

A Robust Finite-Horizon Kalman Filter for Uncertain Discrete Time-Varying Systems with State-Delay and Missing Measurements

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Abstract

In this paper, a robust kalman filter is designed for the uncertainty time-varying discrete systems with state delay in process and output matrices combined with the possibility of missing measurements. The uncertainties are expected in the process, output and white noise covariance matrices. A formula for a candidate upper bound on the actual state estimation error variances for all admissible parameter uncertainties and possible missing measurements is obtained. The filter parameters are optimized to give a minimal upper bound on the state estimation error covariance for all admissible uncertainties and missing measurements.

Keywords: *Uncertain discrete time-varying system; Missing measurements; Robust state estimation; Kalman filtering*

1. Introduction

The performance of estimation is severely affected by the modeling parameters uncertainties and missing measurements. In the past few years, many bodies of work addressed these problems are interesting in to design various filters with a better performance. One of good methods to solve these problems is robust filtering. So far, many approaches including LMI approach, interpolation approach, riccati equation approach and so on, have been successfully proposed and lots of results on this topic have been reported in literature [1-4].

For the case of modeling parameters uncertainties, In terms of LMI approach of the robust H_∞ filtering problem was derived in[5], while in [6] sufficient conditions for the solvability of robust filtering was developed by using two difference riccati equation approach. The problem of robust H_∞ filtering for a class of discrete-time nonlinear systems with state-delay and norm bounded parameter uncertainty was studied in [7]. They solve this problem using linear-matrix inequality approach. In [8], the finite and infinite horizon robust Kalman filter addressed the problem of estimating state variables for discrete-time systems with time-varying norm-bounded uncertainties in both the state and output matrices has presented. In [9], a robust finite-horizon Kalman filtering problem for linear discrete time-varying uncertain systems with additive and multiplicative noises is solved and an optimization approach based on the solutions to discrete Riccati difference equations is used. In [10], Z. Dong and Z. You have proposed a robust finite-horizon Kalman filter for uncertainties discrete time-varying systems with uncertain-covariance white noises. However, all of these works are assumed that the observations are available at the time of estimation.

For the case of missing measurements, commonly referred to as uncertainty in the observation process, the filtering problem was first investigated in [11]. In [12], the problem of variance-constrained filtering for discrete time-varying system with

norm-bounded parameter uncertainties and uncertain measurements has been considered. In [13], A robust finite-horizon filtering problem for a class of discrete time-varying systems with missing measurements and norm-bounded parameter uncertainties was studied. The desired filter parameters can be obtained by solving two discrete Riccati difference equations, which are of a form suitable for recursive computation in online application. In [14], the Robust Finite-Horizon Kalman Filtering is designed for linear discrete time-varying systems with time-varying norm-bounded uncertainties in the state, output and white noise covariance matrices and missing measurements. The filter parameters are obtained over minimized the upper bound for the state estimation error covariance. In [15], a robust finite-time H_∞ filter for uncertain systems subject to missing measurements is developed. The filter parameters can be calculated by solving a sequence of linear matrix inequalities. It should be pointed out that in [11], [12], [13], [14] and [15], state delay in process and output are not taken into account.

In this paper, a robust finite-horizon Kalman filtering problem for the uncertainty time-varying discrete systems with state delay in process and output matrices combined with the possibility of missing measurements is studied. The system under consideration is subject to the state, output and the white noise covariance matrices. The purpose is to design a filter having an estimation error variance with an optimized guaranteed upper bound for all admissible uncertainties and the possibility of missing measurements. The filter parameters are obtained by optimizing the upper-bounds for state estimation error variances.

2. Problem Formulation

Consider the following class of uncertain discrete time-varying systems with state delay:

$$\begin{aligned} x_{k+1} &= (A_k + \Delta A_k) x_k + (D_{k-\tau} + \Delta D_{k-\tau}) x_{k-\tau} + (B_k + \Delta B_k) w_k \\ x_{k-\tau} &= 0, k \in [0, \tau] \end{aligned} \quad (1)$$

where $x_k \in R^n$ is a state vector, $x_{k-\tau} \in R^n$ is a state delay vector, and $w_k \in R^n$ is a zero mean Gaussian white noise sequence. A_k , $D_{k-\tau}$ and B_k are known real time-varying matrices with appropriate dimensions. ΔA_k , $\Delta D_{k-\tau}$ and ΔB_k are real time-varying matrix functions representing the time-varying parameter uncertainties, and they are assumed to have the following form:

$$\Delta A_k = H_{1,k} F_k E_{1,k} \quad (2)$$

$$\Delta D_{k-\tau} = H_{1,k-\tau} F_k E_{k-\tau} \quad (3)$$

$$\Delta B_k = H_{1,k} F_k E_{2,k} \quad (4)$$

where $H_{1,k}$, $H_{1,k-\tau}$, $E_{1,k}$, $E_{1,k-\tau}$ and $E_{2,k}$ are known time-varying matrices of appropriate dimensions, and is the norm-bounded time-varying uncertainty, *i.e.*, $F_k^T F_k \leq I$.

The measurements which may contain missing dates are described by:

$$\begin{aligned} y_k &= \gamma_k (C_k + \Delta C_k) x_k + \xi_{k-\tau} (G_{k-\tau} + \Delta G_{k-\tau}) x_{k-\tau} + (L_k + \Delta L_k) v_k \\ x_{k-\tau} &= 0, k \in [0, \tau] \end{aligned} \quad (5)$$

where the stochastic variables γ_k and $\xi_{k-\tau}$ are Bernoulli distributed white sequence taking the value of 0 or 1 randomly with

$$\text{Prob}(\gamma_k = 1) = \mu_k \quad (6)$$

$$\text{Prob}(\gamma_k = 0) = 1 - \mu_k \quad (7)$$

$$\text{Prob}(\xi_{k-\tau} = 1) = \chi_{k-\tau} \quad (8)$$

$$\text{Prob}(\xi_{k-\tau} = 0) = 1 - \chi_{k-\tau} \quad (9)$$

and $\mu_k \in [0, 1]$ and $\chi_{k-\tau} \in [0, 1]$ are known time-varying positive number representing the

probability that the observation sequence contains the signal of interests, $y_k \in R^m$ is a measured output vector, $v_k \in R^m$ is a zero mean Gaussian white noise sequence, C_k , $G_{k-\tau}$ and L_k are known real time-varying matrices with appropriate dimensions, and the norm-bounded uncertainties in the observation process are expressed as:

$$\Delta C_k = H_{2,k} F_k E_{1,k} \tag{10}$$

$$\Delta G_{k-\tau} = H_{2,k-\tau} F_k E_{k-\tau} \tag{11}$$

$$\Delta L_k = H_{2,k} F_k E_{2,k} \tag{12}$$

Supposing that $w_k, v_k, \gamma_k, \chi_{k-\tau}$ and x_0 are mutually uncorrelated, and w_k, v_k and x_0 have the following statistical properties:

$$E \left\{ \begin{bmatrix} w_k \\ v_k \\ x_0 \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ 0 \\ \hat{x}_0 \end{bmatrix} \tag{13}$$

$$E \left\{ \begin{bmatrix} w_k \\ v_k \\ x_0 \end{bmatrix} \begin{bmatrix} w_k \\ v_k \\ x_0 \end{bmatrix}^T \right\} = \begin{bmatrix} Q_k \delta_{kj} & 0 & 0 \\ 0 & R_k \delta_{kj} & 0 \\ 0 & 0 & \Sigma_{1,0} \end{bmatrix} \tag{14}$$

where $E\{\cdot\}$ stands for the mathematical expectation operator, Q_k , R_k and $\Sigma_{1,0}$ represent covariance matrices of noises and the initial state, and δ_{kj} denotes the Kronecker delta function, which is equal to unity for $k = j$ and zero elsewhere.

The expected filter for systems (1) and (5) is represented as

$$\begin{aligned} \hat{x}_{k+1} &= A_{ok} \hat{x}_k + D_{ok-\tau} \hat{x}_{k-\tau} + K_{ok} (y_k - \mu_k C_k \hat{x}_k - \chi_{k-\tau} G_k \hat{x}_{k-\tau}) \\ \hat{x}_{k-\tau} &= 0, k \in [0, \tau] \end{aligned} \tag{15}$$

where $k \in [0, N]$, $\hat{x}_k \in R^n$ is the estimated state value and $\hat{x}_{k-\tau} \in R^n$ is the estimated state-delay value, A_{ok}, D_{ok} and K_{ok} are the filter parameters to be determined.

The purpose of this paper is to design finite-horizon filter such that there exists a sequence of positive definite matrices Ξ_k satisfying

$$E \left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T \right] \leq \Xi_k \tag{16}$$

and then optimize the filter parameters A_{ok}, D_{ok} and K_{ok} over minimizing the upper bounded Ξ_k .

3. State Covariance and Upper Bounds

The accurate error covariance is impossible find because the systems (1) and (5) involve uncertain and stochastic terms. Therefore, the necessary works are to derive the estimation error and determine a corresponding upper-bounded. For this purpose, an augment state vector is defined

$$\tilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \tag{17}$$

and then the augmented state-space model system following form system (1) and filter (15) can be described as

$$\begin{aligned} \tilde{x}_{k+1} &= (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) x_k + (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) x_{k-\tau} + \\ & (\bar{B}_k + \bar{H}_{2,k} F_k \bar{E}_{2,k}) \bar{W}_k + \bar{A}_{e1} x_k + \bar{A}_{e2} x_k + \bar{D}_{e1} x_{k-\tau} + \bar{D}_{e2} x_{k-\tau} \end{aligned} \tag{18}$$

where

$$\bar{A}_k = \begin{bmatrix} A_k & 0 \\ \mu_k K_{ok} C_k & A_{ok} - \mu_k K_{ok} C_k \end{bmatrix}, \quad \bar{D}_{k-\tau} = \begin{bmatrix} D_{k-\tau} & 0 \\ \chi_{k-\tau} K_{ok} G_{k-\tau} & D_{ok-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau} \end{bmatrix}$$

$$\begin{aligned} \bar{B}_k &= \begin{bmatrix} B_k & 0 \\ 0 & K_{ok} L_k \end{bmatrix}, \quad \bar{H}_{1,k} = \begin{bmatrix} H_{1,k} \\ K_{ok} H_{2,k} \end{bmatrix}, \quad \bar{E}_{1,k} = \begin{bmatrix} E_{1,k} & 0 \end{bmatrix}, \quad \bar{H}_{k-\tau} = \begin{bmatrix} H_{1,k-\tau} \\ K_{ok} H_{2,k-\tau} \end{bmatrix} \\ \bar{E}_{k-\tau} &= \begin{bmatrix} E_{k-\tau} & 0 \end{bmatrix}, \quad \bar{H}_{2,k} = \begin{bmatrix} H_{1,k} & 0 \\ 0 & K_{ok} H_{2,k} \end{bmatrix}, \quad \bar{E}_{2,k} = \begin{bmatrix} E_{1,k} & 0 \\ 0 & E_{1,k} \end{bmatrix} \\ \bar{A}_{e1} &= \begin{bmatrix} 0 & 0 \\ (\gamma_k - \mu_k) K_{ok} C_k & 0 \end{bmatrix}, \quad \bar{A}_{e2} = \begin{bmatrix} 0 & 0 \\ (\gamma_k - 1) K_{ok} H_{2,k} F_k E_{1,k} & 0 \end{bmatrix} \\ \bar{D}_{e1} &= \begin{bmatrix} 0 & 0 \\ (\xi_{k-\tau} - \chi_{k-\tau}) K_{ok} G_{k-\tau} & 0 \end{bmatrix}, \quad \bar{D}_{e2} = \begin{bmatrix} 0 & 0 \\ (\xi_k - 1) K_{ok} H_{2,k-\tau} F_k E_{k-\tau} & 0 \end{bmatrix} \end{aligned}$$

We denote the state covariance matrix of the augmented system (18) as

$$\tilde{\Sigma}_k = \mathbb{E} \left[\begin{bmatrix} \tilde{x}_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} \tilde{x}_k^T \\ \hat{x}_k^T \end{bmatrix} \right] = \mathbb{E} \left\{ \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}^T \right\} \quad (19)$$

Since $\bar{A}_{e1}, \bar{A}_{e2}, \bar{D}_{e1}$ and \bar{D}_{e2} have zero mean stochastic matrix sequences in (18), then we can obtain the Lyapunov equation that governs the evolution of the covariance matrix $\tilde{\Sigma}_k$ as

$$\begin{aligned} \tilde{\Sigma}_{k+1} &= (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) \tilde{\Sigma}_k (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k})^T + \\ &(\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) \tilde{\Sigma}_{k,k-\tau} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau})^T + \\ &(\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) \tilde{\Sigma}_{k-\tau,k} (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k})^T + \\ &(\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) \tilde{\Sigma}_{k-\tau} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau})^T + \\ &(\bar{B}_k + \bar{H}_{2,k} F_k \bar{E}_{2,k}) \tilde{W}_k (\bar{B}_k + \bar{H}_{2,k} F_k \bar{E}_{2,k})^T + \Psi_{1,k} + \\ &\Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \\ \tilde{\Sigma}_{k-\tau} &= 0, \quad \Psi_{1,k-\tau} = 0, \quad \Psi_{2,k-\tau} = 0, \quad k \in [0, \tau] \end{aligned} \quad (20)$$

where

$$\begin{aligned} \tilde{\Sigma}_{k,k-\tau} &= \mathbb{E} \left[\begin{bmatrix} \tilde{x}_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} \tilde{x}_{k-\tau}^T \\ \hat{x}_{k-\tau}^T \end{bmatrix} \right] = \mathbb{E} \left\{ \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix} \begin{bmatrix} x_{k-\tau} \\ \hat{x}_{k-\tau} \end{bmatrix}^T \right\}, \\ \tilde{\Sigma}_{k,k-\tau} &= 0, \quad k \in [0, \tau] \\ \tilde{\Sigma}_{k-\tau,k} &= \tilde{\Sigma}_{k,k-\tau}^T, \quad \tilde{W}_k = \begin{bmatrix} Q_k & 0 \\ 0 & R_k \end{bmatrix} \\ \Psi_{1,k} &= \mathbb{E} \left[\bar{A}_{e1} \tilde{\Sigma}_k \bar{A}_{e1}^T \right] \\ &= \mathbb{E} \left[\begin{bmatrix} 0 & 0 \\ (\gamma_k - \mu_k) K_{ok} C_k & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ (\gamma_k - \mu_k) K_{ok} C_k & 0 \end{bmatrix}^T \right] \\ &= \mathbb{E} \left[(\gamma_k - \mu_k)^2 \begin{bmatrix} 0 & 0 \\ K_{ok} C_k & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ K_{ok} C_k & 0 \end{bmatrix}^T \right] \\ &= \mu_k (1 - \mu_k) \begin{bmatrix} 0 & 0 \\ K_{ok} C_k & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ K_{ok} C_k & 0 \end{bmatrix}^T \end{aligned}$$

$$\begin{aligned}
 \Psi_{2,k} &= \mathbb{E} \left[\bar{A}_{e2} \tilde{\Sigma}_k \bar{A}_{e2}^T \right] \\
 &= \mathbb{E} \left[\begin{bmatrix} 0 & 0 \\ (\gamma_k - 1) K_{ok} H_{2,k} F_k E_{1,k} & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ (\gamma_k - 1) K_{ok} H_{2,k} F_k E_{1,k} & 0 \end{bmatrix}^T \right] \\
 &= \mathbb{E} \left[(\gamma_k - 1)^2 \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k} E_{1,k} & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k} E_{1,k} & 0 \end{bmatrix}^T \right] \\
 &= (\mu_k - 1) \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k} E_{1,k} & 0 \end{bmatrix} \tilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k} E_{1,k} & 0 \end{bmatrix}^T \\
 \Psi_{1,k-\tau} &= \mathbb{E} \left[\bar{D}_{e1} \tilde{\Sigma}_{k-\tau} \bar{D}_{e1}^T \right] \\
 &= \mathbb{E} \left[\begin{bmatrix} 0 & 0 \\ (\xi_{k-\tau} - \chi_{k-\tau}) K_{ok} G_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ (\xi_{k-\tau} - \chi_{k-\tau}) K_{ok} G_{k-\tau} & 0 \end{bmatrix}^T \right] \\
 &= \mathbb{E} \left[(\xi_{k-\tau} - \chi_{k-\tau})^2 \begin{bmatrix} 0 & 0 \\ K_{ok} G_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ K_{ok} G_{k-\tau} & 0 \end{bmatrix}^T \right] \\
 &= \chi_{k-\tau} (1 - \chi_{k-\tau}) \begin{bmatrix} 0 & 0 \\ K_{ok} G_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ K_{ok} G_{k-\tau} & 0 \end{bmatrix}^T \\
 \Psi_{2,k-\tau} &= \mathbb{E} \left[\bar{D}_{e2} \tilde{\Sigma}_k \bar{D}_{e2}^T \right] \\
 &= \mathbb{E} \left[\begin{bmatrix} 0 & 0 \\ (\xi_{k-\tau} - 1) K_{ok} H_{2,k-\tau} F_k E_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ (\xi_{k-\tau} - 1) K_{ok} H_{2,k-\tau} F_k E_{k-\tau} & 0 \end{bmatrix}^T \right] \\
 &= \mathbb{E} \left[(\xi_{k-\tau} - 1)^2 \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k-\tau} E_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k-\tau} E_{k-\tau} & 0 \end{bmatrix}^T \right] \\
 &= (\chi_{k-\tau} - 1) \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k-\tau} E_{k-\tau} & 0 \end{bmatrix} \tilde{\Sigma}_{k-\tau} \begin{bmatrix} 0 & 0 \\ K_{ok} H_{2,k-\tau} E_{k-\tau} & 0 \end{bmatrix}^T
 \end{aligned}$$

In order to show that the error covariance matrix has the upper-bounded, the following lemmas are introduced.

Lemma 1: [16] Given any real matrices Σ_1, Σ_2 and Σ_3 with appropriate dimensions such that $\Sigma_3^T \Sigma_3 \leq I$ and $\lambda > 0$, then the following inequality holds:

$$\Sigma_1 \Sigma_3 \Sigma_2 + \Sigma_2^T \Sigma_3^T \Sigma_1^T \leq \lambda \Sigma_1 \Sigma_1^T + \lambda^{-1} \Sigma_2^T \Sigma_2 \quad (21).$$

Lemma 2: [17] Given matrices A, H, E and F with compatible dimensions such that $FF^T \leq I$. Let X be a symmetric positive-definite matrix and there exist arbitrary $\alpha > 0$ such that $\alpha^{-1} I - EXE^T > 0$, then the following inequality holds:

$$\begin{aligned}
 (A + HFE)X(A + HFE)^T \\
 \leq A(X^{-1} - \alpha E^T E)^{-1} A^T + \alpha^{-1} HH^T
 \end{aligned} \quad (22).$$

Lemma 3: [18] For $0 < k < N$, suppose $X > 0$ and $s_k(X) = s_k^T(X) \in \mathbb{R}^{n \times n}$ and $g_k(X) = g_k^T(X) \in \mathbb{R}^{n \times n}$, if there exists $Y > X$ such that

$$s_k(Y) \geq s_k(X) \quad (23)$$

and

$$g_k(Y) \geq g_k(X) \quad (24)$$

then applying solutions M_k and N_k to the following difference equations:

$$M_{k+1} = s_k(M_k), N_{k+1} = g_k(N_k), M_0 = N_0 > 0 \quad (25)$$

satisfies $M_{k+1} \leq N_{k+1}$.

The following result can be obtained from Lemma 1~Lemma 3.

Theorem 1: If there exists positive scalar sequence $\alpha_k > 0$ such that $\alpha_k^{-1}I - \bar{E}_{1,k} \tilde{\Sigma}_k \bar{E}_{1,k}^T > 0$, positive scalar sequence $\beta_{k-\tau} > 0$ such that $\beta_{k-\tau}^{-1}I - \bar{E}_{k-\tau} \tilde{\Sigma}_{k-\tau} \bar{E}_{k-\tau}^T > 0$ and positive scalar sequence $\eta_k > 0$ such that $\eta_k^{-1}I - \bar{E}_{2,k} \tilde{W}_k \bar{E}_{2,k}^T > 0$, then

$$\begin{aligned} \tilde{\Sigma}_{k+1} &\leq (1+\lambda)\bar{A}_k(\tilde{\Sigma}_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \bar{A}_k^T + \\ &+(1+\lambda^{-1})\bar{D}_{k-\tau}(\tilde{\Sigma}_{k-\tau}^{-1} - \beta_{k-\tau} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} \bar{D}_{k-\tau}^T + \\ &\bar{B}_k(\tilde{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} \bar{B}_k^T + (1+\lambda)\alpha_k^{-1} \bar{H}_{1,k} \bar{H}_{1,k}^T + \\ &(1+\lambda^{-1})\beta_{k-\tau}^{-1} \bar{H}_{k-\tau}^T \bar{H}_{k-\tau} + \eta_k^{-1} \bar{H}_{2,k}^T \bar{H}_{2,k} + \Psi_{1,k} + \Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \\ \tilde{\Sigma}_{k-\tau} &= 0, \Psi_{1,k-\tau} = 0, \Psi_{2,k-\tau} = 0, k \in [0, \tau] \end{aligned} \quad (26)$$

with initial value $\tilde{\Sigma}_0 = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$ and let

$$\begin{aligned} \Sigma_{k+1} &= (1+\lambda)\bar{A}_k(\Sigma_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \bar{A}_k^T + \\ &+(1+\lambda^{-1})\bar{D}_{k-\tau}(\Sigma_{k-\tau}^{-1} - \beta_{k-\tau} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} \bar{D}_{k-\tau}^T + \\ &\bar{B}_k(\tilde{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} \bar{B}_k^T + (1+\lambda)\alpha_k^{-1} \bar{H}_{1,k} \bar{H}_{1,k}^T + \\ &(1+\lambda^{-1})\beta_{k-\tau}^{-1} \bar{H}_{k-\tau}^T \bar{H}_{k-\tau} + \eta_k^{-1} \bar{H}_{2,k}^T \bar{H}_{2,k} + \Psi_{1,k} + \Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \\ \Sigma_{k-\tau} &= 0, \Psi_{1,k-\tau} = 0, \Psi_{2,k-\tau} = 0, k \in [0, \tau] \end{aligned} \quad (27)$$

$$\alpha_k^{-1}I - \bar{E}_{1,k} \Sigma_k \bar{E}_{1,k}^T > 0 \quad (28)$$

$$\beta_{k-\tau}^{-1}I - \bar{E}_{k-\tau} \Sigma_{k-\tau} \bar{E}_{k-\tau}^T > 0 \quad (29)$$

$$\eta_k^{-1}I - \bar{E}_{2,k} \tilde{W}_k \bar{E}_{2,k}^T > 0 \quad (30)$$

with initial value $\Sigma_0 = \tilde{\Sigma}_0$, then $\tilde{\Sigma}_k \leq \Sigma_k$ and Σ_k will be a candidate upper-bound for the estimation error covariance $\tilde{\Sigma}_k$.

Proof: According to Lemma 1, we know that there exists a positive sequence λ satisfying

$$\begin{aligned} &(\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) \tilde{\Sigma}_{k,k-\tau} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau})^T + (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) \tilde{\Sigma}_{k-\tau,k} (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k})^T \leq \\ &\lambda (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) \tilde{\Sigma}_k (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k})^T + \lambda^{-1} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) \tilde{\Sigma}_{k-\tau} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau})^T \end{aligned}$$

then we have

$$\begin{aligned} \tilde{\Sigma}_{k+1} &\leq (1+\lambda)(\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k}) \tilde{\Sigma}_k (\bar{A}_k + \bar{H}_{1,k} F_k \bar{E}_{1,k})^T + \\ &(1+\lambda^{-1})(\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau}) \tilde{\Sigma}_{k-\tau} (\bar{D}_{k-\tau} + \bar{H}_{k-\tau} F_k \bar{E}_{k-\tau})^T + \\ &(\bar{B}_k + \bar{H}_{2,k} F_k \bar{E}_{2,k}) \tilde{W}_k (\bar{B}_k + \bar{H}_{2,k} F_k \bar{E}_{2,k})^T + \Psi_{1,k} + \Psi_{2,k} + \\ &\Psi_{1,k-\tau} + \Psi_{2,k-\tau} \end{aligned} \quad (31)$$

Combining Lemma 2 and (26), we have the following corollary.

$$\begin{aligned} \tilde{\Sigma}_{k+1} &\leq (1+\lambda)\bar{A}_k(\tilde{\Sigma}_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \bar{A}_k^T + (1+\lambda)\alpha_k^{-1} \bar{H}_{1,k} \bar{H}_{1,k}^T + \\ &(1+\lambda^{-1})\bar{D}_{k-\tau}(\tilde{\Sigma}_{k-\tau}^{-1} - \beta_{k-\tau} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} \bar{D}_{k-\tau}^T + (1+\lambda^{-1})\beta_{k-\tau}^{-1} \bar{H}_{k-\tau}^T \bar{H}_{k-\tau} + \\ &\bar{B}_k(\tilde{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} \bar{B}_k^T + \eta_k^{-1} \bar{H}_{2,k}^T \bar{H}_{2,k} + \Psi_{1,k} + \Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \end{aligned} \quad (32)$$

We denote Σ_{k+1} as

$$\begin{aligned} \Sigma_{k+1} &= (1+\lambda)\bar{A}_k(\Sigma_k^{-1} - \alpha_k\bar{E}_{1,k}^T\bar{E}_{1,k})^{-1}\bar{A}_k^T + \\ &+ (1+\lambda^{-1})\bar{D}_{k-\tau}(\Sigma_{k-\tau}^{-1} - \beta_{k-\tau}\bar{E}_{k-\tau}^T\bar{E}_{k-\tau})^{-1}\bar{D}_{k-\tau}^T \\ &\bar{B}_k(\bar{W}_k^{-1} - \eta_k\bar{E}_{2,k}^T\bar{E}_{2,k})^{-1}\bar{B}_k^T + (1+\lambda)\alpha_k^{-1}\bar{H}_{1,k}\bar{H}_{1,k}^T + \\ &(1+\lambda^{-1})\beta_{k-\tau}^{-1}\bar{H}_{k-\tau}^T\bar{H}_{k-\tau} + \eta_k^{-1}\bar{H}_{2,k}^T\bar{H}_{2,k} + \Psi_{1,k} + \Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \\ \Sigma_{k-\tau} &= 0, \Psi_{1,k-\tau} = 0, \Psi_{2,k-\tau} = 0, k \in [0, \tau] \end{aligned} \quad (33)$$

According to (26), we denote

$$\tilde{\Sigma}_{k+1} = s_k(\tilde{\Sigma}_k) \quad (34)$$

where

$$\begin{aligned} s_k(\tilde{\Sigma}_k) &= (1+\lambda)\left(\tilde{A}_k + \tilde{H}_{1,k}F\tilde{E}_{1,k}\right)\tilde{\Sigma}_k\left(\tilde{A}_k + \tilde{H}_{1,k}F\tilde{E}_{1,k}\right)^T + \\ &(1+\lambda^{-1})\left(\tilde{D}_{k-\tau} + \tilde{H}_{3,k-\tau}F\tilde{E}_{3,k-\tau}\right)\tilde{\Sigma}_{k-\tau}\left(\tilde{D}_{k-\tau} + \tilde{H}_{3,k-\tau}F\tilde{E}_{3,k-\tau}\right)^T + \\ &\left(\tilde{B}_k + \tilde{H}_{2,k}F\tilde{E}_{2,k}\right)\tilde{W}_k\left(\tilde{B}_k + \tilde{H}_{2,k}F\tilde{E}_{2,k}\right)^T \end{aligned} \quad (35)$$

According to (27), we can also denote

$$\Sigma_{k+1} = h_k(\Sigma_k) \quad (36)$$

where

$$\begin{aligned} h_k(\Sigma_k) &= (1+\lambda)\tilde{A}_k(\Sigma_k^{-1} - \alpha_k\tilde{E}_{1,k}^T\tilde{E}_{1,k})^{-1}\tilde{A}_k^T + \\ &+ (1+\lambda^{-1})\tilde{D}_{k-\tau}(\Sigma_{k-\tau}^{-1} - \gamma_k\tilde{E}_{3,k-\tau}^T\tilde{E}_{3,k-\tau})^{-1}\tilde{D}_{k-\tau}^T \\ &\tilde{B}_k(\tilde{W}_k^{-1} - \beta_k\tilde{E}_{2,k}^T\tilde{E}_{2,k})^{-1}\tilde{B}_k^T + (1+\lambda)\alpha_k^{-1}\tilde{H}_{1,k}\tilde{H}_{1,k}^T + \\ &(1+\lambda^{-1})\gamma_k^{-1}\tilde{H}_{3,k-\tau}^T\tilde{H}_{3,k-\tau} + \beta_k^{-1}\tilde{H}_{2,k}\tilde{H}_{2,k}^T \end{aligned} \quad (37)$$

It can be checked that functional h and s defined satisfy the conditions in Lemma 2, hence the conclusion $\tilde{\Sigma}_k \leq \Sigma_k$.

Suppose

$$\Xi_{k+1} = [I \quad -I]\Sigma_{k+1}[I \quad -I]^T \quad (38)$$

then

$$E\left[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\right] \leq \Xi_k \quad (39)$$

4. Robust Finite-Horizon Kalman Filter Design

To obtain the upper-bounded Σ_k , we need to find the solution to (27). Hence, we will try to solve (27), and select the filter parameters $A_{ok}, D_{ok-\tau}$ and K_{ok} that optimize the upper-bounded of the actual state estimation error variance. For this purpose, we have the following theorem.

Theorem 2: If let the positive scalar sequences α_k , $\beta_{k-\tau}$ and η_k satisfying inequalities (28), (29) and (30), then the solution to (27) will be of the form

$$\Sigma_k = \begin{bmatrix} \Sigma_{1,k} & \Sigma_{1,k} - \Xi_k \\ \Sigma_{1,k} - \Xi_k & \Sigma_{1,k} - \Xi_k \end{bmatrix} \quad (40)$$

where $\Sigma_{1,k}$ and Ξ_k are defined as (44) and (45).

and $tr(\Xi_k)$ is minimal if there exists an filter (15) with parameters

$$A_{ok} = A_k + (A_k - \mu K_{ok} C_k)\Xi_k E_{1,k}^T (\alpha^{-1}I - E_{1,k}\Xi_k E_{1,k}^T)^{-1} E_{1,k} \quad (41)$$

$$D_{ok-\tau} = D_{k-\tau} + (D_{k-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau})\Xi_{k-\tau} E_{k-\tau}^T (\beta_{k-\tau}^{-1}I - E_{k-\tau}\Xi_{k-\tau} E_{k-\tau}^T)^{-1} E_{k-\tau} \quad (42)$$

and

$$\begin{aligned} K_{ok} &= [(1+\lambda)\mu A_k(\Xi_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T + (1+\lambda^{-1})\chi_{k-\tau} D_{k-\tau}(\Xi_{k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau}) G_{k-\tau}^T \\ &+ (1+\lambda)\alpha_k^{-1} H_{1,k} H_{2,k}^T + (1+\lambda^{-1})\beta_{k-\tau}^{-1} H_{1,k-\tau} H_{2,k-\tau}^T] R_{1,k}^{-1} \end{aligned} \quad (43)$$

where

$$\begin{aligned}
 R_{1,k} &= (1 + \lambda)\mu^2 C_k (\Xi_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T + (1 + \lambda)\alpha_k^{-1} H_{2,k} H_{2,k}^T \\
 &+ (1 + \lambda^{-1})\chi_{k-\tau}^2 G_{k-\tau} (\Xi_{k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau})^{-1} G_{k-\tau}^T + (1 + \lambda^{-1})\beta_{k-\tau}^{-1} H_{2,k-\tau} H_{2,k-\tau}^T \\
 &+ L_k (R_k^{-1} - \eta_k E_{2,k}^T E_{2,k})^{-1} L_k^T + \eta_k^{-1} H_{2,k} H_{2,k}^T + \mu_k (\mu_k - 1) C_k \Sigma_{1,k} C_k^T \\
 &+ (\mu_k - 1) H_{2,k} E_{1,k} \Sigma_{1,k} (H_{2,k} E_{1,k})^T + \chi_{k-\tau} (\chi_{k-\tau} - 1) G_{k-\tau} \Sigma_{1,k-\tau} G_{k-\tau}^T \\
 &+ (\chi_{k-\tau} - 1) H_{2,k-\tau} E_{k-\tau} \Sigma_{1,k-\tau} (H_{2,k-\tau} E_{k-\tau})^T
 \end{aligned}$$

Furthermore, the state covariance matrix will be

$$\begin{aligned}
 \Sigma_{1,k+1} &= (1 + \lambda) A_k (\Sigma_{1,k}^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T + \\
 &+ (1 + \lambda^{-1}) D_{k-\tau} (\Sigma_{1,k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau})^{-1} D_{k-\tau}^T \\
 &+ B_k (Q_k^{-1} - \eta_k E_{2,k}^T E_{2,k})^{-1} B_k^T + (1 + \lambda)\alpha_k^{-1} H_{1,k} H_{1,k}^T + \\
 &(1 + \lambda^{-1})\beta_{k-\tau}^{-1} H_{1,k-\tau}^T H_{1,k-\tau} + \eta_k^{-1} H_{2,k}^T H_{2,k} \\
 \Sigma_{1,k-\tau} &= 0, k \in [0, \tau]
 \end{aligned} \tag{44}$$

and the estimation error covariance will be

$$\begin{aligned}
 \Xi_{k+1} &= -(1 + \lambda)\alpha_k^{-1} H_{1,k} H_{2,k}^T + (1 + \lambda^{-1})\beta_{k-\tau}^{-1} H_{1,k-\tau} H_{2,k-\tau}^T + \\
 &(1 + \lambda)\mu A_k (\Xi_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T + \\
 &(1 + \lambda^{-1})\chi_{k-\tau} D_{k-\tau} (\Xi_{k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau})^{-1} G_{k-\tau}^T] R_{1,k}^{-1} \\
 &[(1 + \lambda)\alpha_k^{-1} H_{2,k} H_{1,k}^T + (1 + \lambda^{-1})\beta_{k-\tau}^{-1} H_{2,k-\tau} H_{1,k-\tau}^T + \\
 &(1 + \lambda)\mu C_k (\Xi_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T + \\
 &(1 + \lambda^{-1})\chi_{k-\tau} G_{k-\tau} (\Xi_{k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau})^{-1} D_{k-\tau}^T] + \\
 &(1 + \lambda) A_k (\Xi_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T + \\
 &(1 + \lambda^{-1}) D_{k-\tau} (\Xi_{k-\tau}^{-1} - \beta_{k-\tau} E_{k-\tau}^T E_{k-\tau})^{-1} D_{k-\tau}^T + \\
 &B_k (Q_k^{-1} - \eta_k E_{2,k}^T E_{2,k})^{-1} B_k^T + \\
 &+ ((1 + \lambda)\alpha_k^{-1} + \eta_k^{-1}) H_{1,k} H_{1,k}^T + (1 + \lambda^{-1})\beta_{k-\tau}^{-1} H_{1,k-\tau} H_{1,k-\tau}^T
 \end{aligned} \tag{45}$$

To prove the Theorem 2, the matrix inversion lemma is used. It is described as follow.

Lemma 4: [19] There exist real conformable matrices A, B, C and D . A and D are reversible. Then the following equation holds

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)CA^{-1}$$

Proof: substituting (40) (41) (42) and (43) into (27) and considering conditions (44) and (45), straightforward algebraic manipulations show that the right-hand side of (27) is given by

$$\begin{aligned}
 &(1 + \lambda)\bar{A}_k (\Sigma_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \bar{A}_k^T + (1 + \lambda^{-1})\bar{D}_{k-\tau} (\Sigma_{k-\tau}^{-1} - \beta_{k-\tau} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} \bar{D}_{k-\tau}^T \\
 &+ \bar{B}_k (\bar{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} \bar{B}_k^T + (1 + \lambda)\alpha_k^{-1} \bar{H}_{1,k} \bar{H}_{1,k}^T + (1 + \lambda^{-1})\beta_{k-\tau}^{-1} \bar{H}_{k-\tau}^T \bar{H}_{k-\tau} \\
 &+ \eta_k^{-1} \bar{H}_{2,k}^T \bar{H}_{2,k} + \Psi_{1,k} + \Psi_{2,k} + \Psi_{1,k-\tau} + \Psi_{2,k-\tau} \\
 &= \begin{bmatrix} \Sigma_{1,k+1} & \Sigma_{1,k+1} - \Xi_{k+1} \\ \Sigma_{1,k+1} - \Xi_{k+1} & \Sigma_{1,k+1} - \Xi_{k+1} \end{bmatrix}
 \end{aligned} \tag{46}$$

$$\Sigma_{k-\tau} = 0, \Psi_{1,k-\tau} = 0, \Psi_{2,k-\tau} = 0, k \in [0, \tau]$$

This means that (40) is a solution to (27). In the following part, we need to prove that upper bound is optimized by using the filter parameters $A_{ok}, D_{ok-\tau}$ and K_{ok} . According to (27) and (38), we have the upper bound for state estimation error variance as:

$$\begin{aligned}
 \Xi_{k+1} &= (1+\lambda)[A_k - \mu K_{ok} C_k \quad \mu K_{ok} C_k - A_{ok}](\Sigma_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \\
 & \llbracket [A_k - \mu K_{ok} C_k \quad \mu K_{ok} C_k - A_{ok}]^T + (1+\lambda^{-1})[D_{k-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau} \quad \chi_{k-\tau} K_{ok} G_{k-\tau} - D_{ok-\tau}] \\
 & \llbracket (\Sigma_k^{-1} - \beta_{k-1} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} [D_{k-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau} \quad \chi_{k-\tau} K_{ok} G_{k-\tau} - D_{ok-\tau}]^T + [B_k \quad -K_{ok} L_k] \\
 & \llbracket (\tilde{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} [B_k \quad -K_{ok} L_k]^T + (1+\lambda) \alpha_k^{-1} (H_{1,k} - K_{ok} H_{2,k}) \\
 & \llbracket (H_{1,k} - K_{ok} H_{2,k})^T + (1+\lambda^{-1}) \beta_{k-\tau}^{-1} (H_{1,k-\tau} - K_{ok} H_{2,k-\tau}) (H_{1,k-\tau} - K_{ok} H_{2,k-\tau})^T \\
 & + \eta_k^{-1} [H_{1,k} \quad -K_{ok} H_{2,k}] [H_{1,k} \quad -K_{ok} H_{2,k}]^T + \mu_k (1-\mu_k) K_{ok} C_k \Sigma_{1,k} (K_{ok} C_k)^T \\
 & + (\mu_k - 1) K_{ok} H_{2,k} E_{1,k} \Sigma_{1,k} (K_{ok} H_{2,k} E_{1,k})^T + \chi_{k-\tau} (1-\chi_{k-\tau}) K_{ok} G_{k-\tau} \Sigma_{1,k-\tau} (K_{ok} G_{k-\tau})^T \\
 & + (\chi_{k-\tau} - 1) K_{ok} H_{2,k-\tau} E_{k-\tau} \Sigma_{1,k-\tau} (K_{ok} H_{2,k-\tau} E_{k-\tau})^T
 \end{aligned} \tag{47}$$

Obviously, Ξ_{k+1} is related to A_{ok} , $D_{ok-\tau}$ and K_{ok} . In order to determine the filter parameters A_{ok} , $D_{ok-\tau}$ and K_{ok} that minimize Ξ_{k+1} , we take the first variation to (47) with respect to A_{ok} , $D_{ok-\tau}$ and K_{ok} and equal them to zero.

$$\frac{\partial \Xi_{k+1}}{\partial A_{ok}} = [A_k - \mu K_{ok} C_k \quad \mu K_{ok} C_k - A_{ok}](\Sigma_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} [0 \quad -I]^T = 0 \tag{48}$$

$$\frac{\partial \Xi_{k+1}}{\partial D_{ok-\tau}} = [D_{k-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau} \quad \chi_{k-\tau} K_{ok} G_{k-\tau} - D_{ok-\tau}](\Sigma_{k-\tau}^{-1} - \beta_{k-1} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} [0 \quad -I]^T = 0 \tag{49}$$

and

$$\begin{aligned}
 \frac{\partial \Xi_{k+1}}{\partial K_{ok}} &= (1+\lambda)[A_k - \mu K_{ok} C_k \quad \mu K_{ok} C_k - A_{ok}](\Sigma_k^{-1} - \alpha_k \bar{E}_{1,k}^T \bar{E}_{1,k})^{-1} \\
 & \llbracket [-\mu C_k \quad \mu C_k]^T + (1+\lambda^{-1})[D_{k-\tau} - \chi_{k-\tau} K_{ok} G_{k-\tau} \quad \chi_{k-\tau} K_{ok} G_{k-\tau} - D_{ok-\tau}] \\
 & \llbracket (\Sigma_k^{-1} - \beta_{k-1} \bar{E}_{k-\tau}^T \bar{E}_{k-\tau})^{-1} [-\chi_{k-\tau} G_{k-\tau} \quad \chi_{k-\tau} G_{k-\tau}]^T + [B_k \quad -K_{ok} L_k] \\
 & \llbracket (\tilde{W}_k^{-1} - \eta_k \bar{E}_{2,k}^T \bar{E}_{2,k})^{-1} [0 \quad -L_k]^T + (1+\lambda) \alpha_k^{-1} (H_{1,k} - K_{ok} H_{2,k}) \\
 & \llbracket (-H_{2,k})^T + (1+\lambda^{-1}) \beta_{k-\tau}^{-1} (H_{1,k-\tau} - K_{ok} H_{2,k-\tau}) (-H_{2,k-\tau})^T \\
 & + \eta_k^{-1} [H_{1,k} \quad -K_{ok} H_{2,k}] [0 \quad -H_{2,k}]^T + \mu_k (1-\mu_k) K_{ok} C_k \Sigma_{1,k} C_k^T \\
 & + (\mu_k - 1) K_{ok} H_{2,k} E_{1,k} \Sigma_{1,k} (H_{2,k} E_{1,k})^T + \chi_{k-\tau} (1-\chi_{k-\tau}) K_{ok} G_{k-\tau} \Sigma_{1,k-\tau} G_{k-\tau}^T \\
 & + (\chi_{k-\tau} - 1) K_{ok} H_{2,k-\tau} E_{k-\tau} \Sigma_{1,k-\tau} (H_{2,k-\tau} E_{k-\tau})^T = 0
 \end{aligned} \tag{50}$$

From (48) (49) and (50) we can obtain (41) (42) and (43) respectively. Thus theorem 2 has been proved. According to the process of testifying theorem 2, we can clear know that filter parameters given in (48) (49) and (50) are indeed optimal and minimize the upper-bound Ξ_k .

Remark 1: Theorem 1 give the upper-bound for state estimation error variance and theorem 2 provide a method to calculate filter parameters A_{ok} , $D_{ok-\tau}$ and K_{ok} by optimizing the upper-bound. The optimization is step by step by calculating (44) and (45) under give the scalar parameters α_k , λ , $\beta_{k-\tau}$ and η_k . Because of the scalar parameters α_k , λ , $\beta_{k-\tau}$ and η_k will affect the positive definiteness of Ξ_k and $\Sigma_{1,k}$, and the performance of filter is depended on the solution of Ξ_k and $\Sigma_{1,k}$, it is necessary to discuss select the scalar parameters α_k , λ , $\beta_{k-\tau}$ and η_k that ensure the Ξ_k and $\Sigma_{1,k}$ is feasible.

5. Numerical Example

To demonstrate the effectiveness of the proposed filter applied to the uncertainty systems with state delay in process and output matrices combined with the possibility of missing measurements, a simulation example has been given and the system parameters for simulation of system (1) and (5) are

$$\begin{aligned}
 A_k &= \begin{bmatrix} 0.2 & -0.1 & 0 \\ 0.004 & 0.4 & 0.1 \\ 0.1\sin(6k) & 0.1 & 0.6 \end{bmatrix}, H_{1,k} = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix}, D_{k-3} = \begin{bmatrix} -0.1 & 0 & 0.6 \\ 0.05 & -0.2 & 0.1 \\ 0 & 0 & -0.3 \end{bmatrix}, H_{1,k-\tau} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \\
 C_k &= \begin{bmatrix} 2 & 0 & 1+\sin(6k) \\ 0 & 1 & 0 \end{bmatrix}, H_{2,k} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, G_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1+\sin(6k) \end{bmatrix}, H_{2,k-\tau} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \\
 L_k &= \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}, B_k = \begin{bmatrix} 0.3 \\ 0.2 \\ 0.6 \end{bmatrix}, E_{1,k} = [0.2 \ 0 \ 0.2], E_{1,k-\tau} = [0.5 \ 0.4 \ 0.2], E_{2,k} = -4, \Sigma_{1,0} = 2I, \\
 &E_0 = I, E[\gamma_k] = 0.8, E[\xi_{k-\tau}] = 0.6
 \end{aligned}$$

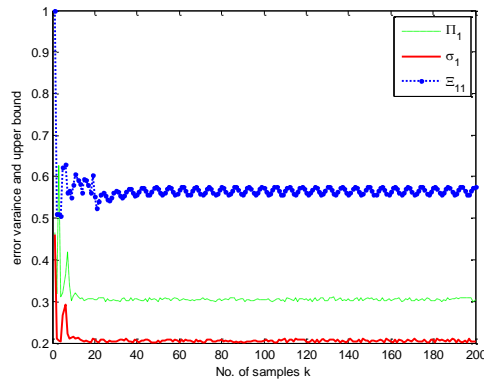


Figure 1. Upper-Bound as Well as Estimation Error Covariance for State 1

In figure 1~ figure 3, we choose the scalar parameters $\alpha_k = 0.8$, $\lambda = 0.8$, $\beta_{k-\tau} = 0.8$ and $\eta_k = 0.8$. We compare the proposed filter to the traditional Kalman filter. The plots of Π_1 , Π_2 and Π_3 are state estimation error variance of conventional Kalman filter. The plots of σ_1 , σ_2 and σ_3 are state estimation error variance. The plots of Ξ_{11} , Ξ_{22} and Ξ_{33} are upper-bound for state estimation error variance. It can be seen that the actual estimation error variances σ_1 , σ_2 and σ_3 of proposed robust kalman filter are high at the beginning of simulation and converge in small value at last. In the complete simulation process, the actual estimation error variances get more steady and stay below their upper bounds. The actual estimation error variances Π_1 , Π_2 and Π_3 of typical Kalman filter are oscillatory in the proceeds and their value are bigger than σ_1 , σ_2 and σ_3 because it dose not consider the uncertainties in the modeling parameters and/or the possibility of missing measurements.

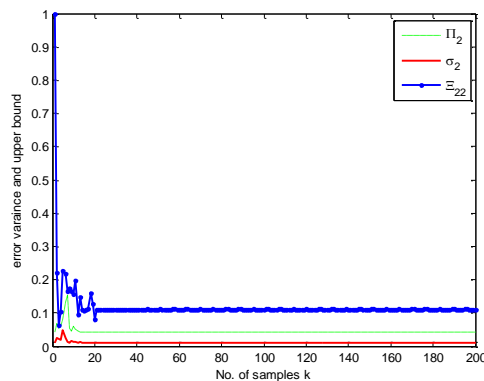


Figure 2. Upper-bound as well as Estimation Error Covariance for State 2

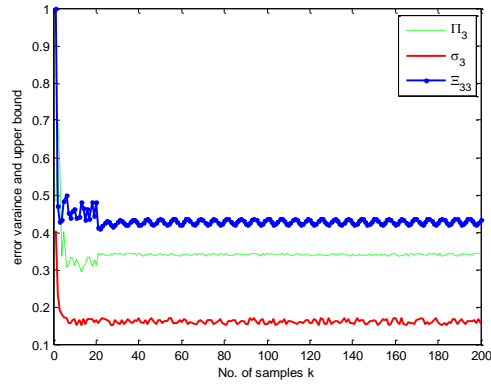
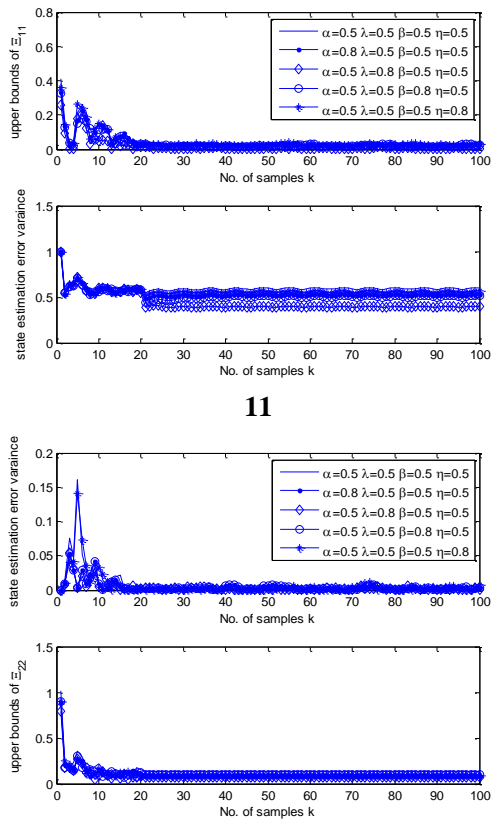


Figure 3. Upper-bound as well as Estimation Error Covariance for State 3



11

Figure 5. Upper-bound as Well as Estimation Error Covariance Comparison for State 2

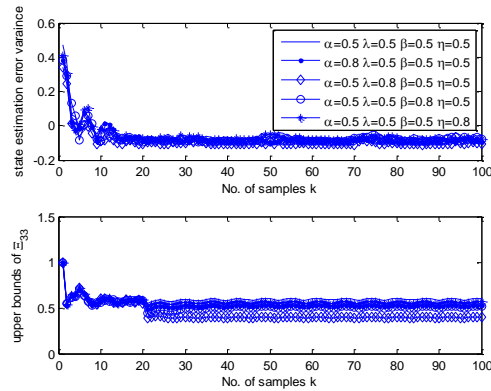


Figure 6. Upper-bound as Well as Estimation Error Covariance Comparison for State 3

In figure 4~ figure 6, we discuss the scalar parameters α_k , λ , $\beta_{k-\tau}$ and η_k how to affect the upper bound Ξ_k and the actual estimation error of variance. It can be concluded that Ξ_k decreases with the increase of α_k . The same conclusion about λ , $\beta_{k-\tau}$ and η_k can also be obtained from figure 4 ~ figure 6. Moreover, the upper bounds for state estimation error variances are the most sensitive to the parameter λ , are the least sensitive to the parameter η_k , and is sensitive to the parameter α_k as well as parameter γ_k . The actual estimation error of variances also have similar conclusion about the scalar parameters α_k , λ , $\beta_{k-\tau}$ and η_k . Therefore, the best choice for α_k , λ , $\beta_{k-\tau}$ and η_k are make them as larger as possible in the conditions of that Ξ_k is positive semi-definite matrix.

6. Conclusion

In this paper, we consider the robust definite-horizon Kalman filtering problem subject to the uncertainty time-varying discrete systems with state delay in process and output matrices combined with the possibility of missing measurements. We proved the existence of upper-bound for state estimation error variance and find a formula for a candidate upper-bound which is minimized over the optimal filter parameters. The simulation results show that the performance of proposed robust definite-horizon Kalman filter for the uncertainty time-varying discrete systems with state delay in process and output matrices combined with the possibility of missing measurements is better than typical Kalman filter.

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