

Second Order Duality in Multiobjective Programming With Generalized Convexity

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Abstract

The purpose of this paper is concerned with a class of multi objective programming problems in which the objective functions and constrained functions are twice differentiable and containing the support functions of a compact convex set. For such mathematic programming, to deal with its dual problem, first, the concept of second order $B - (p, r) -$ invexity is introduced. Further, the Mangasarian type second order dual model associated with the multi objective problem are formulated. Several weak, strong and strict converse dual theorems are established and proved by utilizing the new generalized convexity. The results extend and improve the corresponding results in the literature.

Keywords: *Second order $B - (p, r) -$ invexity, Multiobjective Programming, Mangasarian type second order dual, efficient solution*

1. Introduction

The field of multi objective programming known as vector programming has grown remarkably in different directions in the setting of optimality conditions and duality theory since the 1980's under the assumptions of various generalized convex functions. See, for example, [1-3] and the references cited therein. In recent past, these results have been generalized by many authors. Jayswal, *et al.*, [4] established the sufficient optimality conditions and duality results under the assumptions of $(p, r) - \rho - (\eta, \theta) -$ invexity. In the papers [5-8], the multi objective programming problems were discussed under the generalized convexity assumptions. Slimani, *et al.*, [9] formulated non differentiable multi objective programming involving generalized $d -$ invexity.

Duality plays a fundamental role in mathematics, especially in optimization; see for example Ref. [10-12]. It is an old study, additionally to a given optimization problem, a corresponding dual problem. Duality has resulted in many applications within optimization, and it has provided many unifying conceptual insights into economics and management science. Duality received more attention and many researchers have contributed to the development of duality in optimization. During the past decades, the common duality was extended to some classes of second-order duality under the assumptions of higher-order convexities. More specifically, in [13], the concept of second order generalized $\alpha -$ type I univex function were introduced and several duality theorems were obtained. Gupta, *et al.*, [14] formulated the symmetric duality for multi objective programming problems by utilizing higher-order $(F, \alpha, \rho, d) -$ convexity. Also, we can see the references in Ref. [15].

In this paper, motivated by the above work, we first introduce new generalized invex functions, called second order $B - (p, r) -$ invex functions. Further, we develop the dual theory. Mangasarian type second order dual for a class of multi objective programming

problems involving support functions are considered. Several dual results for the dual problems in which the functions occurring belong to the class of second order $B - (p, r) -$ invex functions

2. Notations and Preliminaries

The following convention of vectors R^n will be followed throughout this paper:
 $x \cong y \Leftrightarrow x_i \cong y_i, i = 1, 2, \dots, n$; $x \geq y \Leftrightarrow x \cong y, x \neq y$; $x > y \Leftrightarrow x_i > y_i, i = 1, 2, \dots, n$.

Let p, r be arbitrary real numbers. $\eta, \theta : X \times X \rightarrow R^n, b : X \times X \rightarrow R_+, \alpha : X \times X \rightarrow R_+ \setminus \{0\}$, and the scalar function $\varphi : X \rightarrow R$ be twice differentiable at $\bar{x} \in X$ and $\rho \in R$.

Definition2.1. The function φ is said to be second order $B - (p, r) -$ invex \bar{x} on X , if for all $x \in X$, there exist $b, \alpha, \eta, \theta, \rho$ and $q \in R^n$, such that

$$\begin{aligned} & \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] \\ & \cong \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) + \rho \|\theta(x, \bar{x})\|^2 \end{aligned} \quad (1)$$

for $p \neq 0, r \neq 0$

If for all $x \in X, x \neq \bar{x}$, the above inequality holds as strict inequality, then φ is said to be strictly second order $B - (p, r) -$ invex at \bar{x} on X .

Definition2.2. The function φ is said to be second order $B - (p, r) -$ pseudoinvex \bar{x} on X , if for all $x \in X$, there exist $b, \alpha, \eta, \theta, \rho$ and $q \in R^n$, such that

$$\begin{aligned} & \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] < 0 \\ & \Rightarrow \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) < -\rho \|\theta(x, \bar{x})\|^2 \end{aligned} \quad (2)$$

for $p \neq 0, r \neq 0$

Definition2.3. The function φ is said to be strictly second order $B - (p, r) -$ pseudoinvex \bar{x} on X , if for all $x \in X, x \neq \bar{x}$, there exist $b, \alpha, \eta, \theta, \rho$ and $q \in R^n$, such that

$$\begin{aligned} & \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) \cong -\rho \|\theta(x, \bar{x})\|^2 \\ & \Rightarrow \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] > 0 \end{aligned} \quad (3)$$

for $p \neq 0, r \neq 0$

Or equivalently

$$\begin{aligned} & \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] \leq 0 \\ \Rightarrow & \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) < -\rho \left\| \theta(x, \bar{x}) \right\|^2 \end{aligned} \quad (4)$$

for $p \neq 0, r \neq 0$

Definition2.4. The function φ is said to be second order $B - (p, r) -$ quasiinvex \bar{x} on X , if for all $x \in X$, there exist $b, \alpha, \eta, \theta, \rho$ and $q \in R^n$, such that

$$\begin{aligned} & \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] \leq 0 \\ \Rightarrow & \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) \leq -\rho \left\| \theta(x, \bar{x}) \right\|^2 \end{aligned} \quad (5)$$

for $p \neq 0, r \neq 0$

Or equivalently

$$\begin{aligned} & \frac{1}{p} \alpha(x, \bar{x}) [\nabla \varphi(\bar{x}) + \nabla^2 \varphi(\bar{x}) q]^T (e^{p\eta(x, \bar{x})} - I) > -\rho \left\| \theta(x, \bar{x}) \right\|^2 \\ \Rightarrow & \frac{1}{r} b(x, \bar{x}) [e^{r(\varphi(x) - \varphi(\bar{x}) + \frac{1}{2} q^T \nabla^2 \varphi(\bar{x}) q)} - 1] > 0 \end{aligned} \quad (6)$$

for $p \neq 0, r \neq 0$

Remark2.1. It should be pointed out that the exponentials appearing on the right-hand sides of inequalities above are understood to be taken component wise, for example $e^{(a_1, a_2, \dots, a_n)} = (e^{a_1}, e^{a_2}, \dots, e^{a_n})$ and $I = (1, 1, \dots, 1) \in R^n$.

Remark2.2. All theorems in the further part of this work will be proved only in the case when $p \neq 0, r \neq 0$ (other cases are dealt with likewise since the only changes arise from the form of inequality defining the class of second-order $B - (p, r) - V -$ type I for given p and r). The proofs in the other cases are easier than in this one. Moreover, without limiting generality of considerations, we shall assume that $r > 0$ (in the case when $r < 0$, the directions some of the inequalities in the proofs of theorems should be changed to the opposite one).

It is well known that if a point $x_0 \in X_0$ is an efficient solution for (MP) and the Slater's constraint qualification is satisfied on X , then the following Karush-Kuhn-Tucker type conditions are satisfied: there exist multipliers $\lambda \in R_+^k, \mu \in R_+^m, w_i \in R^n$ and $v_j \in R^n$ for $i \in K$ and $j \in M$, such that

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \nabla (f_i(x_0) + \langle w_i, x_0 \rangle) + \sum_{j=1}^m \mu_j \nabla (g_j(x_0) + \langle v_j, x_0 \rangle) = 0 \\ & \sum_{j=1}^m \mu_j (g_j(x_0) + \langle v_j, x_0 \rangle) = 0 \\ & \langle w_i, x_0 \rangle = s(x_0 | C_i), w_i \in C_i, i \in K \\ & \langle v_j, x_0 \rangle = s(x_0 | D_j), v_j \in D_j, j \in M \\ & \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \mu_j \geq 0, j \in M \end{aligned}$$

Throughout this paper, we consider the following multi objective programming problem:

$$\begin{aligned} & \text{minimize } F(x) = (f_1(x) + s(x | C_1), f_2(x) + s(x | C_2), \dots, f_k(x) + s(x | C_k)) \\ \text{(MP)} \quad & \text{s.t. } x \in X_0 = \{x \in X : g_j(x) + s(x | D_j) \leq 0, j \in M = \{1, 2, \dots, m\}\} \end{aligned}$$

Where $X \subseteq R^n$ is an open set. $f_i : X \rightarrow R, i = 1, 2, \dots, k$ and $g_j : X \rightarrow R, j \in M$ are twice differentiable on X and C_i, D_j are compact convex sets of R^n for each $i \in K$ and $j \in M$, $s(x | C_i)$ and $s(x | D_j)$ denote the support functions of C_i and D_j evaluated at x , defined by

$$s(x | C_i) = \max \{ \langle w_i, x \rangle | w_i \in C_i \}, i \in K, s(x | D_j) = \max \{ \langle v_j, x \rangle | v_j \in D_j \}, j \in M.$$

Definition2.5. A point $\bar{x} \in X_0$ is an efficient solution for problem (MP), if and only if there exists no $x \in X_0$, such that

$$f_i(x) + s(x | C_i) \leq f_i(\bar{x}) + s(\bar{x} | C_i), \text{ for all } i \in K \quad (7)$$

And

$$f_i(x) + s(x | C_i) < f_i(\bar{x}) + s(\bar{x} | C_i), \text{ for at least one } i \in K \quad (8)$$

Definition2.6. Let $\bar{x} \in X_0$ be an efficient solution for problem (MP), if there exists $(\bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v})$, such that

$$\begin{aligned} & \sum_{i=1}^k \bar{\lambda}_i (\nabla f_i(\bar{x}) + \bar{w}_i) + \sum_{j=1}^m \bar{\mu}_j (\nabla g_j(\bar{x}) + \bar{v}_j) = 0 \\ & \sum_{j=1}^m \bar{\mu}_j g_j(\bar{x}) = 0, \bar{\lambda}_i > 0, \sum_{i=1}^k \bar{\lambda}_i = 1, \bar{\mu}_j \geq 0, j \in M \end{aligned} \quad (9)$$

Then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v})$ is said to be a Karush-Kuhn-Tucker point of (MP).

3. Mangasarian Type Second Duality

In this section, we consider the following Mangasarian type second order dual associated with multi objective problem (MP) and establish several duality theorems under the new function assumptions:

$$\begin{aligned} & \maximize \{ f_1(y) + \langle w_1, y \rangle - \frac{1}{2} q^T \nabla^2 f_1(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q), \\ & \dots, f_k(y) + \langle w_k, y \rangle - \frac{1}{2} q^T \nabla^2 f_k(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q) \} \end{aligned}$$

$$(MD) \text{ s.t. } \lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \mu_j \geq 0,$$

$$w_i \in C_i, v_j \in D_j, i \in K, j \in M$$

$$y \in X, q \in R^n$$

Theorem 3.1. (Weak Duality) Let x and $(y, \lambda, \mu, w, v, q)$ be feasible solutions of (MP) and (MD), respectively such that

(i) $(f_i(\cdot) + \langle w_i, \cdot \rangle + \sum_{j=1}^m \mu_j (g_j(\cdot) + \langle v_j, \cdot \rangle))$ is second-order $B - (p, r)$ -invex at y with respect to b_i, α_i and ρ_i ;

(ii) $b_i(x, y) > 0$ for all $x \in D$;

(iii) $\sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x, y)} \geq 0$ for all $x \in D$;

Then the following can not hold :

$$f_i(x) + s(x|C_i) \leq f_i(y) + \langle w_i, y \rangle - \frac{1}{2} q^T \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q) \quad ,$$

for all $i \in K$

(10)

And

$$f_l(x) + s(x|C_l) < f_l(y) + \langle w_l, y \rangle - \frac{1}{2} q^T \nabla^2 f_l(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q) \quad ,$$

for at least one $l \in K$.

(11)

Proof: Suppose contrary to the result that the inequalities (10) and (11) hold.

In view of $\langle w_i, x \rangle \leq s(x|C_i)$ with $i \in K, w_i \in C_i$, the inequalities (10) and (11) yield

$$f_i(x) + \langle w_i, x \rangle \leq f_i(y) + \langle w_i, y \rangle - \frac{1}{2} q^T \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q) \quad ,$$

for all $i \in K$

(12)

And

$$f_l(x) + \langle w_l, x \rangle < f_l(y) + \langle w_l, y \rangle - \frac{1}{2} q^T \nabla^2 f_l(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2} q^T \nabla^2 g_j(y) q) \quad , \text{ for}$$

at least one $l \in K$.

(13)

The hypothesis (i) implies

$$\begin{aligned}
 & \frac{1}{r} b_i(x, y) [e^{r(f_i(x) + \langle w_i, x \rangle + \sum_{j=1}^m \mu_j (g_j(x) + \langle v_j, x \rangle) - f_i(y) - \langle w_i, y \rangle - \sum_{j=1}^m \mu_j (\nabla g_j(y) + \langle v_j, y \rangle) + \frac{1}{2} q^T \nabla^2 f_i(y) q + \frac{1}{2} q^T \nabla^2 (\sum_{j=1}^m \mu_j g_j(y) q))} - 1] \\
 & \cong \frac{1}{p} \alpha_i(x, y) [\nabla f_i(y) + w_i + \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y) q)]^T (e^{p\eta(x, y)} - I) \\
 & + \rho_i \|\theta(x, y)\|^2, i \in K
 \end{aligned} \tag{14}$$

Since $b_i(x, y) > 0, \mu_j \cong 0, \langle v_j, x \rangle \cong s(x|D_j), j \in M, v_j \in D_j, \langle w_i, x \rangle \cong s(x|C_i)$, with the feasibility of x for (MP), and the inequalities (12, 13), we have

$$\begin{aligned}
 & \frac{1}{r} b_i(x, y) [e^{r(f_i(x) + \langle w_i, x \rangle + \sum_{j=1}^m \mu_j (g_j(x) + \langle v_j, x \rangle) - f_i(y) - \langle w_i, y \rangle - \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle) + \frac{1}{2} q^T \nabla^2 f_i(y) q + \frac{1}{2} q^T \nabla^2 (\sum_{j=1}^m \mu_j g_j(y) q))} - 1] \\
 & \cong 0
 \end{aligned} \tag{15}$$

And the above inequality holds as strictly inequality for at least one $i \in K$.

The inequality (14) along with (15) and $\lambda_i > 0, \sum_{i=1}^k \lambda_i = 1$ implies

$$\begin{aligned}
 & \frac{1}{p} [\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y) q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y) q)]^T (e^{p\eta(x, \bar{x})} - I) \\
 & + \sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x, y)} \|\theta(x, y)\|^2 < 0, i \in K
 \end{aligned} \tag{16}$$

Which together with the first dual constraint condition yields

$$\sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x, y)} \|\theta(x, y)\|^2 < 0 \tag{17}$$

That is

$$\sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x, y)} < 0 \tag{18}$$

A contradiction to the hypothesis (iii). Hence, the proof is completed.

Theorem 3.2. (Weak Duality) Let x and $(y, \lambda, \mu, w, v, q)$ be feasible solutions of (MP) and (MD), respectively such that

- (i) $[\sum_{i=1}^k \lambda_i (f_i(\cdot) + \langle w_i, \cdot \rangle) + \sum_{j=1}^m \mu_j (g_j(\cdot) + \langle v_j, \cdot \rangle)]$ is strictly second-order $B - (p, r) -$ invex at y with respect to b, α and ρ ;
- (ii) $b(x, y) \cong 0$ for all $x \in D$;
- (iii) $\frac{\rho}{\alpha(x, y)} \cong 0$ for all $x \in D$;

Then the following can not hold :

$$f_i(x) + s(x|C_i) \cong f_i(y) + \langle w_i, y \rangle - \frac{1}{2} q^T \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle) - \frac{1}{2} q^T \nabla^2 g_j(y) q,$$

for all $i \in K$

$$\tag{19}$$

And

$$f_i(x) + s(x|C_i) < f_i(y) + \langle w_i, y \rangle - \frac{1}{2}q^T \nabla^2 f_i(y)q + \sum_{j=1}^m \mu_j(g_j(y) + \langle v_j, y \rangle - \frac{1}{2}q^T \nabla^2 g_j(y)q),$$

for at least one $l \in K$.

(20)

Proof: We proceed by contradiction. Suppose that the inequalities (10) and (11) hold.

Hence in view of $\langle w_i, x \rangle \leq s(x|C_i), i \in K, w_i \in C_i$ and $\langle v_j, x \rangle \leq s(x|D_j), j \in M, v_j \in D_j$, we have

$$f_i(x) + \langle w_i, x \rangle \leq f_i(x) + s(x|C_i), i \in K ; g_j(x) + \langle v_j, x \rangle \leq g_j(x) + s(x|D_j), j \in M$$

(21)

Since $\lambda_i > 0, \sum_{i=1}^k \lambda_i = 1, \mu_j \geq 0, j \in M$, which along with the feasibility of x for (MP), the inequalities (19) and (20) yield

$$\sum_{i=1}^k \lambda_i (f_i(x) + \langle w_i, x \rangle) + \sum_{j=1}^m \mu_j (g_j(x) + \langle v_j, x \rangle) - \sum_{i=1}^k \lambda_i (f_i(y) + \langle w_i, y \rangle - \frac{1}{2}q^T \nabla^2 f_i(y)q) - \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2}q^T \nabla^2 g_j(y)q) < 0$$

(22)

Using the hypothesis (ii) and (i), the inequality (22) implies

$$\frac{1}{\rho} \alpha(x, y) [\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y)q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y)q)]^T (e^{p\eta(x,y)} - I) + \rho \|\theta(x, y)\|^2 < 0$$

(23)

Which is a contradiction with the feasibility of $(y, \lambda, \mu, w, v, q)$ for (MD). Hence we get the desired conclusion of theorem.

Theorem 3.3. (Weak Duality) Let x and $(y, \lambda, \mu, w, v, q)$ be feasible solutions of (MP) and (MD), respectively such that

(i) $[\sum_{i=1}^k \lambda_i (f_i(\cdot) + \langle w_i, \cdot \rangle) + \sum_{j=1}^m \mu_j (g_j(\cdot) + \langle v_j, \cdot \rangle)]$ is second-order $B - (p, r) -$ pseudoinvex

at y with respect to b, α and ρ ;

(ii) $b(x, y) > 0$;

(iii) $\frac{\rho}{\alpha(x, y)} \geq 0$ for all $x \in D$;

Then the following can not hold :

$$f_i(x) + s(x|C_i) \leq f_i(y) + \langle w_i, y \rangle - \frac{1}{2}q^T \nabla^2 f_i(y)q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2}q^T \nabla^2 g_j(y)q) ,$$

for all $i \in K$

(24)

And

$$f_i(x) + s(x|C_i) < f_i(y) + \langle w_i, y \rangle - \frac{1}{2}q^T \nabla^2 f_i(y)q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle - \frac{1}{2}q^T \nabla^2 g_j(y)q) ,$$

for at least one $l \in K$.

(25)

Proof: Proceeding as in the proof of the above theorem, we get

$$\frac{1}{r} b(x, y) [e^{r(\sum_{i=1}^k \lambda_i (f_i(x) + \langle w_i, x \rangle) + \sum_{j=1}^m \mu_j (g_j(x) + \langle v_j, x \rangle) - \sum_{i=1}^k \lambda_i (f_i(y) + \langle w_i, y \rangle) - \frac{1}{2} q^T \nabla^2 f_i(y) q) - \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle) - \frac{1}{2} q^T \nabla^2 g_j(y) q)} - 1] < 0 \quad (26)$$

In view of the hypothesis (i), the above inequality yields

$$\frac{1}{p} \alpha(x, y) [\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y) q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y) q)]^T (e^{p\eta(x, y)} - I) + \rho \|\theta(x, y)\|^2 < 0 \quad (27)$$

Hence from the hypothesis (iii), we have

$$\frac{1}{p} [\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y) q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y) q)]^T (e^{p\eta(x, y)} - I) < 0 \quad (28)$$

Which is a contradiction with the feasibility of $(y, \lambda, \mu, w, v, q)$ for (MD). Hence we get the desired conclusion of theorem.

Theorem 3.4. (Weak Duality) Let x and $(y, \lambda, \mu, w, v, q)$ be feasible solutions of (MP) and (MD), respectively such that

(i) $[f_i(\cdot) + \langle w_i, \cdot \rangle + \sum_{j=1}^m \mu_j (g_j(\cdot) + \langle v_j, \cdot \rangle)]$ is strictly second order $B - (p, r) -$ pseudoinvex

at y with respect to b_i, α_i and ρ_i ;

(ii) $b_i(x, y) \cong 0$, for all $x \in D$;

(iii) $\sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x, y)} \cong 0$, for all $x \in D$;

Then the following can not hold :

$$f_i(x) + s(x|C_i) \cong f_i(y) + \langle w_i, y \rangle - \frac{1}{2} q^T \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle) - \frac{1}{2} q^T \nabla^2 g_j(y) q \quad ,$$

for all $i \in K$

(29)

And

$$f_l(x) + s(x|C_l) < f_l(y) + \langle w_l, y \rangle - \frac{1}{2} q^T \nabla^2 f_l(y) q + \sum_{j=1}^m \mu_j (g_j(y) + \langle v_j, y \rangle) - \frac{1}{2} q^T \nabla^2 g_j(y) q \quad ,$$

for at least one $l \in K$.

(30)

Proof: Proceeding as in the proof of the theorem 3.1, by the hypothesis (i), the inequalities (12) and (13) together with the hypothesis (ii) imply

$$\frac{1}{p} \alpha_i(x, y) [\nabla f_i(y) + w_i + \nabla^2 f_i(y) q + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y) q)]^T (e^{p\eta(x, y)} - I) + \rho_i \|\theta(x, y)\|^2 < 0 \quad (31)$$

Since $\lambda_i > 0, \sum_{i=1}^k \lambda_i = 1$, the above inequality implies

$$\begin{aligned} & \frac{1}{p} \left[\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y)q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y)q) \right]^T (e^{p\eta(x,y)} - I) \\ & + \sum_{i=1}^k \frac{\lambda_i \rho_i}{\alpha_i(x,y)} \|\theta(x,y)\|^2 < 0 \end{aligned} \quad (32)$$

In view of the hypothesis (iii), the above inequality yields

$$\frac{1}{p} \left[\sum_{i=1}^k \lambda_i (\nabla f_i(y) + w_i + \nabla^2 f_i(y)q) + \sum_{j=1}^m \mu_j (\nabla g_j(y) + v_j + \nabla^2 g_j(y)q) \right]^T (e^{p\eta(x,y)} - I) < 0 \quad (33)$$

Which is a contradiction with the feasibility of $(y, \lambda, \mu, w, v, q)$ for (MD). Hence we get the desired conclusion of theorem.

Theorem 3.5. (Strong Duality) Assume that \bar{x} is an efficient solution of (MP). Let $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q})$ is a Karush-Kuhn-Tucker point of (MP). Then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q} = 0)$ is a feasible solution of (MD) and the two objectives have the same value. If, in addition, the assumptions of weak duality (theorem 3.1, theorem 3.2) hold for all the feasible solution of (MP) and (MD), then $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q} = 0)$ is an efficient solution of (MD).

Proof: Since $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v})$ is a Karush-Kuhn-Tucker point of (MP), then the first and second constraint conditions hold for $\bar{q} = 0$. Therefore, $(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q} = 0)$ is a feasible solution of (MD) and the corresponding objectives values of (MP) and (MD) are equal. The efficiency of this feasible solution for (MD) thus follows weak duality (theorem 3.1, theorem 3.2).

Theorem 3.6. (Strict Converse Duality) Let \bar{x} and $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q})$ be the feasible solutions of (MP) and (MD) respectively. Further, we suppose that

- (i) $\sum_{i=1}^k \bar{\lambda}_i (f_i(\bar{x}) + s(\bar{x}|C_i))$
 $\cong \sum_{i=1}^k \bar{\lambda}_i (f_i(\bar{u}) + \langle \bar{w}_i, \bar{u} \rangle) - \frac{1}{2} \bar{q}^T \nabla^2 f_i(\bar{u}) \bar{q} + \sum_{j=1}^m \bar{\mu}_j (g_j(\bar{u}) + \langle \bar{v}_j, \bar{u} \rangle) - \frac{1}{2} \bar{q}^T \nabla^2 g_j(\bar{u}) \bar{q}$;
- (ii) $[\sum_{i=1}^k \bar{\lambda}_i (f_i(\cdot) + \langle \bar{w}_i, \cdot \rangle) + \sum_{j=1}^m \bar{\mu}_j (g_j(\cdot) + \langle \bar{v}_j, \cdot \rangle)]$ is strictly second order $B - (p, r) -$ invex

at \bar{u} with respect to b, α and ρ ;

(iii) $b(x, \bar{u}) > 0$, for all $x \in D$;

(iv) $\rho \cong 0$.

Then $\bar{x} = \bar{u}$, that is, \bar{u} is an efficient solution of (MP).

Proof: We assume that $\bar{x} \neq \bar{u}$ and exhibit a contradiction.

In view of the hypothesis (ii), we have

$$\begin{aligned}
 & \frac{1}{r} b(x, u) [e^{p\eta(\bar{x}, \bar{u})} - 1] \\
 & > \frac{1}{p} \alpha(x, u) \left[\sum_{i=1}^k \bar{\lambda}_i (\nabla f_i(\bar{u}) + \bar{w}_i + \nabla^2 f_i(\bar{u})q) + \sum_{j=1}^m \bar{\mu}_j (\nabla g_j(\bar{u}) + \bar{v}_j + \nabla^2 g_j(\bar{u})q) \right]^T (e^{p\eta(\bar{x}, \bar{u})} - 1) \\
 & + \rho \left\| \theta(x, u) \right\|^2
 \end{aligned} \tag{34}$$

Which in view of the feasibility of $(\bar{u}, \bar{\lambda}, \bar{\mu}, \bar{w}, \bar{v}, \bar{q})$ for (MD) and the hypothesis (iv) implies

$$\begin{aligned}
 & \frac{1}{r} b(x, u) [e^{p\eta(\bar{x}, \bar{u})} - 1] \\
 & > 0
 \end{aligned} \tag{35}$$

Which together with the hypothesis (iii), $\langle \bar{w}_i, \bar{x} \rangle \equiv s(x|C_i)$ with $i \in K, \bar{w}_i \in C_i$ and the feasibility of \bar{x} for (MP) yields

$$\begin{aligned}
 & \sum_{i=1}^k \bar{\lambda}_i (f_i(\bar{x}) + s(x|C_i)) - \sum_{i=1}^k \bar{\lambda}_i (f_i(\bar{u}) + \langle \bar{w}_i, \bar{u} \rangle) \\
 & - \frac{1}{2} q^T \nabla^2 f_i(\bar{u}) q - \sum_{j=1}^m \bar{\mu}_j (g_j(\bar{u}) + \langle \bar{v}_j, \bar{u} \rangle) - \frac{1}{2} q^T \nabla^2 g_j(\bar{u}) q > 0
 \end{aligned} \tag{36}$$

Which is a contradiction with the hypothesis (i). Hence, $\bar{x} = \bar{u}$. On the other hand, proceeding as in the proof of the theorem 3.2, we can obtain easily that \bar{u} is an efficient solution of (MP).

4. Discussion and Conclusion

Throughout this paper, we have defined new classes of generalized convexity functions calling second order $B - (p, r)$ -invex functions which extending many well-known classes of generalized convex functions. Furthermore, we have achieved and proved some dual results for Mangasarian type second order dual associated with multi objective problem under the new functions assumptions. There should be further opportunities for exploiting this structure of the multi objective programming problem.

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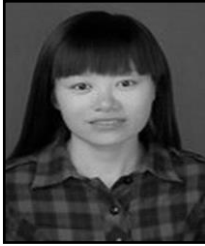
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