Signal Decimation and Interpolation in Fractional Domain using Non-linear Basis Functions

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Abstract

This paper presents signal decimation and interpolation techniques under a multiresolution frame work for both lower and higher dimensional applications. New classes of non-linear basis functions have been derived from the sigmoid activation function extensively used in artificial neural networks (ANN). It has been shown that the proposed non-linear basis functions are well suited for interpolation/approximation of band limited signals. An efficient scheme for band limited signal interpolation has been introduced. Fast IIR digital filters (inverse filters) have been derived from the combinatorial theory in connection with the proposed basis functions. The proposed inverse filters can easily be implemented recursively with three multiplications and additions only. Further, the factorization of higher order filters for easy implementation has also been considered. Frequency response characteristics for the pre-filters and their corresponding interpolators are presented to reveal the quality of interpolation. An experiment has been carried out to interpolate a discrete sequence of length 33 into a sequence of length 257 (with a zooming factor of 8). Second part of the paper presents another efficient scheme for image decimation and interpolation. Experimental results on image data compression have been presented to justify the use of the proposed technique.

Keywords: Decimation, interpolation, combinatorial theory, IIR and FIR digital filters, signal processing, higher dimension applications, image compression.

1. Introduction

Decimation and interpolation of band limited signals play the key role in many digital signal and image processing applications. It is seen from the literature that estimation of intermediate values from discrete samples (data) so as to provide a continuous display or representation of the given signal is crucial in the fields of digital signal and image processing [1,2]. In fact, non-linear signal interpolation/approximation is more precise than a linear scheme since it maintains the equal resolution throughout the entire signal support. Hence, such schemes are more useful for signal processing applications. Due to the advent of fractional signal processing [3,4], there is also a strong need for devising new decimation and interpolation schemes dealing with fractional numbers. In this context, we present an efficient scheme for decimation and interpolation of band limited signals in fractional domain using non-linear basis functions.

In recent years, image interpolation/approximation has become inevitable in both signal and image processing. It is found from the literature that the geometrical transformations are important fields where it is necessary to resample the higher dimensional signals from their discrete samples to an undistorted or reference coordinate system [1, 2]. These techniques are applicable to geometrical corrections in different zooming applications. Due to the advent of technology, image interpolation and approximation operations are also useful for the rapidly emerging digital image processing techniques used for television for the preparation of special effects footage and page make-up activities in the publishing industry. Image decimation and interpolation techniques are also useful for biomedical applications.

The past decade has witnessed renewed research activities in the field of multiresolution signal representations which are very effective for analyzing image details at different resolution [5, 6]. It is interesting enough to note here that multiresolution signal decomposition techniques have become popular due to the developments in wavelet theory and applications [5, 6]. In this paper, we have been motivated to introduce a multiresolution signal decomposition technique for image decimation and interpolation. Early development of image pyramid has been carried out by Burt and Adelson [7]. Image pyramids provide multiple copies of an image at different resolutions and, thus, give us a hierarchical data structure. However, the image pyramid generation scheme (proposed by Burt and Adelson [7]) deals with image representation at a lower sampling rate, which cause image artifacts and loss of resolution in the subsequent lower resolution versions of the image. Recently, Yang and Nguyen proposed the interpolated Mth-band filters as the interpolating filters that are used for image size conversion [8]. However, these methods [7, 8] suffer from loss of resolution and need more computation depending on the length of the filters. Further, these techniques deal with only integers and use linear mathematical models. Lee et al [9] presented wavelet based interpolation scheme for resolution enhancement of medical images using interpolation and decimation filters. Adaptive filtering based on selective decimation and adaptive interpolation is proposed in [10]. These ideas are useful for echo cancellation. An area efficient 4-stream FIR Interpolation/decimation for IEEE 802.11n WLAN is recently proposed in [11] which is useful for wireless applications. Recently, multirate signal processing approaches are presented in [12] which describe usefulness of decimation and interpolation techniques.

This has motivated us to develop a discrete framework for multiresolution signal decomposition and reconstruction in fractional domain using fast recursive filters. The main thrust is to maintain equal resolution throughout the entire signal support. Hence, we have also been motivated to introduce non linear mathematical models for signal decimation and interpolation. Such non linear interpolation models may be useful for different biomedical signal processing applications. Here, we introduce a method which is based on the idea of low-pass filtering followed by down sampling to avoid aliasing in the subsequent lower resolution versions of the signal. The proposed scheme uses a pre-filter to take care of the aliasing problem. The proposed pre-filter offers better bandwidth. Thus, the proposed technique may be used to generate improved image pyramids. The method may also be useful for image decimation and interpolation with improved SNR.

In this paper, we propose a new decimation/interpolation scheme which is a four stage process. First, the signal undergoes through a non-linear mapping to facilitate signal approximation within the interval [0, 1]. In the second stage, the signal is passed through a pre-filter. In the third stage, the signal is reconstructed by interpolating the sampled data with a post filter. Before the third stage, the signal is up sampled by a factor of 'm' (resolution factor) to facilitate signal interpolation and representation. Finally, the signal is again mapped back to the original case through a nonlinear transformation process.

This paper presents a discrete framework for signal decimation and interpolation under non linear setting. That is why we are using the sigmoid function for dynamic compression at the first stage. And in the last stage, we expand back the signal dynamic by using the inverse operation. In between, we use linear digital filters for pre filtering and post filtering. Since pre-filtering is being done recursively, the output from the post-filter may fall outside the range [0, 1] (range of 'y'). So rescaling has been done (before the last stage) to restrict the output within the range [0, 1]. Then we expand back the signal dynamic by means of inverse operation.

The single-dimensional sigmoid function is defined as:

$$y = \frac{1}{1 + e^{-\omega x}} \tag{1}$$

where w is the weight . The non-linear activation function (sigmoid function) has been extensively used in artificial neural networks [13, 14] derived from its utility in Bayesian estimation of classification of probabilities [15, 16]. This sigmoid function with three different weights has been shown in Figure 1.
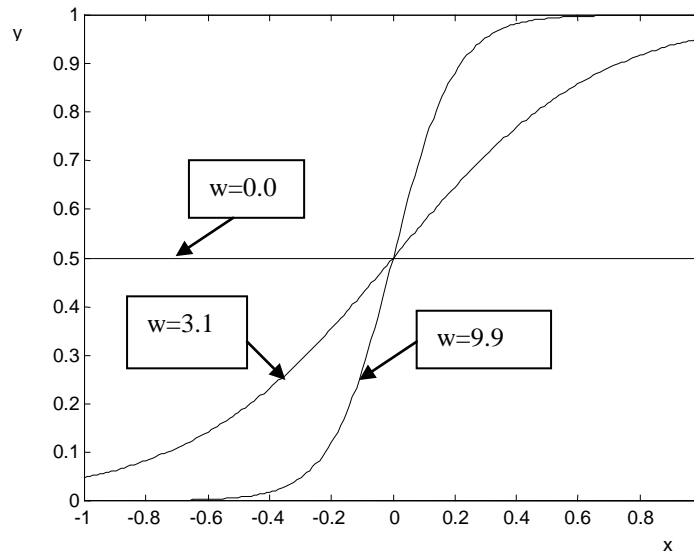

Figure 1. The Sigmoid Function for Three Different Weights W=0.0, W=3.1 And W=9.9

New classes of non-linear basis functions have been derived from this sigmoid function. Digital IIR and FIR filters have been derived from the coefficients of non-linear basis functions. These filters are used for decimation and interpolation of discrete signals with an expansion factor 'm'. The present paper also provides interesting results showing factorization of the transfer function of the higher order digital filters. Frequency response of the proposed analysis filter has been shown to justify the quality of signal decimation and interpolation.

It has been shown that the basic symmetrical filters can be implemented recursively with only three multiplications and additions per sample point, which is an additional advantage of the proposed scheme. Higher order filters are decomposed into a set of basis symmetrical filters and are implemented recursively. The method of decomposition of higher order filters has been explained in this paper. Experimentally it has been shown that the decimation and

interpolation of signals (fractional domain) with reduced amount of error can be achieved using the proposed technique.

The organization of the paper is as follows. Section 2 deals with the problem formulation. Function approximation on [0, 1] using nonlinear basis functions has been explained in Section 3. An example for signal interpolation has been discussed in Section 4. The recursive implementation of proposed filters is shown in Section 5. An efficient scheme for image decimation and interpolation has been described in Section 6. Section 7 presents results on image interpolation and image data compression. Section 8 is the concluding part of this paper.

2. Problem Formulation

In this Section, we present the problem formulation. Explicitly we construct the nonlinear basis functions by using the properties of lower order derivatives of the sigmoid function discussed in the introduction section. The first derivative of the sigmoid function 'y' is written as

$$\frac{dy}{dx} = w\frac{e^{-\omega x}}{(1 + e^{-\omega x})^2} = w\,y(1 - y), \tag{2}$$

The second derivative of 'y' is:

$$\frac{d}{dx}[y^k(1-y)^l] = wky^k(1-y)^{l+1} - wly^{k+1}(1-y)^l. \tag{3}$$

Following Eqs. (1), (2) and (3), one can compute higher order derivatives easily. From the knowledge of these derivatives, we define the following non-linear basis functions.

Definition 1: The non-linear basis functions of degree 'n' are defines as

$$B^{(n)}(x) = w^n \sum_{k=1}^{n} C_k^{(n)} y^k (1-y)^{n+1-k}, \tag{4}$$

where $C_k^{(n)}$ are coefficients.

These coefficients for different higher order basis functions for odd 'n' are displayed in Table 1.

Table 1. The Coefficients $C_k^{(n)}$

Degree (n)	k								
	1	2	3	4	5	6	7	8	9
3	1	4	1						
5	1	26	66	26	1				
7	1	120	1191	2416	1191	120	1		
9	1	502	14608	88234	156190	88234	14608	502	1

From these coefficients $C_k^{(n)}$, we have developed different symmetrical FIR and IIR digital filters. Filter transfer functions for symmetrical IIR digital filters for different odd values of 'n' are displayed in Table 2. Note that $(E^{(n)})(z)$ represents the transfer function of the digital FIR filter and $(E^{(n)})^{-1}(z)$ denotes the transfer function of the digital IIR filter. Some important observations of the filters of Table 2 are as follows. These filters (displayed in Table 2) are linear phase IIR filters. Therefore they have some poles outside the unit circle and , thus, are unstable when used as causal filters. However they can be used in a non causal way. This will be clearer when we see the implementation of the basic symmetrical IIR filters later in this paper.

Table-2. Filter Transfer Functions $(E^{(n)})^{-1}(z)$

Degree (n)	$(E^{(n)})^{-1}(z)$
3	$1/(z+4+z^{-1})$
5	$1/(z^2+26z+66+26z^{-1}+z^{-2})$
7	$1/(z^3+120z^2+1191z+2416+1191z^{-1}+120z^{-2}+z^{-3})$
9	$1/(Z^4+502Z^3+14608\ Z^2+88234Z+156190+\ 88234Z^{-1}+14608Z^{-2}+502Z^{-3}+Z^{-4})$

Theorem 1 : The coefficients $C_k^{(n)}$ in Eq. (4) are Eulerian numbers.

Proof : From definition 1, we observe that the coefficients $C_k^{(n)}$ of the power pair $y^k(1-y)^{n+1-k}$ can be obtained by using the following recursion.

$$C_k^{(n)} = 0 \quad \text{for all } n \text{ if } n<0 \text{ or } k<1;$$
$$C_1^{(1)} = 1;$$
$$C_k^{(n)} = k\ C_k^{(n-1)} + (n+1-k)\ C_{k-1}^{(n-1)} \tag{5}$$

Using the above recursion, one can generate the numbers given in Table 1. On the other hand, from the combinatorial theory [16,17], the Eulerian numbers C_{l-j} are written as

$$C_{l-j} = \sum_{i=0}^{l-j-1} (-1)^i \binom{n+1}{i} (l-j-i)^n \tag{6}$$

where $\binom{n+1}{i}$ are binomial coefficients, $l = \dfrac{n+1}{2}$ and j varies from $-l+1$ to $l-1$. Interestingly, both Eq. (5) and Eq. (6) are equivalent. In fact, Eq. (5) is the recursive relation to generate Eulerian numbers given in Eq. (6). Thus, the coefficients $C_k^{(n)}$ of the power pair $y^k(1-y)^{n+1-k}$ are nothing but Eulerian numbers. This proves Theorem 1.

Theorem 2: The higher order filter transfer functions can be expressed in factored form given by

$$\left(E^{(n)}\right)^{-1}(z) = \frac{1}{\prod_{i=1}^{\ell-1}\left(z + r_i + z^{-1}\right)} \quad . \tag{7}$$

Proof: Here, we provide a proof for Theorem 2. The main objective of the present paper is to provide a non linear function approximation technique. It may be noted here that any function can be approximated in terms of the linear combination of the proposed shifted non linear basis functions defined in Eq.(4). It is well known from the literature [16,17] that the approximated function $\hat{f}_k$ (discrete samples) can be written as

$$\hat{f}_k = \sum_{j=-l+1}^{l-1} s_{k+j} C_{l-|j|} \tag{8}$$

where $\mathbf{C}$ is the system matrix generated by non-linear basis functions, s_k's are coefficients of approximation, $l = \dfrac{n+1}{2}$ and $-l+1 \le j \le l-1$. Let c_k are the coefficients of the system matrix $\mathbf{C}$. Due to the symmetry of Eulerian numbers, we define d_k as

$$d_k = s_{k-1} + r s_k + s_{k+1} \tag{9}$$

where 'r' is a variable. From Eqs. (8) and (9), we obtain a recursive relation of the form given by

$$p_i = c_i - r p_{i-1} - p_{i-2}, \text{ for } i = 3,4,5,...,l-1 \tag{10}$$

with $p_1 = c_1 = 1$ and $p_2 = c_2 - r$. Note that c_i are the elements of the system matrix $\mathbf{C}$. In general, p_k can be expressed as

$$p_k = \sum_{i=0}^{k-1}(-1)^i r^i \sum_{j=0}^{j \le \frac{l-i-2}{2}}(-1)^j \binom{i+j}{j} c_{l-2j-i-1} \tag{11}$$

Eq.(11) is a polynomial in r of degree $(\ell-1)$ and can be written as :

$$g_1 r^{\ell-1} - g_2 r^{\ell-2} + g_3 r^{\ell-3} - ......g_\ell = 0 \tag{12}$$

By equating the coefficients of $r^{\ell-K}$ in Eq. (12) with that of Eq. (11), one can obtain a recursive relation for g_k given by

$$g_k = c_k - \sum_{i=1}^{i \le \frac{k-1}{2}} \binom{\ell-k+zi}{i} g_{k-2i} \quad \text{for } 3 \le k \le l \tag{13}$$

Eq. (13) has $(\ell-1)$ real and positive roots $r_i (i = 1,2,3,...,\ell-1)$.

The coefficients c_k (same as C_k) are tabulated in Table 1. The coefficients g_k are evaluated by using Eq. (13). Then the roots of polynomial given in Eq. (12) can be determined by any standard root finding subroutine or by using MATLAB. The coefficients c_k, g_k and the roots r_i are displayed in Table 3. For a display purpose only, we have

provided truncated values of the roots r_i for different higher order filters. However one can find out exact values through nay standard root finding algorithm.

By using the roots r_i displayed in Table 3, the higher order filter transfer functions can be expressed in factored form given by

$$\left(E^{(n)}\right)^{-1}(z) = \frac{1}{\prod_{i=1}^{\ell-1}\left(z + r_i + z^{-1}\right)} \tag{14}$$

This proves Theorem 2.

Interestingly enough, the generic symmetrical IIR digital filter $\left(E^{(n)}\right)^{-1}(z)$ can be decomposed into a cascade of elementary (basic) IIR filters ($\frac{1}{\left(z + r_i + z^{-1}\right)}$) with i = 1,2,…,$\ell-1$. Each elementary (basic) IIR filter can be easily implemented recursively with three multiplications and additions. The denominator of the basic symmetrical IIR filter is a quadratic polynomial z. Hence, we can find two distinct roots. We have to consider a pole which is inside the unit circle to provide stability. This basic IIR filter   may be again decomposed into a sum of a causal and an anti causal filters. This kind of decomposition is used for efficient implementation of digital IIR filters [18,19]. The recursive implementation of such symmetrical IIR filters will be discussed later in Section 6.

Table 3

Degree (n)	c_k [k = 1,…,2,l]	g_k [k =1,…,2,l]	r_i [i =1,2,…,l-1]
5	66 26 1	64 26 1	2.753049 23.246951
7	2416 1191 120 1	2176 1188 120 1	2.403464 8.282187 109.314356
9	156190 88234 14608 502 1	126976 86728 14604 502 1	2.252749 5.158374 23.179267 471.409632
11	15724248 9738114 2203488 152637 2036 1	11321344 9280208 2195344 152632 2036 1	2.173521 3.946214 11.230634 60.006012 1958.643638
13	2275170000 1505620000 423282000 45533500 1479730 8178 1	1431570000 1369060000 417363000 4549600 1479720 8178 1	2.126663 3.335037 7.338326 23.183971 148.417999 7993.597656

Using Eq.(4) one can generate non linear basis functions of any degree 'n'. The Eulerian numbers $C_k^{(n)}$ can be generated using the recursive relations given in Eq.(5). The newly introduced non-linear basis functions $B^{(n)}(x)$ of four different degrees have been displayed in Figure 2 (a)-(d). For a display purpose only we have considered the value of $w=2.0$. There is no indication that the chosen value is optimal. The magnitude and shape of the non linear functions varies with the value of the weight 'w'. To the best of our knowledge, these non linear basis functions are not reported in the literature.
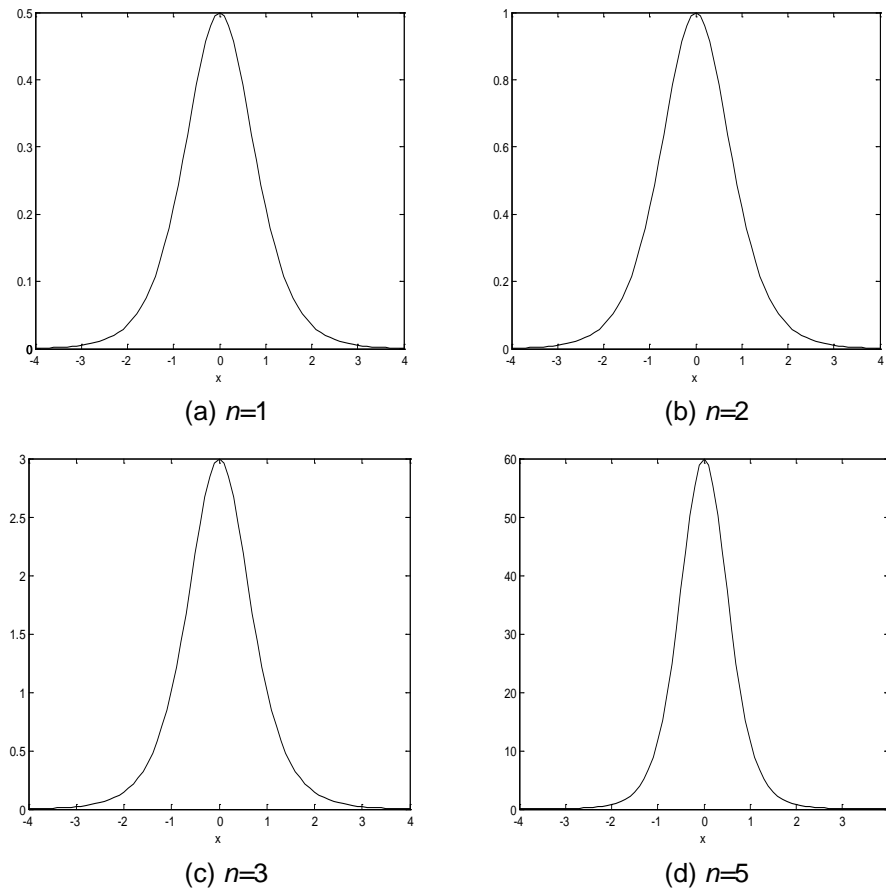

(a) $n=1$ (b) $n=2$

(c) $n=3$ (d) $n=5$

Figure 2. Non-linear Basis Functions of Different Degree. Note that the weight $w=2.0$

3. Function Approximation on [0, 1] using Non-Linear Basis Functions

In this section, the approximation of a signal $f(x)=x$ with a linear combination of the shifted non-linear basis functions of degree $'n = 3'$ has been considered. Figure 3 shows the plot of the approximated signal $f(x) = \sum_{k=0}^{N-1} f(k)\, B^{(3)}(x-k).$ Note that 'N' denotes the number of samples.
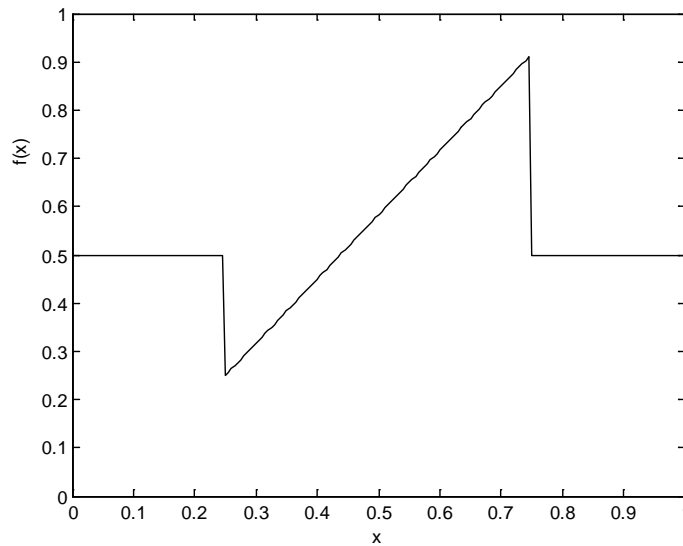
Figure 3. Approximation of the Monomial x with a linear combination of Shifted non-linear basis Functions of Degree $'n = 3'$. Note that the Weight Factor *w*=1.5

Note that we have chosen the value of weight w=1.5 to get the well approximated results. It is observed from Figure 3 that there is no exemplification at the beginning and end data points although we consider a typical example here. In this sense, the proposed non linear approximation technique seems to be better than other linear approximation techniques reported in the literature. The proposed technique can be used to maintain equal resolution throughout the entire signal support as shown in Figure 3. Further, this method can also be used for signal approximation in fractional domain. The significance of the proposed scheme can be further explored if we see the error plot. An error plot for the approximation example has been displayed in Figure 4.
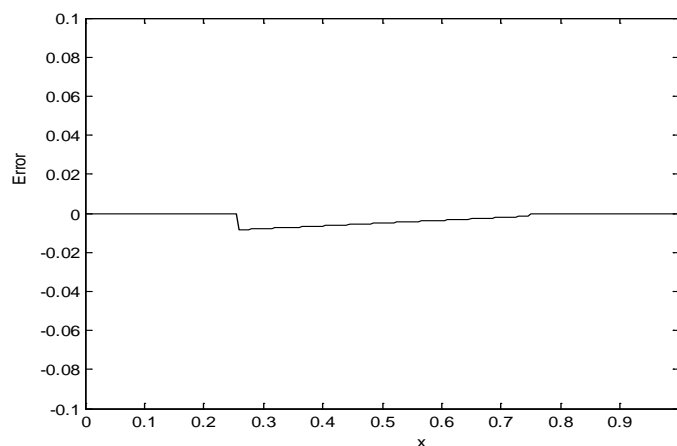

Figure 4. Error in Approximation of the Monomial x with a Linear Combination of Shifted Non-linear Basis Functions of Degree $'n = 3'$. Note that the Weight Factor *w*=1.5

It is noteworthy to mention here that the proposed scheme may be efficiently used for uniform approximation of signals by linear combinations of shifted non-linear basis functions displayed in Figure 2. To the best of our knowledge, efficient non-linear signal approximation techniques using nonlinear basis functions are not available till date. On the other hand, non-linear signal approximation is more precise than a linear approximation technique in practice, because it maintains the equal resolution throughout the entire signal support. Hence, it is useful for both signal and image processing applications. The above approximation example (Figures 3 and 4) depicts the fact that the proposed scheme may be useful for signal approximation in fractional domain. The best approximation can be achieved by properly choosing the weight factor 'w'.

4. Signal Interpolation Example

In this Section, an interpolation example has been presented. In this experiment, we use the proposed method shown in Figure 5 for interpolation of discrete data. First, the discrete samples x_k are mapped unto y_k through the non-linear transformation. Then it is passed through the symmetric IIR digital filter $(E^{(n)})^{-1}(z)$. After that the signal is up sampled with a factor of 'm'. Then the signal is passed through a moving average filter for $(n+1)$ times. The Eulerian numbers are also written as a generalized binomial kernel $u^{(n)}(k)$ whose Z-transform is given as $/1+z^{-1}/^{(n+1)}$. This tells us to implement a moving average filter for $(n+1)$ times. The output from the moving average filter is then convolved with the sequence b_{mk}^n. Note that the sequence b_{mk}^n has been derived from the non-linear basis functions with degree 'n' and an expansion factor 'm' by sampling. Finally, the signal is mapped back to the continuous domain through the inverse mapping shown in Figure 5.

Here, we consider an example to interpolate a sequence f(m), m=0,1,…,N, of length (N+1) into a sequence of length (PN+1) using the proposed technique. Note that P is a positive integer to be decided by the user. Eight different test functions have been considered for performance evaluation. Usually, the following test functions are considered for interpolation experiments. The test functions considered are – (1) f(x) = cos(5πx/32+1) exp(-x/12); (2) f(x) = 2 cos (3.1πx/32+1)+ cos (6.8πx/32-2); (3) f(x) = -0.5 cos(3πx/32): (4) f(x) = x/32; (5) f(x) = 4-(x-16)^2/64 ; (6) f(x) = (x+1) ; (7) f(x) = 1.7 sin(3πx/32); (8) f(x) = log(1+x). These test functions have been sampled at unit intervals in between $x=0$ to $x=32$. The interpolation of a discrete sequence of length 33 into a sequence of length 257 (for this particular experiment) has been considered. The following performance measure has been proposed to evaluate the performance of the method used. This is given as

$$
\text{NMSE [dB]} = 10\log_{10}\left[\frac{\sum_{x=0}^{256}\ [in(\frac{x}{8}) - out(x)]^2}{\sum_{x=0}^{256}\ in(\frac{x}{8})^2}\right] \tag{15}
$$

where NMSE [dB] is the normalized mean square error expressed in decibels (dB).The performance results are presented in Table 4.
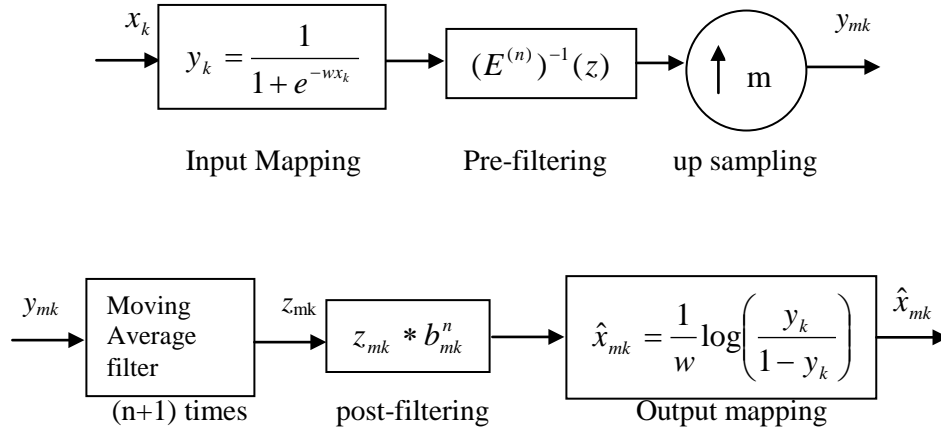
Figure 5. Block Diagram of an Interpolator

Table 4. Performance Evaluation

Test function	1	2	3	4	5	6	7	8
NMSE [dB]	-37.73	-39.22	-52.22	-53.43	-85.55	-55.21	-71.92	-69.46

5. Fast Recursive Implementation

The transfer function of the symmetric stable IIR filter of order 3 $(E^{(3)})^{-1}(z)$ is decomposed as :

$$\left(E^{(3)}\right)^{-1}(z) = l\left(\frac{1}{(1-az^{-1})} + \frac{az}{(1-az)}\right) \tag{16}$$

The net impulse response is the superposition of a causal sequence $c^+(k)$ and an anti causal sequence $c^-(k)$, given by

$$
\begin{cases}
c^+(k) = x(k) + a\, c^+(k) & (k = 2, \ldots\ldots k) \\
c^-(k) = a\,(x(k) + c^-(k)) & (k = k-1, \ldots\ldots, 1) \\
c(k) = l(c^+(k) + c^-(k)) & (k = 1,2,\ldots.k)
\end{cases} \tag{17}
$$

where

$l = -a/(1-a^2)$ and $a = -0.2679$. The proposed boundary conditions are

$$
\begin{cases}
C^+(k) = \delta \\
C^-(k) = C^+(k)
\end{cases} \tag{18}
$$

where δ is some predefined value. Note that the proposed method can be extended to higher dimension. Thank God, these recursive filtering can be successively applied along the row and column coordinates of a digital image. Similar decomposition of B-spline filter is also reported in [20].

6. Image Decimation and Interpolation

The image pyramid generation for computing image details at multiple resolutions was reported by Burt and Adelson [7]. Burt's unimodal function [7] used for image approximation at a lower sampling rate cause loss of resolution in the subsequent lower resolution copies of the image. Here, we introduce a discrete framework (shown in Figure 6) for image decimation and interpolation. This approach may provide us decimation and interpolation techniques applicable to higher dimension. The important feature of the proposed method is to use antialiasing filters in the analysis stage so as to reduce aliasing in the subsequent sub-sampled signals. This feature enables us to reduce image artifacts in the lower resolution copies of the image under consideration. Thus, the proposed method is more useful in a signal processing point of view. In this scheme, such an arrangement of filters is mainly based on the idea of low-pass filtering followed by a down sampling to avoid aliasing in the lower resolution copies of the image.
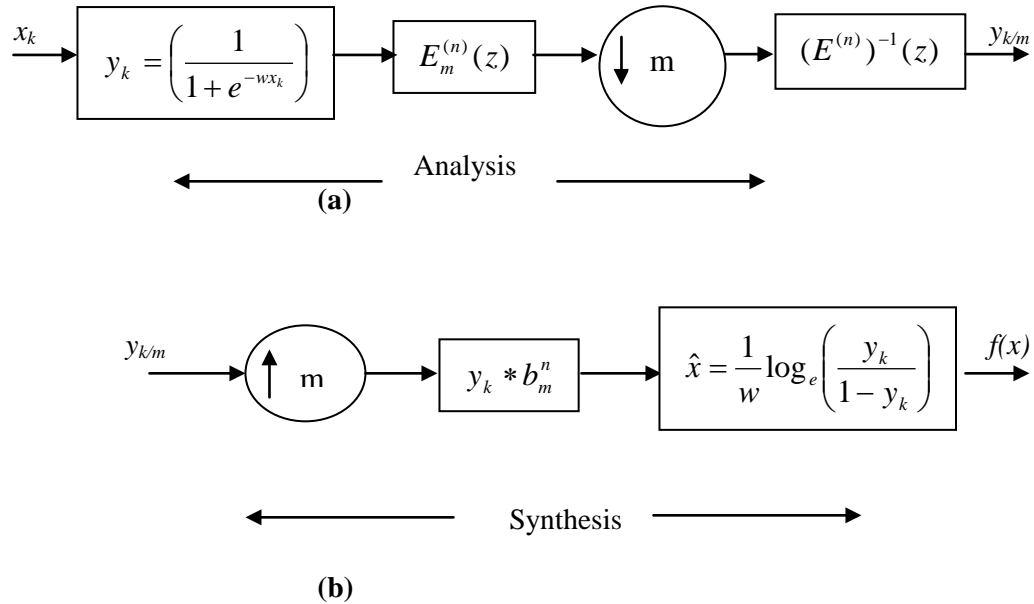

$$x_k \rightarrow \boxed{y_k = \left(\frac{1}{1+e^{-wx_k}}\right)} \rightarrow \boxed{E_m^{(n)}(z)} \rightarrow \bigotimes_{\downarrow m} \rightarrow \boxed{(E^{(n)})^{-1}(z)} \rightarrow y_{k/m}$$

Analysis

(a)

$$y_{k/m} \rightarrow \bigotimes_{\uparrow m} \rightarrow \boxed{y_k * b_m^n} \rightarrow \boxed{\hat{x} = \frac{1}{w}\log_e\left(\frac{y_k}{1-y_k}\right)} \rightarrow f(x)$$

Synthesis

(b)

Figure 6. Block Diagram of the Filter based Scheme

In the proposed scheme, symmetrical IIR filters derived from the coefficients of non-linear basis functions have been used in the first stage as shown in Figure 6. Different filter transfer functions for different odd values of 'n' have been displayed in Table 2.

First, the signal undergoes through the nonlinear transformation. Then it is passed through the FIR digital filter $E_m^{(n)}(z)$ with an expansion factor of 2, *i.e.*, m=2. After that the signal is down sampled by a factor of 2. The lower version copy of the low-pass filtered image is then passed through an IIR digital filter $(E^{(n)})^{-1}(z)$. In the second stage (synthesis), the signal is up sampled by a factor of 2. Then it is convolved with the nonlinear sequence b_m^n. The digital filter $(E_m^{(n)}(z))$ with an expansion factor of 2 (m=2) is given by [20]

$$(E_m^{(n)}(z)) = \frac{1}{2^m} E_1^{(n)}(z)(M_m(z))^{n+1} \tag{19}$$

where $M_m(z)$ is a moving average filter expressed as

$$M_m(z) = \frac{1-z^{-m}}{1-z^{-1}} = \sum_{k=0}^{m-1} z^{-k} \tag{20}$$

The transfer function of the pre-fitter (analysis filter) is given by

$$\tilde{H}_2^{(n)}(z) = E_2^{(n)}(z)[(E_1^{(n)})^{-1}(z)]_{\uparrow m} \tag{21}$$

Note that the operator $[.]_{\uparrow m}$ represents up-sampling by a factor of m. The corresponding interpolator (synthesis filter) is the convolution operation shown in the Figure 6 (b).

Normalized gain versus frequency responses of the pre-filter order *n=3* has been shown in Figure 7. It is observed from Figure 7 that the pre-filter is a low- pass filter which acts as an antialiasing filter. It exhibits better bandwidth and cutoff rate. Interestingly enough, maximally flat frequency response can be achieved by considering higher order filters. The first stage (shown in Figure 6(a) can be used as the analysis filter for decomposition of images into multiple resolution, *i.e.*, fine-to-course resolution. In the second stage, *i.e.*, the synthesis stage, the proposed non-linear basis functions can be used for interpolating the lower resolution copy back to the original image size.
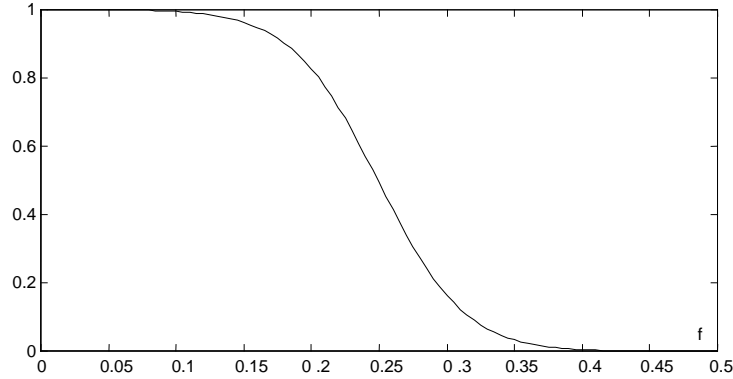

Figure 7. Normalized Frequency Response of the Analysis Filter

7. Experimental Results

In this section, two different examples have been presented to show the potential of the proposed scheme.

7.1. Image Interpolation Example

This example provides interpolation results using zero order, bilinear and the proposed interpolation scheme. In this experiment, the magnification of a 32×32 detail with a zooming factor of 8 has been considered. The test image of size 256×256 used for this experiment has been shown in Figure 8(a). The result obtained by zero order interpolation has been displayed in Figure 8(b). Figure 8(c) shows the result obtained by using bilinear interpolation scheme. Finally, Figure 8(d) display the result obtained by the proposed scheme.
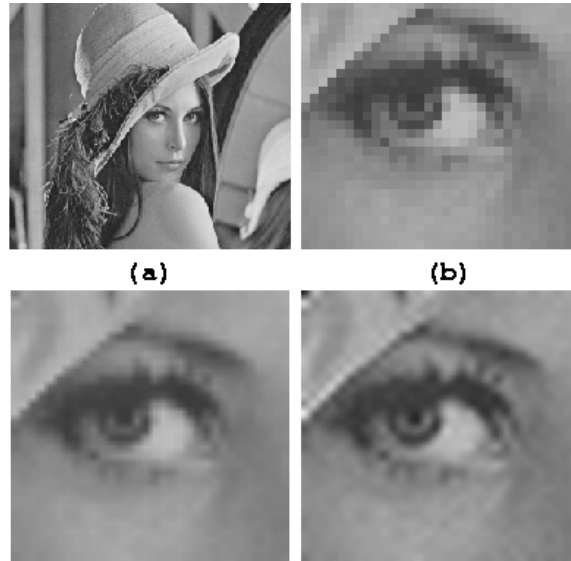
Figure 8. Image Interpolation Results. (a) Test Image of Size 256 × 256, (b) Result Obtained by Zero Order Interpolation, (c) Result Obtained by Bilinear Interpolation Method, (d) Result Obtained by the Proposed Scheme

7.2. Image Compression Examples

In this Section, image data compression examples using the proposed method have been presented. Two different gray level images of size 256x256 have been considered for this experiment. Note that the images are subsampled after pre-filtering (*i.e.*, the analysis stage shown in Figure 6) and then interpolated with the post filter (*i.e.*, the synthesis stage shown in Figure 6). A detailed performance measure using a wider range of compression ratios has been displayed in Table 5 in terms of the signal-to-noise ratio (SNR) in decibels. It is observed from Table 5 that the performance of digital filters increases with the increase in degree '*n*'.

Table 5. Performance Of The Proposed Scheme Using Digital Filters Of Different Order 'N' For Image Compression In Terms Of Their Signal-To-Noise Ratio (Decibels) For Gray Level Images "Brain" And "Lena"

Image	Compression Ratio	Burt's Method	Proposed filters of order 'n'		
			n=3	n=5	n=7
Brain	1:4	22.31	27.97	28.22	29.64
	1:16	16.82	22.81	21.71	21.94
	1:64	13.72	17.82	18.32	18.90
Lena	1:4	20.21	25.98	26.35	27.91
	1:16	14.83	24.56	25.92	26.40
	1:64	12.11	21.88	21.98	23.23

8. Conclusion

In this paper, we have presented a novel scheme for signal decimation and interpolation in fractional domain using non-linear basis functions. This approach is simpler and effective for signal decimation and interpolation. Factorization of the denominator polynomial of higher order filters has been shown. This part of the algorithm makes it easy to implement higher order filters. It is observed from Figure 7 that the cut-off rate and the band-width of the filter increase with the increase in the degree *'n'*. These higher order filters possess the maximally flat characteristics. Interpolation results presented in Table 4 reveal the suitability of the method used.

A discrete frame work for multiresolution signal decomposition using fast recursive filters has been presented. New types of symmetrical FIR and IIR digital filters have been presented utilizing combinatorial theory. From the frequency response characteristics of the analysis filter, it is observed that the proposed method is well suited for image decimation and interpolation. Figure 8 depicts the fact that the present technique may be useful for image zooming applications. Experimental results produced on image data compression (Table 5) reveal the suitability of the proposed scheme. The future extension of the work include the construction of wavelet bases [21-22] using proposed central basis functions and their higher dimensional applications. Finally, we conclude that there is enough scope for extensions and higher dimensional applications of the proposed method.

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