Multiple Reduced Hypercube(MRH): A New Interconnection Network Reducing Both Diameter and Edge of Hypercube

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Abstract

In this paper, Multiple Reduced Hypercube(MRH), which is a new interconnection network based on a hypercube interconnection network, is suggested. Also, this paper demonstrates that MRH(n) proposed in this study is superior to the previously proposed hypercube interconnection networks and the hypercube transformation interconnection networks in terms of network cost(diameter x degree). In addition, several network properties (connectivity, routing algorithm, diameter, broadcasting) of MRH(n) are analyzed.

Keywords: *interconnection network, routing algorithm, diameter, broadcasting.*

1. Introduction

An interconnection network system to link multicomputer processors greatly influences performance and scalability of the whole system. Therefore studies on an interconnection network are a base for parallel processing computer development, and the need is continuously increasing. Interconnection networks that have been proposed to date are classified based on the number of nodes into meshes($n \times k$), hypercube($2n$) and star($n!$), and network scales to evaluate interconnection networks are degree, connectivity, scalability, diameter, network cost, etc[3-9]. In an interconnection network, degree related to hardware cost and diameter related to message passing time is correlated with each other. In general, as degree of an interconnection network is increased, diameter is decreased, which can increase throughput in the interconnection network, however, it increases hardware cost with the increased number of pins of the processor when a parallel computer is designed. An interconnection network with less degree reduces hardware cost but increases message passing time, which adversely affects latency or throughput of an interconnection network. Network scales being typically used for comparative evaluation of an interconnection network due to the said characteristic include network cost[3-9] defined as degree x diameter of an interconnection network. A typical phase of an interconnection network is a hypercube interconnection network. A hypercube interconnection network is a representative interconnection network being broadly used in commercial systems in addition to existing studies by virtue of its merit of easily providing a communication network system required in applications of all kinds. Hypercube is node- and edge-symmetric, has a simple routing algorithm with maximal fault tolerance and a simple reflexive system, and also has a merit

* This work is financially supported by the Ministry of Education, Science and Technology(MEST), the Ministry Knowledge Economy(MKE) through the fostering project of HUNIC.

that it may be readily embedded with the proposed interconnection networks[10,11]. However, it involves weak points that network cost increases due to increase of degree with the increased number of nodes, and that a mean distance between diameter and node is not short as compared with degree. To improve such weak points, Reduced Hypercube[12] that reduced the number of edges of a hypercube interconnection network, Gaussian Hypercube[13], and Exchanged Hypercube[14] have been suggested, and in addition, Crossed Cube[4] that improved diameter of a hypercube interconnection network, Folded Hypercube[5], MRH[6], HFN[3], etc. have been proposed. Many interconnection networks that have been proposed until now demonstrated that they have superior network cost to hypercube by reducing just one network scale of degree or diameter of hypercube. In this paper, a Multi-Reduction Hypercube interconnection network $MRH(n)$ of a new interconnection network with superior network cost to hypercube-class interconnection networks is proposed while degree, diameter, and two network scales of hypercube are entirely reduced. And several properties of $MRH(n)$ - connectivity, routing algorithm, diameter, etc. - are analyzed. Also, this paper demonstrates that network cost of $MRH(n)$ is superior through comparative analysis of network cost between the proposed hypercube-class interconnection network and $MRH(n)$. This paper is composed as follows:

Section 2 introduces $MRH(n)$, Section 3 analyzes the several properties of $MRH(n)$, Section 4 performs comparative analysis of network cost between a hypercube-class interconnection network and $MRH(n)$, and finally, conclusion is given.

2. Preliminaries

An interconnection network can be expressed as an undirected graph, which indicates each process in nodes and a communication channel among processors in edges. An interconnection network is expressed as an undirected graph $G=(V,E)$ as mentioned below. Here, $V(G)$ is a set of nodes that is, $V(G)=\{0,1,2,....,N\text{-}1\}$, $E(G)$ is a set of edges, and a necessary and sufficient condition where an edge (v,w) is to be present as a pair(v,w) of two nodes v and w in $V(G)$ is that a communication channel exists between the node v and the node w. Interconnection networks that have been proposed to date are classified based on the number of nodes into meshes having $n\times k$ nodes, hypercube having 2^n nodes, and a star graph having $n!$ nodes. Hypercube Q_n consists of 2^n nodes and $n2^{n-1}$ edges.

The addresses of each node can be expressed in an n-bit binary number, and when the addresses of two nodes are exactly one bit different, an edge exists between them. The n-dimensional hypercube Q_n is a regular graph whose network cost is n^2 while degree and diameter are n, respectively. Hypercube has a strong point that it can easily provide a communication network system required in applications of all kinds since it is node- and edge-symmetrical and has a simple reflexive system, and is being used in Intel iPSC, nCUBE[12], Connection Machine CM-2[13], SGI Origin 2000, etc[9]. In terms of embedding, it also has a strong point that other interconnection network systems can be efficiently embedded such as tree, pyramid, mesh, etc., however, it has a weak point that a mean distance between diameter and node is not short as compared to degree. This indicates that hypercube does not efficiently use edges. New interconnection networks that improved such weak point include Multiply-Twisted-Cube, Folded Hypercube[4], and Extended Hypercube.

Folded-Hypercube FQ_n is that one edge is added to nodes where addresses of each node are in complement relation in existing hypercube, and in this interconnection network, degree increases by 1 compared to hypercube but diameter of hypercube is improved by about a half.

3. Design of Multiple Reduced Hypercube (*MRH*(*n*))

3.1 Definition of Multiple Reduced Hypercube

The nodes of a Multiple Reduced Hypercube *MRH*(*n*) are expressed as *n* bit strings $s_n s_{n-1} ... s_i ... s_2 s_1$ consisting of binary numbers {0,1} ($1 \le i \le n$). The edges of *MRH*(*n*) are expressed in three forms according to connection method, they are called hypercube edge, exchange edge, and complement edge, respectively, and are indicated as *h*-edge, *x*-edge, and *c*-edge, respectively ($\lfloor n/2 \rfloor + 1 \le h \le n$). Each edge is defined into when *n* is an even number and *n* is an odd number.

Case 1) When *n* is an even number: It is assumed that for edge definition, $s_n s_{n-1} ... s_{i+1}$ is α and a bit string $s_i ... s_2 s_1$ is β in the bit string of a node $U(=s_n s_{n-1} ... s_i ... s_2 s_1)$. Therefore the bit string of a node $U(=s_n s_{n-1} ... s_i ... s_2 s_1)$ can be simply expressed as $\alpha\beta$. Assuming that the nodes *U* and *V* are adjacent with each other, adjacent edges are as follows:

i) Hypercube edge : This edge indicates an edge linking two nodes $U(=s_n s_{n-1} .... s_j ... s_{i+1} s_i ... s_2 s_1)$ and $V(=s_n s_{n-1} .... s_{i+1} s_i ... s_2 s_1)$ of *MRH*(*n*) ($n/2 \le j \le n$).

ii) Exchange edge : This edge indicates an edge linking two nodes $U(=\alpha\beta)$ and $V(=\beta\alpha)$ of *MRH*(*n*) if $\alpha \ne \beta$ in the bit string of the nodes.

iii) Complement edge : This edge indicates an edge linking two nodes $U(=\alpha\beta)$ and $V(=\overline{\alpha\beta})$ of *MRH*(*n*) if $\alpha \ne \beta$ in the bit string of the nodes.

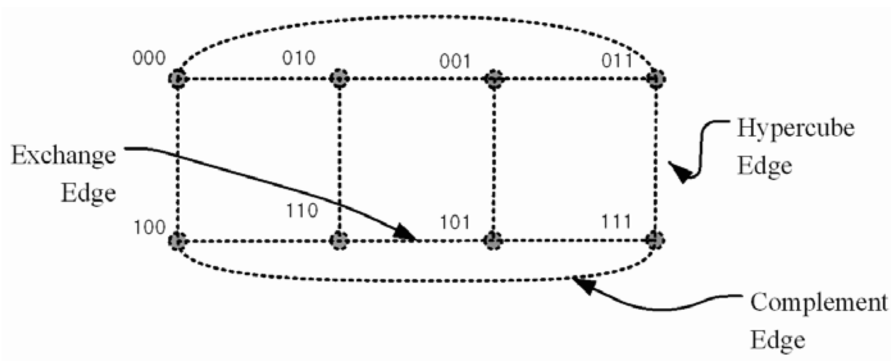


Figure 1. *MRH(3)*

Case 2) When *n* is an odd number: It is assumed that for edge definition, $s_{n-1} ... s_{i+1}$ is α' and a bit string $s_i ... s_2 s_1$ is β' in the bit string of a node $U(=s_n s_{n-1} ... s_i ... s_2 s_1)$. Then the number of bit strings of α' and β' is each $\lfloor n/2 \rfloor$. Therefore a node *U* can be indicated as $U(=s_n \alpha'\beta')$

i) Hypercube edge : This edge indicates an edge linking two nodes $U(=s_n s_{n-1} .... s_j ... s_{i+1} s_i ... s_2 s_1)$ and $V(=s_n s_{n-1} ... \overline{s_j} .... s_{i+1} s_i ... s_2 s_1)$ of *MRH*(*n*)($\lfloor n/2 \rfloor \le j \le n$).

ii) Exchange edge : This edge indicates an edge linking two nodes $U(=s_n \alpha'\beta')$ and $V(=s_n \beta'\alpha')$ of *MRH*(*n*) in the bit string of a node.

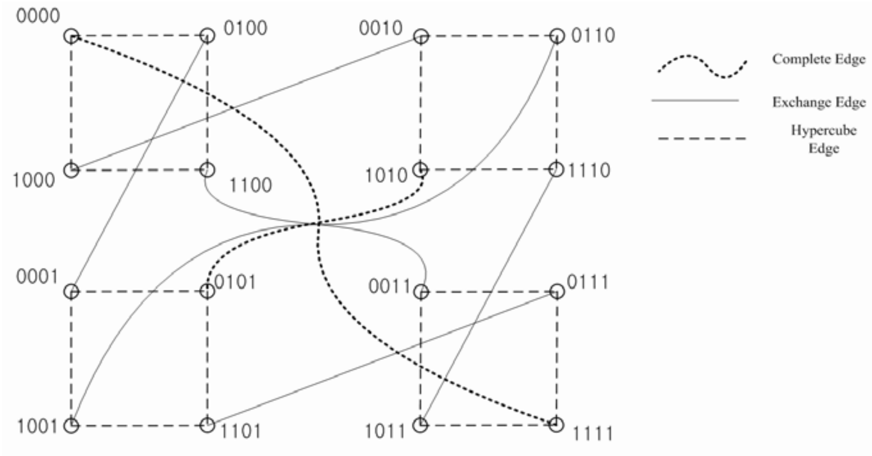
iii) Complement edge : This edge indicates an edge linking two nodes $U(=s_n\alpha'\beta')$ and $V(=s_n \alpha'\beta')$ of $MRH(n)$ if $\alpha'=\beta'$ in the bit string of a node.

By the following definition, it is found that the number of nodes is 2^n as the nodes of $MRH(n)$ are n bit strings $s_n s_{n-1}...s_i...s_2 s_1$ consisting of binary numbers {0,1}, and that $MRH(n)$ is a regular network whose degree is $\lceil n/2 \rceil+1$ since each node has $\lceil n/2 \rceil$ hypercube edges and one exchange or complement edge.

Node(edge) connectivity is the least number of nodes(edges) that are required to be eliminated to divide an interconnection network into two or more parts without common nodes. Even if k-1 or less nodes are eliminated from a given interconnection network, an interconnection network is linked, and once the interconnection network is separated when proper k nodes are eliminated, connectivity of the interconnection network is called k. An interconnection network having the same node connectivity and degree means that it has maximal fault tolerance[1]. It is known that node connectivity, edge connectivity, and degree of an interconnection network G are called $\kappa(G)$, $\lambda(G)$, and $\delta(G)$, respectively, and $\kappa(G)=\lambda(G)=\delta(G)$[1]. This paper demonstrates that node connectivity and degree of $MRH(n)$ are same in order to prove that $MRH(n)$ has maximal fault tolerance, and based on the result, $MRH(n)$ has maximal fault tolerance.

Theorem 1. *The connectivity of MRH(n) is* $\kappa(MRH(n))=\lceil n/2 \rceil+1$ $(n\geq 2)$.

Proof. Degree of each node composing $MRH(n)$ is $\lceil n/2 \rceil+1$. It is demonstrated that even if any $\lceil n/2 \rceil$ nodes are eliminated from $MRH(n)$, $MRH(n)$ is not divided. It is assumed that in $MRH(n)$, X is a set of fault nodes, and the number of elements of the set X is $\lceil n/2 \rceil$. Assuming that in $MRH(n)$, a graph eliminating a fault node set X is a connected graph, node connectivity of $MRH(n)$ is proven to be $\lceil n/2 \rceil+1$. Assuming that a node of $MRH(n)$ is S, an interconnection network eliminating a fault node set X from $MRH(n)$ is expressed as $MRH(n)-X$. It is assumed that $\lceil n/2 \rceil$ nodes are included in X among nodes linked to a node S of $MRH(n)$. Degree of the node S is $\lceil n/2 \rceil$ therefore it can be found that S is linked to one node. It is assumed that $\lceil n/2 \rceil$ or less nodes are included in X among nodes linked to the node S. Then it is found that S is linked to nodes of $\lceil n/2 \rceil+1-\lceil n/2 \rceil$ or less. Thus $\kappa(MRH(n))\geq\lceil n/2 \rceil$. In addition, $MRH(n)$ is a regular interconnection network whose degree is $\lceil n/2 \rceil+1$, therefore $\kappa(MRH(n))\leq\lceil n/2 \rceil+1$. Accordingly, $\kappa(MRH(n))=\lceil n/2 \rceil+1$. Similarly, $\lambda(MRH(n))=\lceil n/2 \rceil+1$ can be demonstrated. □



3.2 Routing Algorithm and Diameter

Assuming that in *MRH*(n), a certain node of the initial node $U(=u_n u_{n-1} \ldots u_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ is $(\alpha\beta)(\alpha=u_n u_{n-1} \ldots u_j$, $\beta=u_{i+1} u_i \ldots u_2 u_1)$ and a certain node of the destination node $V(=v_n v_{n-1} \ldots v_j \ldots v_{i+1} v_i \ldots v_2 v_1)$ is $(\gamma\delta)(\gamma=v_n v_{n-1} \ldots v_j$, $\delta=v_{i+1} v_i \ldots v_2 v_1)$, a routing algorithm is considered($\lfloor n/2 \rfloor+1 \le j \le n$). When two nodes are present in the same cluster, that is, the shortest routing when $\beta=\delta$ is determined by a hypercube algorithm because α and γ exist in the same internal module, node movement using a hypercube edge is indicated as $(\alpha\beta) \Rightarrow (\gamma\delta)$. In case of two nodes $(\alpha\beta)$ and $(\gamma\delta)$ $(\beta \neq \delta)$, the following three routing algorithms are proposed. Routing of the node$(\alpha\beta)$ in case of $(\alpha \neq \beta)$ is linked to the node $(\beta\alpha)$ by means of an exchange edge. If $\alpha=\beta$, the node$(\alpha\beta)$ is the node$(\beta\alpha)$ or the node $(\alpha\,\alpha)$, and then routing is linked to the node$(\beta\beta)$ or the node $(\alpha'\alpha')$. $\Rightarrow$ indicates an hypercube edge and $\rightarrow$ indicates routing between clusters as an exchange or complement edge.

Figure 2. MRH(4)

Case 1) The routing algorithm *A* first moves α to β by means of a hypercube edge in order to move to clusters in which the destination node is included in setting a route to move from the initial node $U(\alpha\beta)$ to the destination node $(\gamma\delta)$, because in a graph of *MRH*(n), a bit may be changed just at the α position. Movement $(\gamma\beta) \rightarrow (\beta\gamma)$ is done with an exchange edge to move from the current $(\gamma\beta)$ node to the cluster in which the destination node is located. Since the current nodes $(\beta\gamma)$ and $(\gamma\beta)$ are located in internal modules of the same cluster, movement $(\beta\gamma) \Rightarrow (\gamma\beta)$ is done with a hypercube edge to change β into γ. That is, a routing algorithm *A* to move from an initial node $U(=u_n u_{n-1} \ldots u_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ to a destination node $V(=v_n v_{n-1} \ldots v_j \ldots v_{i+1} v_i \ldots v_2 v_1)$ is as follows: After movement from $(u_n u_{n-1} \ldots u_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ to the node$(v_{i+1} v_i \ldots v_2 v_1 \ldots u_{i+1} u_i \ldots u_2 u_1)$ with a hypercube edge, the node $(v_{i+1} v_i \ldots v_2 v_1 \ldots u_{i+1} u_i \ldots u_2 u_1)$ is moved to the node$(u_{i+1} u_i \ldots u_2 u_1 \ldots v_{i+1} v_i \ldots v_2 v_1)$ with an exchange edge in order to move to clusters in which the *V* node is included. Since the current node$(u_{i+1} u_i \ldots u_2 u_1 \ldots v_{i+1} v_i \ldots v_2 v_1)$ is located in the same module as the V node, it is moved to $(v_n v_{n-1} \ldots v_j \ldots v_{i+1} v_i \ldots v_2 v_1)$ with a hypercube edge. Therefore it is found that the routing algorithm *A* is $(\alpha\beta) \Rightarrow (\gamma\beta) \rightarrow (\gamma\beta) \Rightarrow (\gamma\delta)$.

Case 2) The routing algorithm *C* is used when the number of different bits of γ and δ is least, meeting the conditions of $\beta \neq \gamma$ and $\gamma \neq \delta$. For algorithm method, an internal module of the destination node is preferentially matched in setting a route to move from the initial node *U* $(\alpha\beta)$ to the destination node $(\gamma\delta)$. To change α into γ with a hypercube edge, movement $(\alpha\beta) \Rightarrow (\gamma\beta)$ is done. Now, $(\gamma\beta)$ is moved to $(\beta\gamma)$ with an exchange edge to change β into the cluster in which the destination node is located. *B* is moved to γ with a hypercube edge that is, $(\gamma\beta) \Rightarrow (\delta\beta)$. To move to the cluster in which the destination node is located, movement $(\delta\gamma) \rightarrow (\gamma\delta)$ is done with an exchange edge, again. That is, the routing algorithm *C* moving from the initial node $U(=u_n u_{n-1} \ldots u_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ to the destination node $V(=v_n v_{n-1} \ldots v_j \ldots v_{i+1} v_i \ldots v_2 v_1)$ is as follows: After movement from the initial node $(u_n u_{n-1} \ldots u_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ to the node$(v_n v_{n-1} \ldots v_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ with a hypercube edge, the node$(v_n v_{n-1} \ldots v_j \ldots u_{i+1} u_i \ldots u_2 u_1)$ is moved to the node$(u_{i+1} u_i \ldots u_2 u_1 \ldots v_n v_{n-1} \ldots v_j)$ with an exchange edge to move to the cluster in which the *V* node is included. The current node$(u_{i+1} u_i \ldots u_2 u_1 \ldots v_n v_{n-1} \ldots v_j)$ is moved to $(v_{i+1} v_i \ldots v_2 v_1 v_n v_{n-1} \ldots v_j)$ with a hypercube edge. To move to the destination node, the node $(v_{i+1} v_i \ldots v_2 v_1 v_n v_{n-1} \ldots v_j)$ is moved to the node $(v_n v_{n-1} \ldots v_j \ldots v_{i+1} v_i \ldots v_2 v_1)$ with an exchange edge. Thus it is found that the routing algorithm *B* is $(\alpha\beta) \Rightarrow (\gamma\beta) \rightarrow (\beta\gamma) \Rightarrow (\delta\gamma) \rightarrow (\gamma\delta)$.

Case 3) When $M=\beta$ in the route of the routing algorithm C, part 1 is the node($\beta\beta$), and when $M=\beta'$, part 2 is the node($\delta\delta$). The routing algorithm C has the shortest distance if the route of ($\alpha\beta$) and ($\gamma\delta$) uses a complement edge. For example, assuming that the initial node is 000000 and the departure node is 110111, if the routing algorithm A or the routing algorithm B is used, a routing distance n is 6. However, if a complement edge is used, 000000 → 111111 ⇒ 110111, therefore a routing distance $n/2$ is 3. Like this, the routing algorithm C is used if ($\alpha\beta$) and ($\gamma\delta$) use complement edges.

To find out the shortest route using a complement edge, a Hamming distance is used as mentioned below. $H(A,B)$ means the number of bits of different binary numbers for A and B, which is called a Hamming distance. It is also called $H(Q,P) = H(\overline{Q}, \overline{P})$ according to properties of a Hamming distance. Assuming that two n bit numbers are indicated as $A = A_n \ldots A_1$ and $B = B_n \ldots B_1$ as Hamming distance, the following equation is obtained:

$$H(A,B)=\sum_{i=1}^{n} A_i \oplus B_i$$

$\oplus$ indicates exclusive-or operator. The distances of R_A , R_B and R_C following the three routing algorithms below are as follows:

i) $R_A = H(\alpha\delta) + H(\gamma\delta) +1$

ii) $R_B = H(\alpha\gamma) + H(\beta\delta) +2$

iii) $R_C = H(\alpha M)+H(\beta M)+H(\delta\delta M')+H(\gamma\gamma M') +\theta$

If $M = \beta = \delta'$, $\theta=1$, and if $M = \beta$ or $M = \delta'$ ($\beta\delta'$), $\theta=2$. Otherwise, $\theta=3$.

M is a minimized cluster used to make a routing distance R_C to be the shortest distance, and M can be found as mentioned below. The cluster M, which makes a routing distance $Q = H(\alpha M)+H(\beta M)+H(\delta M')+H(\gamma M')$ to be the shortest distance, is called a Q-minimized cluster, and if the node P consisting of n bits exists, the bit of the order i of P is indicated as P_i. According to properties of a Hamming distance, $H(Q,P) = H(\overline{Q}, \overline{P})$, therefore $H(M'\gamma)=H(M\gamma')$ and $H(M'\delta)=H(M\delta')$, and based on this, the following equation can be obtained:

$$Q=H(\alpha M)+H(\beta M)+H(\delta' M)+H(\gamma' M)= \sum_{i=1}^{n}\{(M_i \oplus \alpha_i)+(M_i \oplus \beta_i)+(M_i \oplus \gamma_i)+(M_i \oplus \delta_i)\}$$

A set of Q-minimized clusters satisfies the following conditions, and it can be obtained by searching for the bit string M_i to minimize the equation $(M_i \oplus \alpha_i)+(M_i \oplus \beta_i)+(M_i \oplus \gamma'_i)+(M_i \oplus \delta'_i)$:

· If $\alpha_i\beta_i\gamma'_i\delta'_i$ □{0111, 1011, 1101, 1110, 1111} then $M_i = 1$. · If $\alpha_i\beta_i\gamma'_i\delta'_i$ □{0000, 0001, 0010, 0100, 1000} then $M_i = 0$. · If $\alpha_i\beta_i\gamma'_i\delta'_i$ □{0011, 0101, 0110, 1001, 1010, 1100} then $M_i = X$.

Here, X means that it can have any values of the 'Don't care' terms that is, (0 or 1). For example, two nodes are ($\alpha\beta$)=(010001) and ($\gamma\delta$)=(111101) in $MRH(3)$, and three 4-bit values $\alpha_i\beta_i\gamma'_i\delta'_i$ are 0000, 0110, and 1000. M=0X0, and then X value may be 0 or 1, therefore Q-minimized clusters become {000, 010}.

Lemma 1. *Assuming that two nodes of MRH(n) are (αβ) and (γδ)(on condition of (β≠δ) and that a route from the node (αβ) to the node (γδ) is P, if the route P includes 3 or more exchange edges, the route P is not the shortest distance.*

Proof. It is assumed that the departure node is ($\alpha\beta$) and the destination node is ($\gamma\delta$). If

$\alpha=V_{-1}$, $\delta=V_0$, $\gamma=V_{x+1,}$ and $\delta=V_x$, and the route P linking two nodes includes x exchange edges($x\leq3$), the route P is composed as follows:

$P = (V_{-1}V_0) \square (V_1V_0) \rightarrow (V_0V_1) \square (V_2V_1) \rightarrow (V_1V_2) \square \ldots \rightarrow (V_{x-1}V_x) \square (V_{x+1}V_x)$.

A routing distance of the route P : $R_p = \sum_{i=1}^{x+1} H(V_iV_{i-2}) + x$.

It is divided into the following two cases according to x value.

Case 1) When x is an odd number : The route Q including one exchange edge is composed as follows:

$Q = (V_{-1}V_0) \square (V_1V_0) \square (V_3V_0) \square \ldots \square (V_xV_0) \rightarrow (V_0V_x) \square (V_2V_x) \square (V_4V_x) \square \ldots \square (V_{x+1}V_x)$.

A routing distance of the route Q : $R_q = \sum_{i=1}^{x+1} H(V_iV_{i-2})+1$. $R_Q<R_P$, therefore it is found that length of the route Q is shorter than that of the route P.

Case 2) When x is an even number : The route Q including two exchange edges is composed as follows:

$Q = (V_{-1}V_0) \square (V_1V_0) \square (V_3V_0) \square \ldots \square (V_{x+1}V_0) \rightarrow (V_0V_{x+1}) \square (V_2V_{x+1}) \square (V_4V_{x+1}) \square \ldots \square (V_xV_{x+1}) \rightarrow (V_{x+1}V_x)$.

A routing distance of the route Q : $R_q = \sum_{i=1}^{x+1} H(V_iV_{i-2})+2$. $R_Q<R_{P,}$ therefore it is found that length of the route Q is shorter than that of the route P. If the route P includes 3 or more exchange edges, the route P is found not to be the shortest distance.

Lemma 2. *If two nodes are present in the same cluster and the route P includes 2 or more complement edges, the route P is not the shortest distance.*

Proof. If two nodes are present in the same cluster and the route follows the routing algorithm A, it indicates routing in hypercube. If the route P linking two nodes includes x complement edges(x=2), the route P can be composed as follows:

$P = (\alpha\beta) ==> (V_1V_1) \rightarrow (V'_1V'_1) ==> (V_2V_2) \rightarrow (V'_2V'_2) ==> \ldots \rightarrow \ldots ==> (V_xV_x) \rightarrow (V'_xV'_x) ==> (\gamma\delta)$.

A routing distance of the route P is as follows:

$R_P = H(\alpha V_1)+H(\beta V_1)+ \sum_{i=1}^{x-1} 2H(V_{i+1}V_i) +H(\gamma V'_x)+H(\delta V'_x)+ \theta$,

If $\geq\delta$, at least, x complement edges are included. It is divided into the following two cases according to x value:

Case 1) When x is an odd number: The route Q including only one complement edge is composed as follows:

$Q = (\alpha\beta) ==> (V_1V_1) \rightarrow (V'_1V'_1) \square (V_2V'_1) \square \ldots \square (V_{2j}V'_1) \square (V'_{2j+1}V'_1) \square \ldots \square (V'_xV'_1) \square (\delta V'_1) \rightarrow (V'_1\delta) \square (V_2\delta) \square \ldots \square (V_{2j}\delta) \square (V_{2j+1}\delta) \square \ldots \square (V'_x\delta) \square (\gamma\delta)$.

A routing distance of the route Q is as follows:

$R_Q = H(\alpha V_1)+H(\beta V_1)+ \sum_{i=1}^{x-1} 2H(V_{i+1}V_i) +H(\gamma V'_x)+H(\delta V'_x)+ \theta$,

If $V_1=\beta$, θ is 1. Otherwise, it is 2.

It is found that if x=3, $R_Q<R_P$, therefore the route Q is shorter than the route P.

Case 2) When x is an even number : The route Q that does not include any complement edge is composed as follows:

$Q = (\alpha\beta) \square (V_1\beta) \square \ldots \square (V_{2j\text{-}1}\beta) \square (V'_{2j}\beta) \square \ldots \square (V'_x\beta \square (\delta\beta) \rightarrow (\beta\delta) \square (V_1\delta) \square \ldots \square (V_{2j\text{-}1}\delta) \square (V'_{2j}\delta) \square \ldots \square (V'_x\delta) \square (\gamma\delta)$.

A routing distance of the route Q is as follows:

$R_Q = H(V_1\alpha)+H(V_1\beta)+ \sum_{i=1}^{x-1} 2H(V_{i+1}V_i) +H(\delta V'_x)+H(\gamma V'_x)+1$,

$R_Q<R_P$, therefore it is found that the route Q is shorter than the route P. Thus if the route P includes two or more complement edges, the route P is not the shortest distance.

Lemma 3. In the routing algorithm *A,* the shortest route including one exchange edge exists.

Proof. An exchange edge linking the two nodes $(\alpha\beta)$ and $(\gamma\delta)$ is a certain edge to link the clusters β and δ, therefore a route including one exchange edge must pass through an edge linking two nodes, certainly. A route following the routing algorithm A is a certain route including one exchange edge. Therefore the routing algorithm A is the shortest distance among routes including one exchange edge.

Lemma 4. In the routing algorithm *B,* the shortest route including two exchange edges exists.

Proof. The route P including two exchange edges is composed as follows:

$P = (\alpha\beta) \square (V\beta) \rightarrow (\beta V) \square (\delta V) \rightarrow (V\delta) \square (\gamma\delta)$, ($V\neq\beta$ and $V\neq I$).

A routing distance of the route P is as follows:

$R_P = H(V\alpha)+H(\delta\beta)+H(\gamma V)+2$

$R_B = H(\gamma\alpha)+H(\delta\alpha)+2$ and $H(V\alpha)+H(\gamma) = H(\gamma\alpha)$, accordingly $R_P=R_B$. Therefore the routing algorithm B is the shortest distance among routes including two exchange edges. If $H(V\alpha)+H(\gamma V) = H(\gamma\alpha)$, the route P is same as the route length of the routing algorithm B, and also available as a substitute route of B.

Lemma 5. In the routing algorithm *C,* the shortest route including one complement edge exists.

Proof. Demonstration can be easily done based on definition of the routing algorithm C (case *C)* of the previous 4.

Based on the theorems 1-5, an optimal routing algorithm linking two nodes $(\alpha\beta)$ and $(\gamma\delta)$ on condition of $\neq$ in $MRH(n)$ could be found. The optimal routing algorithm is an algorithm having the shortest route among three routing algorithms A, B, and C. Distance d between two nodes $(\alpha\beta)$ and $(\gamma\delta)$ is the shortest distance among the three route lengths.

$d = min(R_A, R_B, R_C)$

For example, a distance d between two nodes U=(000111101010) and V=(011001101010) is obtained. The minimized cluster M between two nodes is 000111. The following three routing lengths are obtained.

$R_A = 3 + 3 + 1 = 7$.

$R_B = 4 + 0 + 2 = 6$.

$R_C = 3 + 1 + 2 + 2 = 8$.

R_B has the least value, therefore it is found that the routing algorithm B is a regular routing algorithm for two nodes (000111101010) and (011001101010).

Table 1. The values R_{Ai} , R_{Bi} , R_{Ci} and M_i

Group	$\alpha_i\beta_i\gamma_i\delta_i$	R_{Ai}	R_{Bi}	R_{Ci}	M_i
1	0000, 1111	0	2	0	$M_i = X$
2	0110, 1001	0	2	2	$M_i = X$
3	0101, 1010	2	2	0	$M_i = X$
4	0011, 1100	2	0	2	$M_i = \beta_i = \delta'_i$
5	0010, 1101, 1000, 0111	1	1	1	$M_i = \beta_i = \delta'_i$
6	0001, 1110	1	1	1	$M_i = \beta_i \neq \delta'_i$
7	0100, 1011	1	1	1	$M_i = \delta_i \neq \beta_i$

A distance between two nodes inside the interconnection network G indicates length of the shortest route between two nodes, and diameter of the interconnection network G means a maximal distance of the shortest route between two nodes.

Theorem 2. *Diameter of MRH(n) is* $\lceil n/2 \rceil + \lfloor (\lceil n/2 \rceil + 1)/3 \rfloor$.

Proof. It is assumed that the departure node is ($\alpha\beta$) and the destination node is ($\gamma\delta$). If β=γ , two nodes are present in one internal module, therefore a hypercube edge is used and maximal distance is $n/2$. If $\beta \neq \delta$, a distance between two nodes shall be the least value among three routing lengths R_A, R_B, and R_C. The three routing lengths can be expressed as follows:

M is a minimized cluster, and if M=β==δ ', θ =1, and if M=β or M=δ', $\theta \leq 2$. Otherwise, θ =3. R_{Ai}, R_{Bi}, and R_{Ci} are indicated as follows:

$R_{Ai} = (\alpha_i \oplus \delta_i) + (\beta_i \oplus \gamma_i)$

$R_{Bi} = (\alpha_i \oplus \gamma_i) + (\beta_i \oplus \delta_i)\ \delta$

$R_{Ci} = (M_i \oplus \alpha_i) + (M_i \oplus \beta_i) + (M_i \oplus \gamma'_i) + (M_i \oplus \delta'_i)$

Diameter D of *MRH(n)* is as follows:

$D = max\{\alpha\beta,\gamma\delta\}\{\alpha'\} = max\{\alpha\beta,\gamma\delta\}\{min(R_A, R_B, R_C)\}$.

Since it is not easy to exactly express relation among R_A, R_B, and R_C expressed as exclusive-or operator($\oplus$), relation among the three equations of R_A, R_B, and R_C is expressed with a plus(+) operator.

The values of R_{Ai}, R_{Bi}, R_{Ci} and M_i based on the values of 4-bit $\alpha_i\beta_i\gamma_i\delta_i$ ($0 \leq i \leq n$-1) are indicated in Table 1. $\alpha_i\beta_i\gamma_i\delta_i$ of ($n/2$) 4-bit values are divided into 7 groups by the values of R_{Ai}, R_{Bi}, R_{Ci} and M_i. It is assumed that is the number of 4-bit values obtained from $\alpha_i\beta_i\gamma_i\delta_i$ of the group k. For example, assuming that two nodes are ($\alpha\beta$)=(0100110110) and ($\gamma\delta$) =(1111000011), the five obtained 4-bit values are 0101, 1001, 0101, 0100, and 1010, (a_2=1, a_3= 3, a_7= 1), and =0 for other k values. When all values are not negative numbers, the sum of all a_k values is $a_1 + a_2 + \ldots + a_7 = n$. If a_6=0, $M_i = \delta'_i$ or $M_i = X$ for I, therefore $M = \delta$ ' and if a_7=0, $M_i = \beta_i$ or M_i=X for I, therefore $M = \beta$. Thus, if a_6=0 or a_7=0, $\theta \leq 2$.

The three distances can be expressed for in the following equation:

If $A = a_5 + a_6 + a_7$, a_6=0 , a_7=0 or , $\theta \leq 2$. Otherwise, =3. The sum of all a_k is $n/2$, therefore $a_1 + a_2 + a_3 + a_4 + a_5 + A = n/2$. The upper limit of a distance is as considered below for four cases in order to find out diameter.

If $a_3 \geq a_2+1$ and $a_4 \leq a_1+a_3$, the routing algorithm B is an optimal algorithm, and an equation of a distance $min(R_A, R_B, R_C)=R_B=2a_4+2a_2+A+2\leq 2a_1+4a_3+A$ when $a_3=a_2+1$ and $a_4=a_1+a_3$.

Once the equation is concluded, $n = a_1+a_2+a_3+a_4+A=2a_1+3a_3+A-1$ and $a_3\leq\lfloor(n+1)/3\rfloor+1$.

Therefore $R_C\leq 2a_1+4a_3+A = n/2+a_3+1\leq n/2+\lfloor(\lceil n+2\rceil+1)/3\rfloor+1$.

The maximum value of the distance drawn from the result is $n/2+\lfloor(\lceil n+2\rceil+1)/3\rfloor+1$. Then diameter is pertinent to dimensions of an even number therefore diameter of a $MRH(n)$ graph including dimensions of an odd number is $\lceil n/2\rceil+\lfloor(\lceil n+2\rceil+1)/3\rfloor+1$. Thus $MRH(n)$ corresponds to about one third of existing n-dimensional hypercube diameter.

4. Comparative analysis with other interconnection networks

Network cost is indicated by a multiple of diameter and degree. Diameter indicates a maximum distance of the shortest route linking two nodes, which can be an effective reference to measure message passing as a lower limit of latency required to disseminate information in the whole interconnection network, and degree is the number of pins composing the processor when a parallel computer is designed with a given interconnection network as a factor to determine the complexity of routing control logic, which is a reference to measure the cost of hardware used to implement an interconnection network. Therefore network cost is the most critical factor to measure an interconnection network. To demonstrate that $MRH(n)$ suggested in this paper based on the results of previous studies is suitable for implementation of a large-scale system for parallel processing, it is proven to be superior to the previously proposed hypercube classes of Hypercube $H(n)$, Folded Hypercube $FH(n)$, Multiply-twisted Cube, and recursive circulant class in terms of network cost as mentioned in Table 1. For analysis of network cost for an interconnection network, cases of the same number of nodes are compared.

Table 2. Hypercube variants vs Modified Hypercubes Interconnection Network Costs

	Nodes	Drgree$_i$	Diameter	Network Cost
H(n)	2^n	n	n	n^2
FH(n)	2^n	$n+1$	$\left\lceil \frac{n}{2} \right\rceil$	$\approx \frac{n^2}{2}$
MTC	2^n	n	$\left\lceil \frac{n+1}{2} \right\rceil$	$\approx \frac{n^2}{2}$
MRH(n)	2^n	0	$\left\lceil \frac{n}{2} \right\rceil + \left\lfloor (\left\lceil \frac{n}{2} \right\rceil + 1)/3 \right\rfloor + 1$	$\approx \frac{n^2}{3}$
RC	2^n	1	$\left\lceil \frac{3n}{4} \right\rceil$	$\approx \frac{3n^2}{4}$

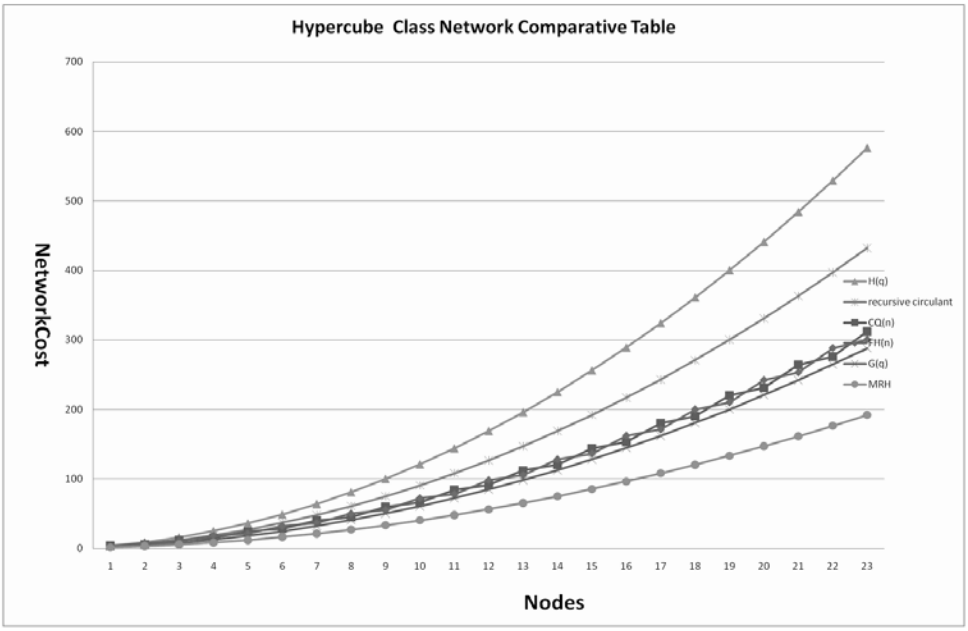


Figure 3. Hypercube Class Network Comparative Table

5. Conclusion

In this paper, an interconnection network $MRH(n)$ with superior network cost while degree, diameter, and two network scales of hypercube are all reduced is suggested. Also, $MRH(n)$ connectivity is proven to be $_k(MRH(n))= \lceil n/2 \rceil +1$, an optimal routing algorithm is suggested, and diameter is demonstrated to be $\lceil n/2 \rceil + \lfloor (\lceil n/2 \rceil +1)/3 \rfloor +1$. In addition, it is demonstrated that $MRH(n)$ is a more superior interconnection network through comparative analysis of network cost if hypercube classes have the same number of nodes. This result proves that $MRH(n)$ is a very suitable interconnection network in implementing a large-scale system for parallel processing.

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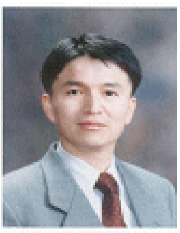

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