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A new multiplierless discrete cosine transform based on the Ramanujan ordered numbers for image coding

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Abstract

The performance of image coding can be improved upon by using a special class of multiplierless discrete cosine transform using Ramanujan numbers termed as Ramanujan DCT (RDCT). Ramanujan ordered numbers are those which approximate 2π/N by $2^{-l}+2^{-m}$ *, where l and m are integers. The cosine angles can then be computed using Chebyshev type of recursion using only shifters and adders. Fast forward and backward transformation may be achieved. Analysis and simulations show that the proposed RDCT maintains good de-correlation and energy compaction properties of the DCT and the error due to approximation is almost zero at lower spectral components and relatively low at higher spectral components. Simulation experiments are provided to justify that the proposed algorithm is best suited for image compression.*

Keywords: *Multiplierless, Ramanujan ordered Numbers.*

1. Introduction

THE discrete cosine transform (DCT) [1], [2] is a robust approximation of the optimal Karhunen-Loève transform (KLT) for a first-order Markov source with large correlation coefficient. It has satisfactory performance in terms of energy compaction capability, and many fast DCT algorithms with efficient hardware and software implementations have been proposed. The DCT has found wide applications in image/video processing and other fields. It has become the heart of many international standards such as JPEG, H.26x, and the MPEG family.
Many different fast algorithms for the DCT computation have been developed for image and video applications. Some of them take advantage of the relationships between the DCT and various existing fast transforms, including the FFT [1], [3]–[9], the Walsh-Hadamard transform (WHT) [5], [6], and the Discrete Hartley transform (DHT) [9]. Some algorithms are based on the sparse factorizations of the DCT matrix [10]–[11], and many of them are recursive [12], [13], [14]. Besides one-dimensional (1-D) algorithms, two-dimensional (2-D) DCT algorithms have also been investigated extensively [11], [12]–[14], generally leading to less computational complexity than the row-column application of the 1-D methods. However, the implementation of the direct 2-D DCT requires much more effort than that of the separable 2-D DCT. The theoretical lower bound on the number of multiplications required for the 1-D eight-point DCT has been proven to be 11 [15], [17]. In this sense, the method proposed by Loeffler *et al.* [15], with 11 multiplications and 29 additions, is the most efficient solution. However, in image and video processing, quantization is often required to compress the data. In these circumstances, significant algorithmic savings can be achieved if some operations of the DCT are incorporated into the quantization step. This leads to a class of fast 1-D and 2-D DCTs that are generally referred to as the *scaled* DCT [16], [8], [21], [23]–[25]. For example, the Arai's method needs only five multiplications [3], [8].

All of the aforementioned fast algorithms still need floating point multiplications, which are slow in both hardware and software implementations. To achieve faster implementation, coefficients in many algorithms such as [8], [9], [10], and [17] can be scaled and approximated by integers such that floating-point multiplications can be replaced by integer multiplications [20], [20]–[24]. The resulting algorithms are much faster than the original versions and, therefore, have wide practical applications.

The fast bi-orthogonal Binary DCT (BinDCT) [18] and Integer DCT (IDCT) [20, 22] belong to a class of multiplierless transforms which computes the coefficients of the form $k / 2^b$. They compute the integer to integer mapping of coefficients through the lifting matrices. The performances of these transforms depend upon the lifting schemes used and the round off functions. In general, these algorithms require the approximation of the decomposed DCT transformation matrices by proper diagonalisation. Thus the complexity is shifted to the techniques used for decomposition. Among the direct implementations, Wang [25] and Chen [26] have given the fast DCT algorithms whose computational complexity is same. Lee [12] proposed a fast algorithm of IDCT, but the inversion and division of the cosine values give rise to the instability problems. The papers [12, 13, 24, 26] have given the fast algorithms of computational complexity $\frac{N}{2}\log_2 N$ multiplications and $\frac{3}{2}N\log N - N + 1$ additions. But, the algorithms require integer multiplications and complicated lifting steps. Computation of DCT coefficients involves evaluation of cosine angles of multiples of 2π/N. If N is chosen such that it could be represented as $2^{-l} + 2^{-m}$, where *l* and *m* are integers, then the trigonometric functions can be evaluated recursively by simple shift and addition operations. Such integers are called Ramanujan numbers, defined by Bhatnagar [27, 28] after the great mathematician, S.Ramanujan. Use of Ramanujan ordered Number for computing DCT was outlined by the author in [29]. Matrix factorization of the transformation matrix reduced the complexity to $\frac{N}{2}\log_2 N$ shifts and $\frac{3}{2}N\log N - N + 1$ additions [30] thereby eliminating the use of multipliers.

In this article, we show that the Ramanujan Discrete Cosine Transform satisfies all the properties of unitary image transforms and hence could be considered as an alternative to the commonly used multiplierless transforms like IDCT for image and video compression.

This paper is organized as follows: Ramanujan ordered Numbers are discussed in Section II. Evaluation of the cosine values using Ramanujan ordered Number is explained in section III. The properties of the DCT kernel are also discussed in Section III. In Section IV, we discuss the performance of the proposed Ramanujan Discrete Cosine Transform (RDCT) through the experimental results, and finally concluding remarks are described in Section V.

2. Ramanujan ordered numbers

Ramanujan ordered numbers are related to π and its decimal point accuracy. They are integers of power of 2. Ramanujan ordered number of type-1 was used in [27, 28] to compute the Discrete Fourier Transform.

Ramanujan ordered numbers of type-1 $\Re_1(a)$ are defined as follows:

$$\Re(a) = \left[\frac{2\pi}{N}\right] \square\, 2^{-a} \qquad (1)$$

where *a* is a non-negative integer. The numbers could be easily computed by simple binary shifts. Consider the binary expansion of π which is 11.00100100001111…… If *a* is chosen as 2, then $\iota_1(2) = 2^{-2}$, and $\Re_1(2) = [11001.001000.......] = 11001$. i.e., $\Re_1(2)$ is equal to 25. Likewise $\Re_1(4)$ =101. Thus the right shifts of the decimal point (a+ 1) time yields $\Re_1(a)$. Let this approximated value be $\hat{\pi}$ and let the relative error of approximation be ε,

$$\hat{\pi} = \frac{1}{2}\left[\Re_1(a)\iota(a)\right] \qquad (2)$$
$$\hat{\pi} = (1+\in)\pi$$

These errors are used to evaluate the degree of accuracy obtained in computation of DCT coefficients. The accuracy of the algorithm can be further increased by increasing the order of the number and are defined as Ramanujan ordered number of type-2.

Ramanujan ordered Number of type-2 [10] are defined such that $2\pi/N$ is approximated by sum or difference of two numbers which are negative powers of 2. Thus Ramanujan Numbers of degree-2 are,

$$\Re_{2\iota}(l,m) = \left[\frac{2\pi}{\iota_{2i}(l,m)}\right] \quad for\ i=1,2$$
$$\iota_{21}(l,m) = 2^{-l} + 2^{-m} \quad for\ m > l \geq 0 \qquad (3)$$
$$\iota_{22}(l,m) = 2^{-l} - 2^{-m} \quad for\ (m-1) > l \geq 0$$

Where l and m are integers, Hence

$$\Re_{21}(3,5) = 40 \quad \Re_{21}(1,3) = 10 \qquad (4)$$

These higher-order numbers give better accuracy at the expense of additional shifts and additions. The relative error in the approximation defined by equation (2) can be extended to Ramanujan ordered number of type-2 and is listed in the table1 below.

Table 1. Ramanujan Number of Order-2

(l,m)	$\Re(l,m)$	$\hat{\pi}$
0,2	5	3.125
1,2	8	3.0
4,5	67	3.140

These Ramanujan Numbers are used for the computations of the DCT coefficients wherein the computational complexity for a 1-D DCT is $O\left(N^2/2\right)$ additions and shift operations. Parallel implementation of the algorithm can be executed with only $O\,(N/2)$ number of adders.

3. Ramanujan discrete cosine transforms (RDCT)

The DCT matrix D_{DCT} of order N is defined as [1]

$$D_{DCT} = \begin{cases} \dfrac{1}{\sqrt{N}} & k=0, 0 \leq n \leq N-1 \\ \sqrt{\dfrac{2}{N}}\cos\dfrac{\pi(2n+1)k}{2N} & 1 \leq k \leq N-1, 0 \leq n \leq N-1 \end{cases} \qquad (5)$$

Neglecting the scaling factors, the DCT kernel could be simplified as

$$c_n = \cos\left(\frac{2\pi}{N} 2^{-2}\left(2n+1)\right)k\right) \quad \textit{for} \quad 0 \le n \le N-1, 0 \le k \le N-1, \qquad (6)$$

3.1 Evaluation of transform coefficients using Chebyshev Recursion [30]

Computation of DCT coefficients requires evaluation of sequences of type $\{c_n \mid c_n = p\cos\left(2\pi / N\right) \quad n = 0,1....N-1, \quad p \in \mathbb{R}\}$ where $\mathbb{R}$ is the set of real numbers. The evaluation of these sequences is done via Chebyshev-type recursion. Let us define

$$\begin{aligned} W\left(M,p\right) &= \left\{w_n \mid w_n = pcos\left(2\pi n / M\right)\right\} \\ n &= 0,1......\Psi, \quad p \in \mathbb{R} \qquad (7) \\ \Psi &= \left(\frac{M}{4}-1\right), \qquad M = \beta N \end{aligned}$$

Where, β is equal to 1, if N is divisible by 4. It is equal to 2, if N is divisible by 2, but not by 4. Otherwise, it is equal to 4(N is not divisible by 2). The use of β facilitates the computation of $W\left(M,p\right)$ by considering cosine values from the first quadrant of the circle. Then the sequence $\{c_n\}$ is obtained by evaluating recursively $\{w_n\}$ .The $W\left(M,p\right)$ sequence is estimated as follows.

Let us define $\frac{2\pi}{N}2^{-2}$ of the DCT kernel as x and hence $c_n = \cos\left(nx\right)$. We approximate x by $\hat{x} = 2^{-l} + 2^{-m}$ with l and m being non-negative integers. Thus we approximate c_n 's by $t_n\left(\alpha\right)$'s, where α is equal to $\hat{x}^2/2$. Since $\hat{x}$ is a Ramanujan ordered number of type-2, α is of the form $2^{-c} + 2^{-d}$ where c and d are integers. Then

$$\begin{aligned} & t_0\left(\alpha\right) = 1 \\ & t_1\left(\alpha\right) = \left(1-\alpha\right) \\ & \vdots \qquad\qquad\qquad\qquad\qquad\qquad (8) \\ & t_{n+1}\left(\alpha\right) = 2\left(1-\alpha\right)t_n\left(\alpha\right) - t_{n-1}\left(\alpha\right) \\ & n = 1,2.....,\left(\Psi-1\right) \end{aligned}$$

It is observed that the above recursive equations are closely related to Chebyshev polynomials of the first kind. $t_n\left(\alpha\right)$'s and therefore c_n 's can be computed by simple shift and addition operations. The sequence $W\left(M,p\right)$ can now be evaluated readily. The approximated value of $\{w_n\}$ will be $\left\{y_n\left(\alpha\right) = pt_n\left(\alpha\right)\right\}$. The DCT coefficients can be evaluated by computing sequences $W\left(M, p_{kj}\right)$ for *j*=1,2 and $k = 0,1,2\ldots,(N-1)$. Since the cosine values are evaluated by the sequences of $W\left(M, p_{kj}\right)$, the inverse transform is also guaranteed. To compute 1-D DCT the proposed algorithm takes $O\left(N^2/2\right)$ additions and shift operations. Parallel implementation of the algorithm can be executed with only $O\left(N/2\right)$ number of adders. The proposed algorithm being recursive ensures that the storage of the trigonometric values is not required.

3.2 Properties of the Ramanujan ordered Number DCT:

Consider the transform of an $N \times 1$ data vector, $X = \left[x(0), x(1).....x(N-1)\right]'$, ("$'$"denotes the transpose) to its corresponding spectrum $N \times 1$ vector, $Y = \left[y(0), y(1).....y(N-1)\right]'$ using the transformation matrix *W*. The *i*th spectral component $\mathrm{y}\left(i\right), i \in \left[0, N-1\right]$, is given by

$$y\left(i\right) = \sum_{j=0}^{N-1} W(i,j)X(i,j) \qquad (9)$$

The transforming matrix performs linear transformation from the input vector X to produce the N spectral outputs Y.

1) Inverse Transformation:

The 2-D inverse DCT transformation is defined as

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \alpha(u)\alpha(v)C\left(u,v\right)\cos\left(\frac{\pi\left(2x+1\right)u}{2N}\right)\cos\left(\frac{\pi\left(2y+1\right)v}{2N}\right)$$

$$for\ x, y = 0,1,2.....N\text{-}1$$

$$where\ \alpha(u) = \begin{cases} \sqrt{\dfrac{1}{N}} & for\ u = 0 \\ \sqrt{\dfrac{2}{N}} & for\ u \neq 0 \end{cases}$$

The transformation matrix is independent of the input signal, thus can be evaluated using RDCT as $W\left(M, p_{kj}\right)$ for *j*=1,2 and $k = 0,1,2\ldots,(N-1)$. This property ensures that the algorithm supports image compression applications with complexity remaining equal both at the decoder and the encoder.

2) Orthogonality:

Since the proposed RDCT algorithm evaluates the DCT coefficients recursively, the structure of the transformation is maintained. This is ensured by

$$W^{-1}(M,p) = W^{T}(M,p) \tag{10}$$

3) Frequency Response:

The transformation could be characterized by an impulse response, $h_i = [h_i(0), h_i(1) \dots h_i(N-1)]$, where $h_i(j) = W(i,j)$. The corresponding transfer function is given by

$$H_i(w) = \sum_{j=0}^{N-1} h_i(j) e^{-Jwj} \tag{11}$$

where $J = \sqrt{-1}$. The frequency response for different spectral components for different length DCT's are shown in fig (1). The frequency response using Floating-point DCT, RDCT and IDCT are plotted in fig (1). It is clear from the figure that RDCT complies with that of floating-point DCT.

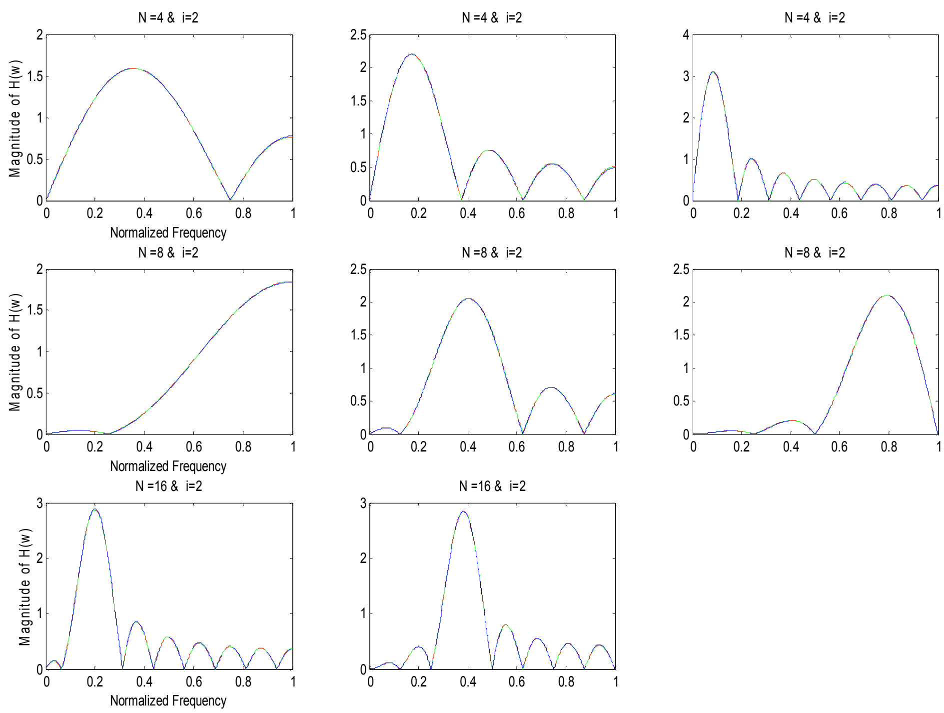

Figure 1: Frequency response for i=2, i=5 of Floating-point DCT (green--), RDCT (blue :), Integer DCT (red)

4) Spectral Characteristics:

The phase plane (real part of $H_i(w)$ plotted against imaginary part of $H_i(w)$) for the floating-point DCT, RDCT and Integer DCT for various spectral components and different lengths are illustrated in fig (2). Fig 2 shows that the RDCT(shown in green) has the phase plane almost very close to that of the floating-point DCT, but the IDCT (shown in blue) although has the same shape deviates from DCT .

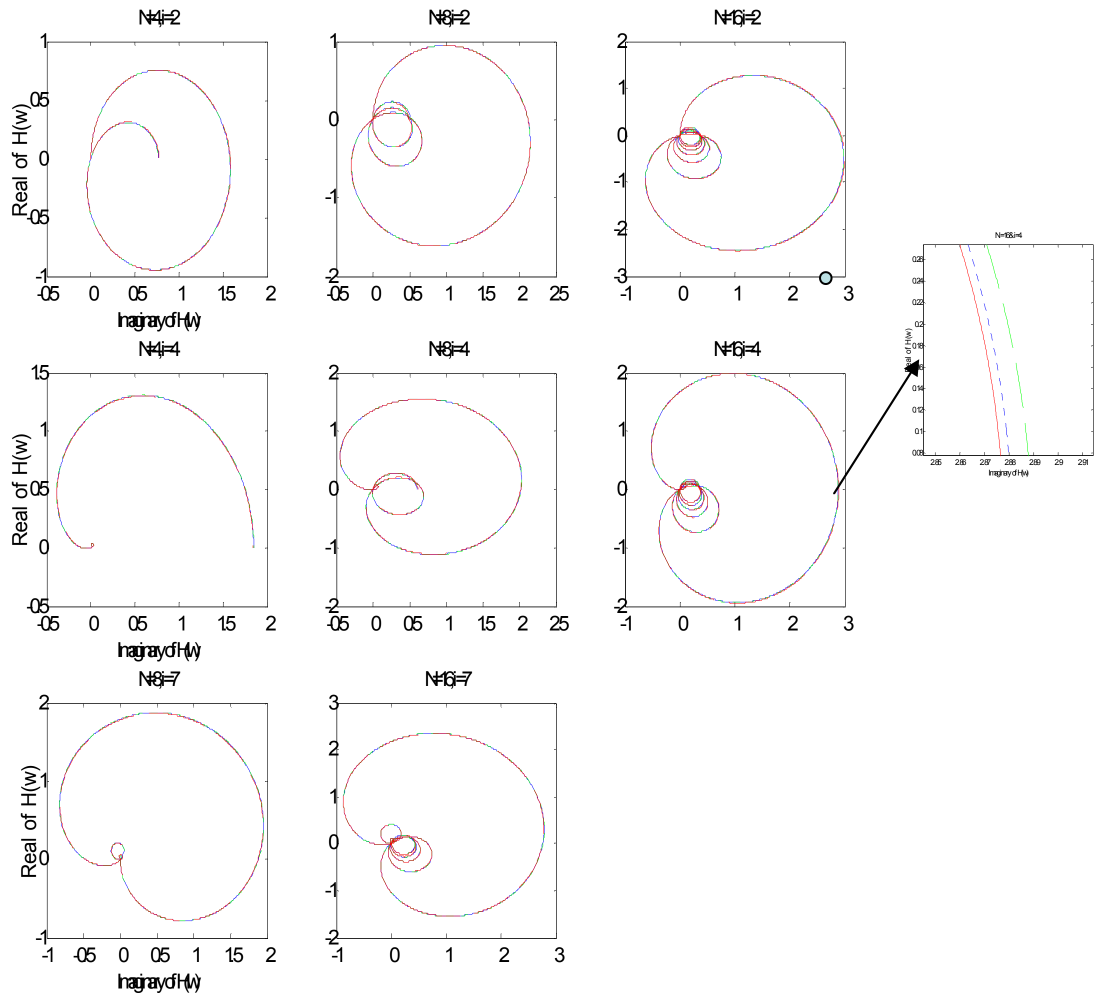

Figure 2: Phase plane for i =2,5 of Floating-point DCT (red--), RDCT (blue:),Integer DCT (green)

The deviation in the DCT coefficients from the floating-point DCT can be computed as the magnitude square error illustrated in Figs (3). The corresponding squares of the magnitudes of the difference between the transfer functions of the floating-point DCT and RDCT, between floating-point DCT and Integer DCT is defined respectively as

$$
\begin{aligned}
D &= | H_i(w)_{DCT} - H_i(w)_{RDCT} |^2 \\
E &= | H_i(w)_{DCT} - H_i(w)_{IDCT} |^2
\end{aligned}
\qquad (12)
$$

From the fig (3) it is clear that the error D between floating-point DCT and the RDCT is almost zero for lower spectral components and is much lower than the error E for higher spectral components. This clearly shows that the error due to the approximation in RDCT is almost zero at lower order frequency components and is very minimal at high frequency components. This property is very useful in using RDCT for image compression, where the emphasis would be to retain the lower spectral components.

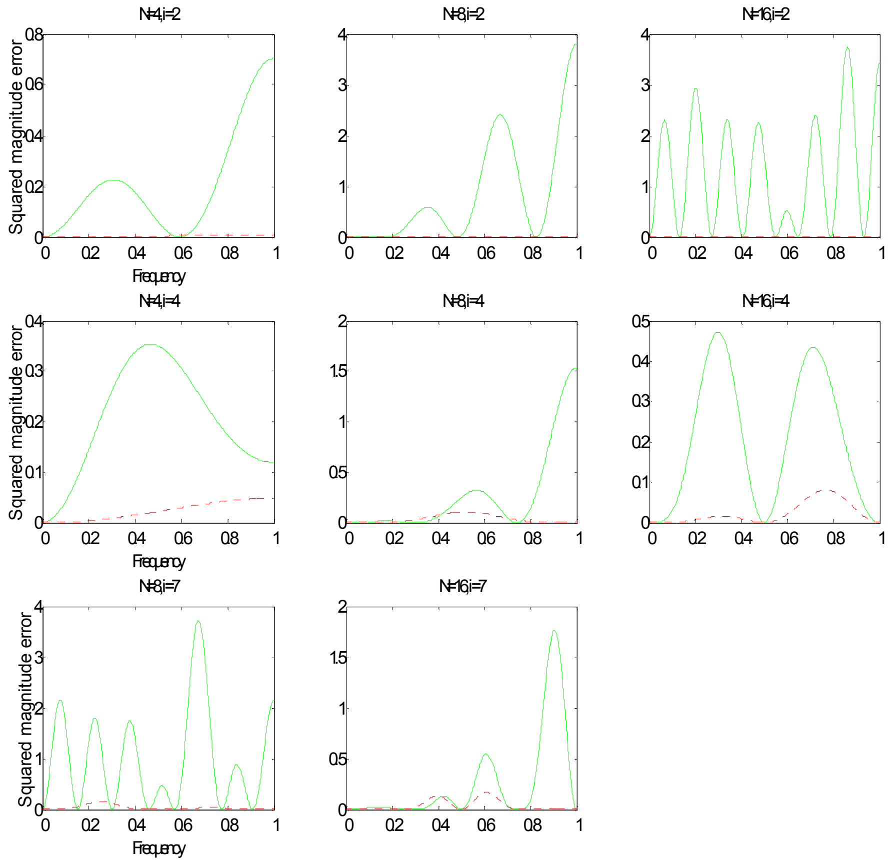

Figure 3: Squared magnitude error for i=5, i=7 of DCT-Integer DCT (green), DCT-RDCT (red :)

The plots clearly depicts that the RDCT has similar spectral characteristics with that of the floating-point DCT in spite of the approximation using Ramanujan Ordered Number of type-2. Due to the spectral closeness, the floating-point DCT and RDCT have similar de-correlation and compaction properties.

5) De-correlation:

The principle advantage of the image transform is the removal of adjacent pixel redundancy. This leads to efficient coding of the uncorrelated transform coefficients independently. To outline the de-correlation properties of the proposed DCT we have considered the normalized autocorrelation of the highly correlated images before and after DCT as shown in Figure 4. Clearly, the amplitude of the autocorrelation after the DCT operation is very small at all lags. The figure clearly shows that the proposed multiplierless RDCT and IDCT also exhibit the same de-correlation property as does the floating-point DCT.

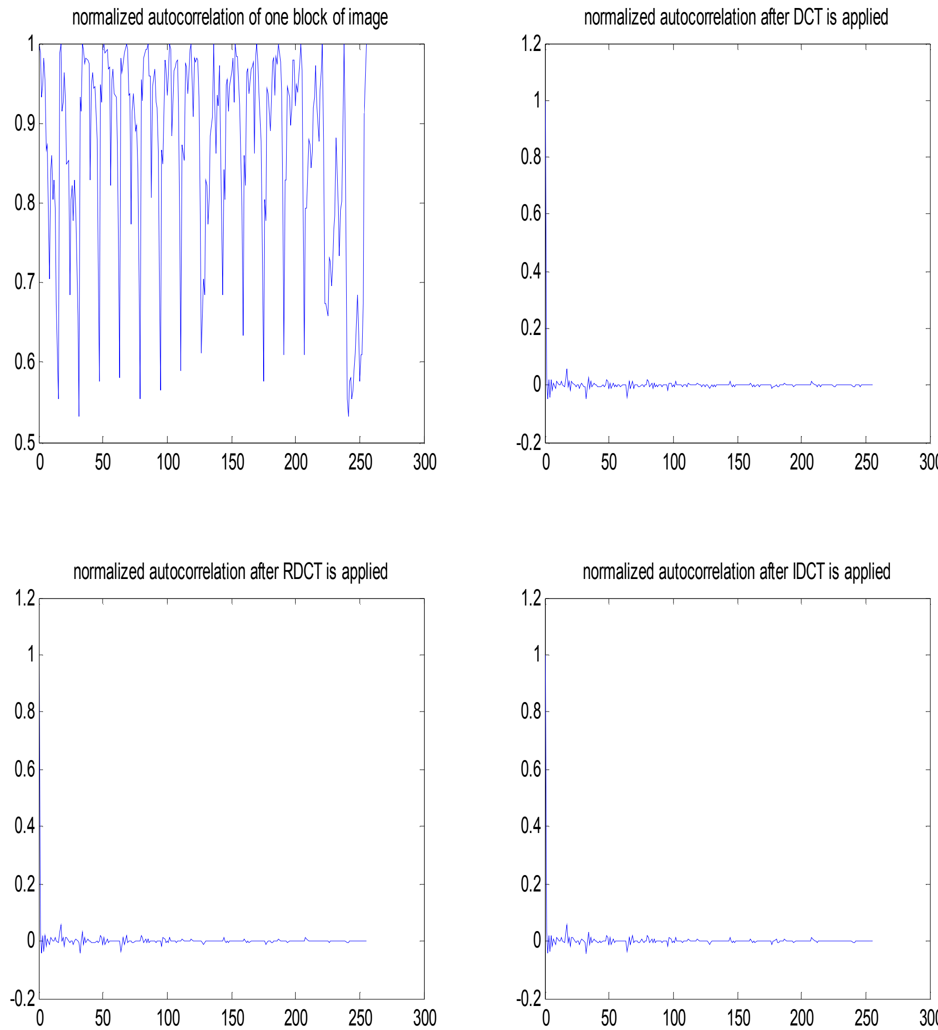

Figure 4: Normalized autocorrelation of one block of image before and after DCT

6) Energy Compaction:

To illustrate the compaction properties of the RDCT compared with floating-point DCT and Integer DCT, a 100 sample sequence derived from zero mean and unit variance first order Markov process have been transformed by the RDCT and floating-point DCT and Integer DCT, each of the order of N =100. The statistical variances of the spectral components have been estimated over independent realizations arranged in the descending order. The normalized cumulative variance (NCV) has been computed. The NCV for the i^{th} spectral component, $i \in [0, N-1]$ is defined [31] as

$$NCV(i) = \frac{\sum_{m=0}^{i} \sigma_y^{\,2}(m)}{\sum_{m=0}^{N-1} \sigma_y^{\,2}(m)} \tag{13}$$

where $\sigma_y^{\,2}(m)$ is the variance of the m^{th} spectral component.

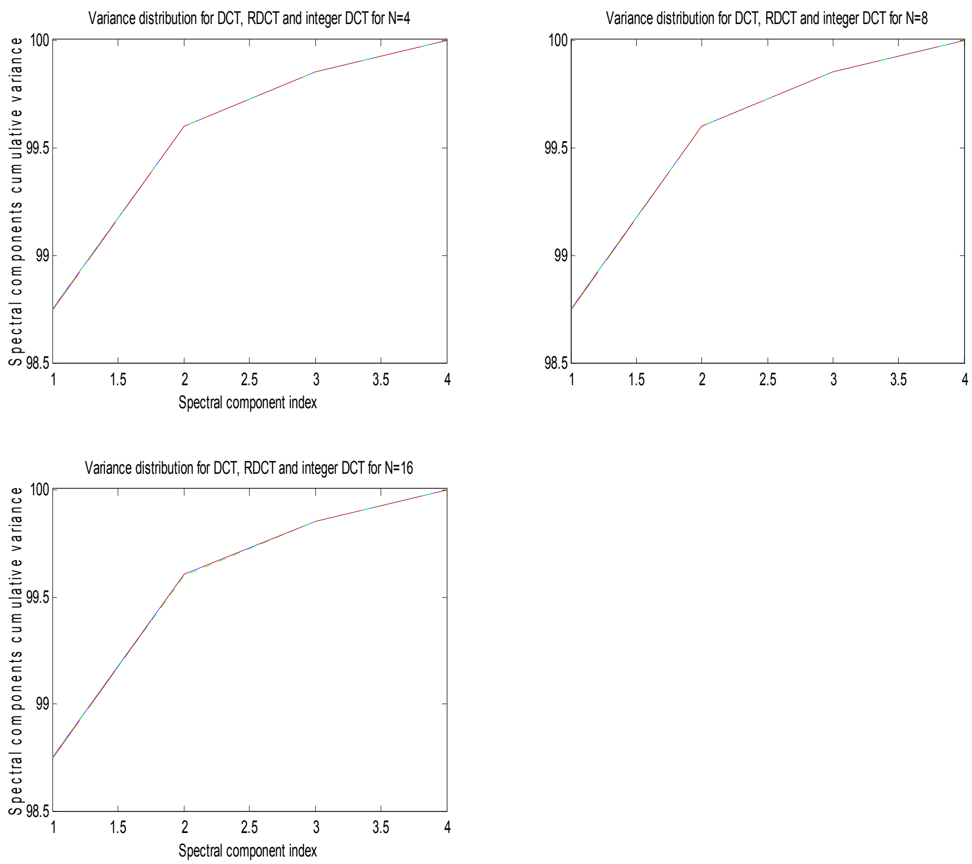

Figure 5: Normalized cumulative variance for different values of N

Fig (5) shows the percentage NCV for floating-point DCT, RDCT and Integer DCT. NCV (i) provides a measure of the compression ratio that can be achieved (eqn. 13). From the fig (5) it is clear that the compression capabilities of the multiplierless transforms (RDCT and Integer DCT) are close to the floating-point DCT. The good energy compaction properties are very evidently seen in the figure (5). Only 10% of each of the floating-point DCT, RDCT and Integer DCT spectral components contain about 97.96%, 97.96%, and 97.93% of the total signal power respectively.

4. Performance of the proposed RDCT

The performance of the proposed RDCT is demonstrated in the context of the visual data compression applications. The performance of the algorithm depends also on the complexity involved in computation. The computational complexity is measured in terms of the number of addition and shifting operations involved in computing the DCT coefficients.

4.1 Computational Complexity

The proposed Ramanujan ordered DCT is a fast algorithm. Since α is a Ramanujan ordered number of type-2, the computations of y_n 's would require only shifters and adders unlike the integer DCT which requires additional lifting steps along with shifters and adders. This avoids the necessity of floating-point multipliers.

Table 2 below compares the proposed RDCT with the floating-point DCT and the Integer DCT in terms of the number of multipliers/shifters and adders based on the operations required for computation of the DCT coefficients.

Table 2. Comparison of computational complexity

Operations	Floating-point DCT	Integer DCT	Proposed RDCT
Multiplications	$N/2\log_2 N$ (Floating-point)	N (Integer)	$N/2\log_2 N$ (shifts)
Additions	$(3N/2\log_2 N)-N+1$	$(2N\log_2 N)-2N+2$	$(3N/2\log_2 N)-N+1$
Lifting Steps	Nil	$(3N/2\log_2 N)-3N+3$	Nil

when N=8, Floating-point DCT requires 12 floating-point multiplications and 29 additions. The need of the floating-point multiplications is eliminated by multiplierless integer transforms which takes 8 integer multiplications, and 35 additions, but requires 15 additional lifting operations implemented using shift registers. RDCT requires 12 shifts and 29 additions with no additional requirement of the lifting steps.

4.2 Image compression employing RDCT.

To prove the energy compaction properties of the proposed RDCT, the two-dimensional RDCT concentrates most of the visually significant information about the image in just a few coefficients. The international standard lossy image compression algorithm JPEG employs floating-point DCT. The input image is divided into 8×8 , and the two-dimensional DCT is computed for each block. The DCT are then quantized, Huffman coded and transmitted. In the receiver, the inverse two-dimensional DCT for each block is computed and then the blocks are set back together to form the original image.

4.2.1 The experiment

The proposed RDCT is tested by replacing the two-dimensional DCT block in the baseline JPEG standard algorithm with the 2-D RDCT block. The performance is then compared by using commonly used multiplierless 2-D Integer DCT. The block size used to scan the image is 8×8 . DC coefficient is quantized and coded separately and transmitted. The AC coefficients are encoded with very few coefficients removing the completely zero coefficients block. The compression ratio and PSNR achieved is shown in the Table below.

Table 3. Performance comparison of RDCT with Integer DCT and Floating-point DCT

Image	Floating-point DCT		Integer DCT		Proposed RDCT	
	Compression Ratio	PSNR (dB)	Compression Ratio	PSNR (dB)	Compression Ratio	PSNR (dB)
Cameraman	5.2541:1	31.6428	5.2505:1	31.6243	5.2518:1	31.6353
Circuit	5.6:1	37.6382	5.5989:1	37.6286	5.6005:1	37.6343
Football	5.9776:1	31.5956	5.9780:1	31.5914	5.9772:1	31.5926
Lena	5.7262:1	35.5718	5.7262:1	35.5591	5.7247:1	35.5686

Table II clearly shows that the RDCT is only around 0.004-0.0075 dB below the Floating-point DCT in the PSNR's of the reconstructed images and around 0.01-0.02dB above the commonly used Integer DCT. The original and the reconstructed images using floating-point DCT, RDCT and the Integer DCT are shown in Fig (6). It is clear that the quality of the reconstructed image with the proposed RDCT is close to but slightly inferior to that employing the floating-point DCT. However considerable savings in computations is achieved using RDCT.

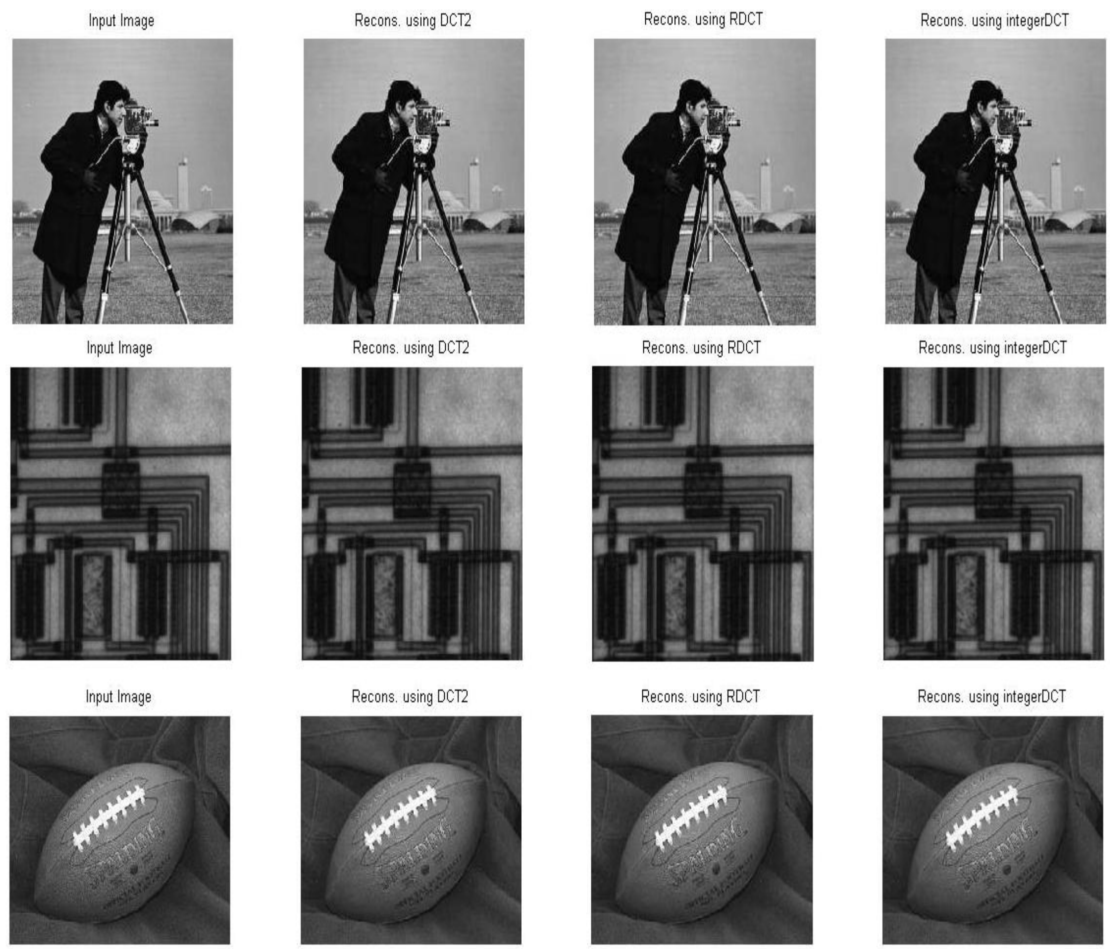

Figure 6: The original image and the reconstructed image using floating-point DCT2, RDCT and IDCT algorithms

5. Conclusion

Multiplierless Ramanujan ordered DCT (RDCT) is fast, computationally simple algorithm. Representing N, the order of the DCT as Ramanujan ordered numbers and using recursive procedure, the DCT coefficients can be computed using only shifts and addition operations. The orthogonality and the frequency response of the DCT coefficients ensure that the algorithm is a unitary transformation. The spectral characteristics of the algorithm clearly bring out the advantage of the RDCT over the other multiplierless INTDCT and the floating-point DCT. The DCT coefficients thus obtained using RDCT algorithm retains the DCT periodicity and spectral structure and hence retains its good de-correlation and power compaction properties. Further RDCT is a fast algorithm as it requires $N/2\log_2 N$ shifts and $(3N/2\log_2 N) - N + 1$ additions. The performance of the RDCT has been investigated in compressing the images. Simulation results shows that the RDCT performance is superior to that of the more commonly used integer DCT and is close to that of the floating-point DCT.

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