1$ is the time index. We can write the discrete S-transform as

$$S[p,m]=\sum_{n=0}^{N-1}x(kT)\frac{|m|}{M\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{m(p-n)}{M}\right)^2}e^{-j2\pi\frac{nm}{M}} \tag{18}$$

Using (15), letting $f=n/(NT)$, $\tau=jT$ and $\alpha=m/(NT)$, the S-transform of a discrete time series[8] $x[kT]$ also can be given by

$$S\left(jT,\frac{n}{NT}\right)=\sum_{m=0}^{N-1}X\left[\frac{m+n}{NT}\right]e^{-\frac{2\pi^2m^2}{n^2}}e^{\frac{i2\pi mj}{N}} \qquad n\neq 0 \tag{19}$$

Using (16), it is equal to the constant defined as:

$$S(jT,0)=\frac{1}{N}\sum_{m=0}^{N-1}x\left(\frac{m}{NT}\right) \tag{20}$$

where $j,n,m=0,1,2,...,N-1$.

The discrete inverse of the S-transform is.

$$x[kT]=\frac{1}{N}\sum_{n=0}^{N-1}\left\{\sum_{j=0}^{N-1}S\left(jT,\frac{n}{NT}\right)\right\}e^{i\frac{2\pi}{N}nk} \tag{21}$$

3.2. Deduce S-transform from Continue Wavelet Transform

The calculation procedure of S-transform as follows:

1) To calculating the Discrete Fourier Transform of $x[kT]$ with a time sampling interval of T, it can get $X(m)$. And making a frequency shifting n, it can get $X(m+n)$.

2) To calculating the Discrete Fourier Transform of the Gaussian window at a specific n, which called a voice Gaussian. The voice Gaussian is $W(n,m)$.

3) Multiply $W(n,m)$ by $X(m+n)$ to obtain $B(n,m)$.

4) The S-transform can be restored by the fast Fourier inverse transform of $B(n,m)$. Then the S-transform can be obtained.

The implementation procedure of discrete S-transform as follows:

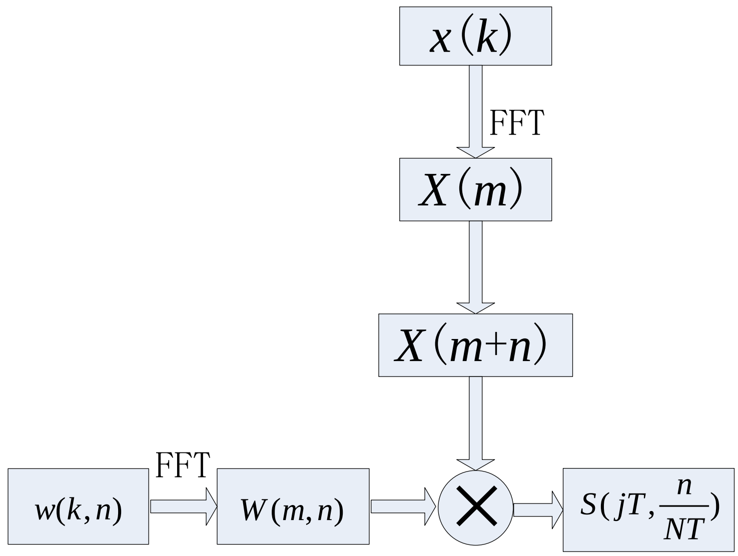

Figure 1. The S-transform Schematic Diagram

4. Comparison Task

4.1. The Time-frequency Comparison Task

The frequency conversion signal is defined as:

$$y(n) = 4\cos\left(\frac{8\pi n}{N}\right) + \frac{N}{5} \tag{22}$$

$$x(n) = \cos\left[2\pi \times y(n) \times \frac{n}{N}\right] \tag{23}$$

Where $n = 1, 2, ..., 512, N = 512$, and n is the sampling point.

Figure 2 shows the time domain graph of frequency conversion signal, the time-frequency spectrum of S-transform, the time-frequency spectrum of Wigner-Ville distribution and the time-frequency spectrum of short-time Fourier transform respectively. It is evident from Figure 2 that the S-transform has a very good time-frequency resolution. The Wigner-Ville distribution is distracted by the cross term [5]. It couldn' t express the time-frequency feature clearly. The short-time Fourier transform is the least flexible with the limit of window width.

In summary, the S-transform possesses the merits of short-time Fourier transform, and overcomes the defects of the short-time Fourier transform and the Wigner-Ville distribution. It gives a fundamentally time-frequency representation.

4.2. The Anti-noise Property Comparison Task

Figure 3 shows that the anti-noise property of three time frequency transforms under Gauss white noise. The Gauss white noise is added to the signal when signal to noise ratio is 5dB. As can be seen from Figure 3, the anti-noise property of short-time Fourier transform is the worst, followed by the Wigner-Ville distribution. The S-transform has a better noise immunity, because it inherited the merits of the continue Wavelet transform.

In conclusion, compared with the other time-frequency analysis method, the S-transform has high stability in noise conditions. It could improve the value of non-stationary signal analyzing.

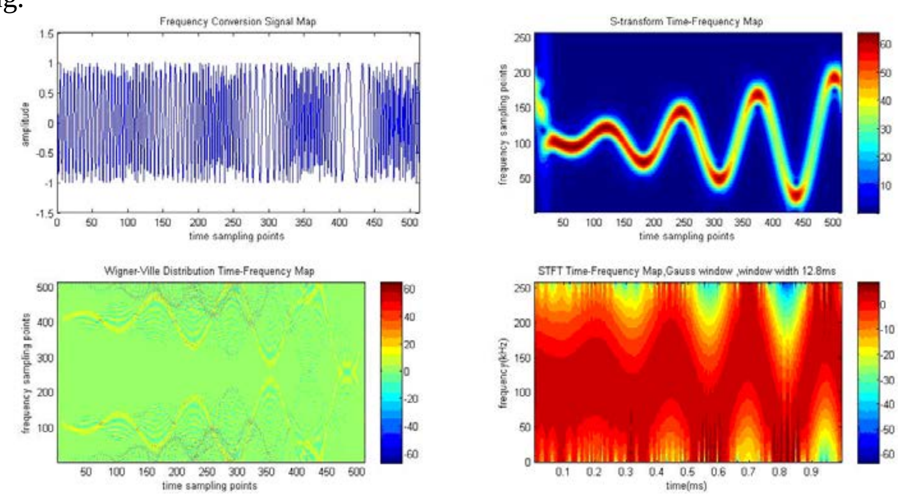

Figure 2. The Time-frequency Domain Comparison

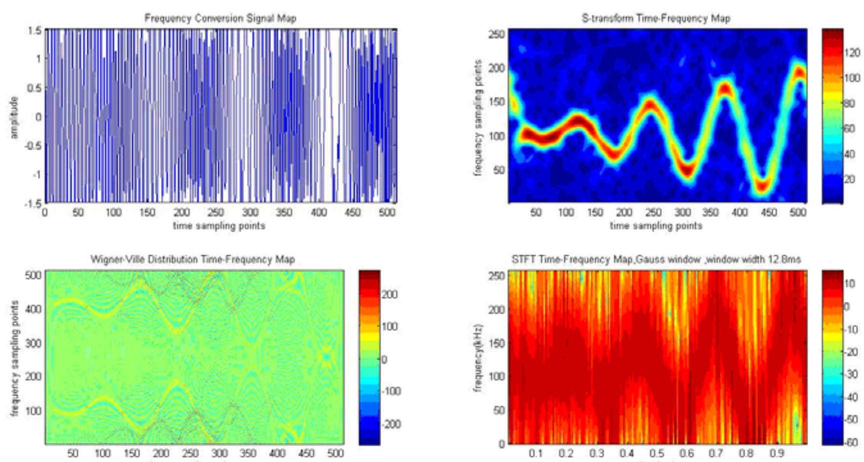

Figure 3. The Time-frequency Domain Comparison under the Gauss White Noise

Figure 4 shows that the anti-noise property of three time frequency transforms under high-frequency disturbances. In the front-end of the signal, the high-frequency disturbances are added. As is illustrated in the figure, the S-transform could represent the different frequency components precisely in the time-frequency plane. The time-frequency spectrum of short-time Fourier transform is overlapping. And it couldn't find the high-frequency signal obviously in the Wigner-Ville distribution.

We can see above, the S-transform has a progressive time-frequency resolution. It is well suited for the local spectral nature of the observations [9].

4.3. The Ability to Distinguish Different Signals

Because the S-transform has a good time-frequency resolution, the time-frequency spectral could express different signals precisely. Compared with the short-time Fourier transform and the Wigner-Ville distribution, the S-transform overcomes the effect of cross terms and fixed resolution [10].

The assumption was that the incoming signal was a superposition of sinusoidal signal and LFM signal. The incoming signal is defined by

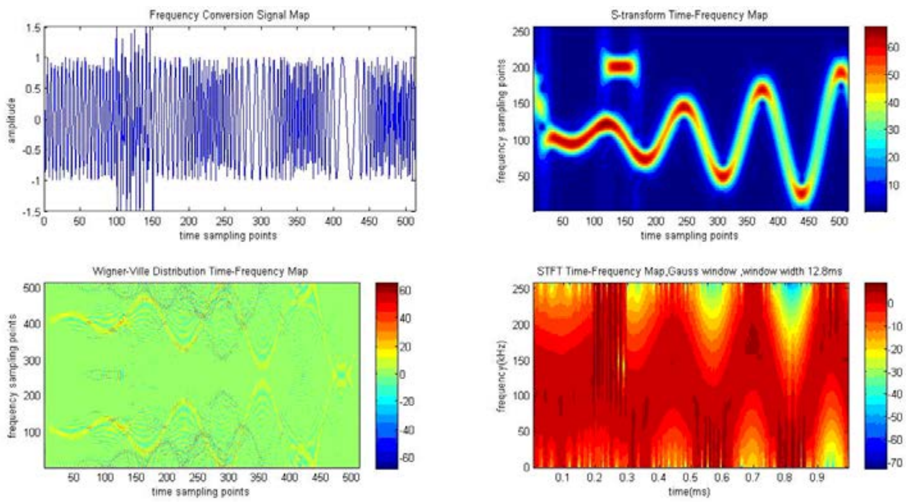

Figure 4. The Time-frequency Domain Comparison Under High-frequency Disturbances

$$x_1(t) = \cos\left(2\pi \times 20t\right) \tag{24}$$

$$x_2\left(t\right) = \cos\left(2\pi \times \left(20 + 50t\right) \times t\right) \tag{25}$$

$$y\left(t\right) = x_1\left(t\right) + x_2\left(t\right) \tag{26}$$

Figure 5 shows the ability of distinguish different signals with different time-frequency methods. As we have seen, the S-transform time-frequency could represent precisely the different characteristic signals. But the short-time Fourier transform has poor performance. Figure 6 shows the ability of distinguish different signals with different time-frequency methods under the Gauss white noise.

As we can see from Figure5-6, the S-transform representation has a better anti-noise performance. So the S-transform is an effective way for the synthesized signal. The S-transform combines a frequency dependent resolution of the time-frequency space. The time-frequency spectrum could describe the structure of incoming signal effectively.

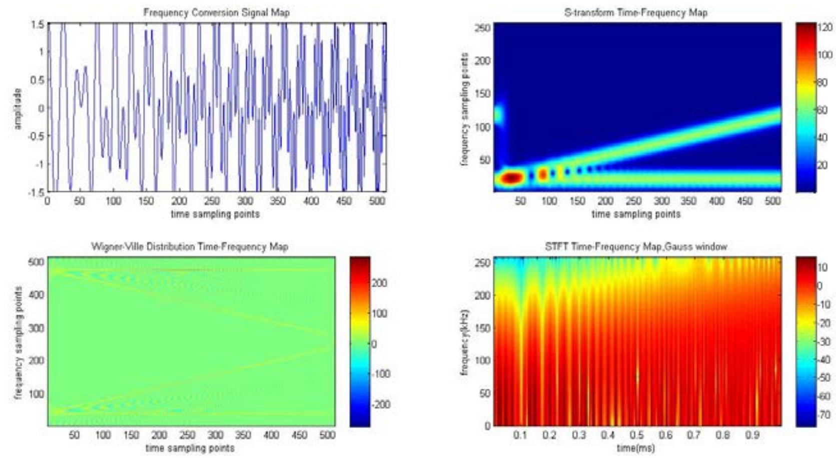

Figure 5. The Ability to Distinguish Different Signals with Different Time-frequency Analyze Methods

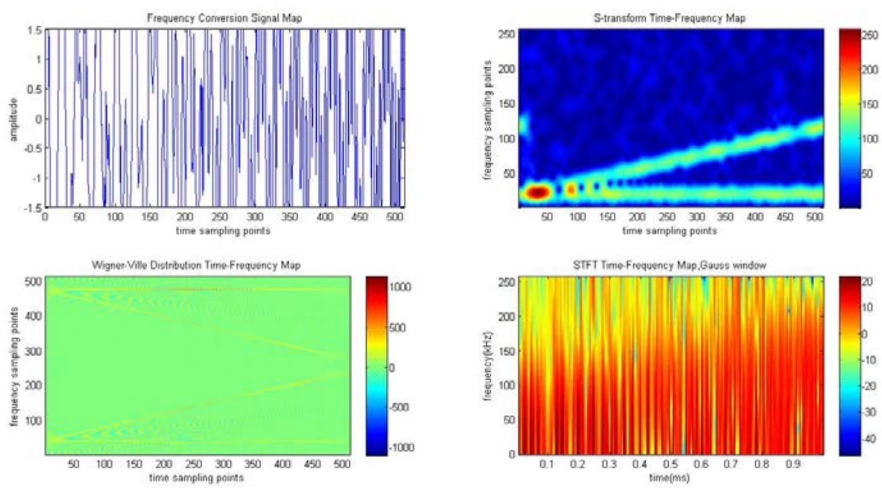

Figure 6. The Ability to Distinguish Different Signals with Different Time-frequency Analyze Methods under the Gauss White Noise

5. Conclusion

The S-transform is based on a moving and scalable localizing Gaussian window. It has some desirable characteristics that are absent in the continuous wavelet transform, and it is unique in that it provides frequency-dependent resolution while maintaining a direct relationship with the Fourier spectrum. The wavelet does not satisfy the condition of zero mean for an admissible wavelet; therefore, it is not strictly a continue Wavelet transform. The simulation shows that the S-transform has a high stability in noise conditions. The S-transform has a progressive time-frequency resolution, so it could describe the non-steady signals effectively.

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