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Unified Error Probability Analysis for Error Correcting Codes with Different Decoding Algorithms

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Abstract

The error rate of error correcting codes with soft-decision-decoding rarely has a closed-form expression. Bounding techniques are widely used to evaluate the performance of maximum-likelihood decoding algorithm. But the existing bounds are not tight enough especially for a low signal-to-noise ratios region and become looser when a suboptimum decoding algorithm is used. The radius of decision region is applied to evaluate the word error rate (WER) of error correcting codes with different decoding algorithms. Simulation results show that this method can effectively evaluate the WER of different decoders with 0.05dB maximum error.

Keywords*: error correcting codes, radius of decision region, word error rate*

1. Introduction

The performance evaluation of error correcting codes with soft-decision-decoding has long been a problem in coding theory and practice. For the popular codes such as turbo codes [1] and Low-density parity check (LDPC) codes [2], the evaluation of word error rate (WER) is generally done by computer simulation. Another method is the bounding techniques [3, 4], which are either very loose or very complicated for the practical use. More importantly, most bounding techniques are based on the maximum-likelihood (ML) decoding, which can not reflect the difference of the commonly used suboptimal decoders such as Log-MAP/Max-Log-MAP [5] in decoding turbo codes and sum-product algorithm (SPA)/min-sum algorithm (MSA) [6] in decoding LDPC codes.

Based on [7], radius of decision region can be applied to calculate the WER of any error correcting code with any decoding algorithm at signal-to-noise ratios (SNRs) of interesting. The basic premise is that when the encoder and decoding algorithm are given, the decision region is fixed, and the WER is completely determined by the decision region. Moreover, for popular long codes such as Turbo and LDPC codes, their probability distributions of squared radii in decision regions are close to Gaussian curves, which implies that the exhausting measurement of their radii is not needed, and only the mean and variance of squared radii are enough to get an approximated probability distribution for them. By using the approximate expression of WER, we can easily obtain relatively accurate WER of error correcting codes with different decoders.

The rest of this paper is organized as follows: Section 2 gives the existing and the radius of decision region method for evaluating WER respectively. Section 3 describes some examples to prove the validity of the radius method for different coding algorithms. The conclusion is drawn in section 4.

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To clarify the notations in the paper, we denote an uppercase bold italic character as a random vector, and a lowercase bold italic one represents its realization. Correspondingly, a random variable is written as an uppercase italic character, and its realization is a lowercase italic type.

2. Error Rate Evaluation

2.1 Existing Method

Due to the lack of a relative accurate expression, bounding techniques are widely used for performance evaluation of ML decoding. The most popular upper bound is the union bound. When the weight enumerating function of a code is known, the union bound presents a tight upper bound at SNRs above the cutoff rate limit, but becomes useless at SNRs below the cutoff rate limit [3]. Based on Gallager's first bounding technique [8], some tight upper bounds are presented [4, 9]. These bounds are tighter relative to the union bound. But even Poltyrev's tangential sphere bound [10], which has been believed as the tightest bound for binary block codes, still has a gap to the real value of WER [9]. Additionally, these bounds are all based on ML decoding, which is too complex to implement in practice. When a suboptimum decoder is used, the WER will change but these bounds still keep their original values.

2.2 Using the Radius of Decision Region

Suppose binary information bits are first encoded, then the encoded bits are modulated into symbols by using a bit-to-symbol map, and transmitted over an additive white Gaussian noise (AWGN) channel. The received signal is

$$\boldsymbol{Y} = \boldsymbol{S} + \boldsymbol{Z} \tag{1}$$

where $\boldsymbol{S} = [S_1, \cdots, S_n]^{\mathrm{T}} \in \mathcal{S}$ ($\mathcal{S}$ is the signal set), $\boldsymbol{Z} = [Z_1, \cdots, Z_n]^{\mathrm{T}}$ is the additive white Gaussian noise with zero-mean and variance σ^2. In this paper, signal-to-noise ratio (SNR) is defined as $\beta = 1/\sigma^2$.

It is also assumed that the signal $\boldsymbol{S}$ is detected coherently and decoded with some algorithm at the receiver, and the channel side information is known if it is required. The decision region $V\boldsymbol{s}$ of the signal $\boldsymbol{s}$ is a set in n dimensional Euclidean space $\boldsymbol{R}_n$. Consider the decoder as a function that maps the received vector $\boldsymbol{y}$ to a transmitted signal $\boldsymbol{s} \in \boldsymbol{S}$, f_{dec} : $\boldsymbol{R}_n \to \boldsymbol{S}$, then the decision region can be defined as $V\boldsymbol{s} = \{\boldsymbol{y} \mid \boldsymbol{y} \in \boldsymbol{R}_n, f_{\mathrm{dec}}(\boldsymbol{y}) = \boldsymbol{s}\}$. Whenever the decoder is specified, the decision region is fixed. When the received vector $\boldsymbol{y}$ lies inside the $V\boldsymbol{s}$, it will be correctly decoded to $\boldsymbol{s}$, otherwise a decoding error occurs. For linear block codes, all of the codewords have the same decision region.

The radius random variable R of decision region can be measured with simulation. For the system model above, generate a white Gaussian noise vector $\boldsymbol{z} = (z_1, \ldots, z_n)$, then send $\boldsymbol{y} = \boldsymbol{s} + \lambda \boldsymbol{z}/\|\boldsymbol{z}\|$ to the decoder. There exists a $r>0$ to meet $f_{\mathrm{dec}}(\boldsymbol{s} + \lambda \boldsymbol{z}/\|\boldsymbol{z}\|) = \boldsymbol{s}$, $\forall \lambda \le r$ and $f_{\mathrm{dec}}(\boldsymbol{s} + \lambda \boldsymbol{z}/\|\boldsymbol{z}\|) \neq \boldsymbol{s}$, $\forall \lambda > r$. Consequently, r is the radius of decision region in the direction of $\boldsymbol{z}$. r^2 is the squared radius and denote its mean as $c_1 = \mathbf{E}[R^2]$, variance as $c_2 = \mathbf{Var}[R^2]$. In this paper, we will assume that the decision region is the typical decision region. That is, for any $\boldsymbol{z}$ and any $0 \le \lambda < r$, there always exists $\boldsymbol{s} + \lambda \boldsymbol{z}/\|\boldsymbol{z}\| \in V\boldsymbol{s}$, thus, the $V\boldsymbol{s}$ is named as a typical decision region. Therefore, a typical decision region is a single connected region, and a convex set is a typical decision region, conversely, may not be true. For a ML decoder, obviously, the decision region is the typical decision region, however, which can't be proved correctly in present for Turbo-like code such as Turbo and LDPC codes with iterative decoding algorithms. But in practice, due to the probability of this exception is much smaller than the WER of interest, we can omit the case that the decision region is not typical. The decoding error is basically determined by the decision region of each codeword. Next, we will discuss how to apply the radius of decision region to evaluate the WER for error control codes.

The decision region and the corresponding distribution of the radii in the decision region is completely determined whenever the decoder is specified, no matter whether it is a maximum likelihood (ML) decoder like the Viterbi decoding for Convolution codes or a suboptimal decoder such as the iterative decoders for Turbo and LDPC codes. The word error rate is the probability that the received vector $\boldsymbol{y} \notin V\boldsymbol{s}$ conditioned on a transmitted codeword $\boldsymbol{s} \in S$, that is

$$\begin{aligned} P_{e|\boldsymbol{s}} &= \Pr(\hat{\boldsymbol{s}} \neq \boldsymbol{s} \mid \boldsymbol{S} = \boldsymbol{s}) \\ &= \Pr(\boldsymbol{Y} \notin V\boldsymbol{s} \mid \boldsymbol{S} = \boldsymbol{s}) \\ &= \Pr(\boldsymbol{s} + \boldsymbol{Z} \notin V\boldsymbol{s}) \end{aligned} \tag{2}$$

where $\hat{\boldsymbol{s}}$ is the decoded signal. The random event "$\boldsymbol{s} + \boldsymbol{Z} \notin V\boldsymbol{s}$" is equivalent to the another random event"the length of noise $\|\boldsymbol{Z}\|$ is greater than the radius of decision region R", therefore, Equation (2) can also be written as [7]

$$\begin{aligned} P_{e|\boldsymbol{s}} &= \Pr(\|\boldsymbol{Z}\| > R) \\ &= \Pr(\|\boldsymbol{Z}\|^2 > R^2) \\ &= \mathsf{E}\left[1 - \frac{1}{\Gamma(n/2)}\gamma\left(\frac{n}{2}, \frac{1}{2}\left(\frac{r}{\sigma}\right)^2\right)\right] \end{aligned} \tag{3}$$

where $\|\boldsymbol{z}\| = \sqrt{\sum_{i=1}^{n} z_i^2}$, $\Gamma(\cdot)$ and $\gamma(\cdot)$ are the Gamma function and the incomplete Gamma function [11], respectively. We know that an average error rate is the average of all error rates which come from different signals, therefore, an average error rate is obtained by taking expectation over all possible sending signals, that is

$$P_e = \mathsf{E}\left[P_{e|\boldsymbol{s}}\right] \tag{4}$$

For a linear block code in an AWGN channel, linearity of the code guarantees that the distances from one code to all other codewords are independent of the choice of this code, so all the codewords have the same error rates [12]. Therefore, an average error rate is equal to the conditioned error rate, i.e. $P_e = P_{e|\boldsymbol{s}}$. Generally, we think this also can be applies to Turbo-like codes. In the following, for simplicity, we use P_e instead of $P_{e|\boldsymbol{s}}$.

Equation (3) involves the integration of incomplete gamma function for every sample of radius of decision region. Moreover, accurate measurement of radii requires a large number of decoding tests. Thus, it is inconvenient for practical use, and an approximate formula will be welcome. According to the central limit theorem, as n is large enough, $\|\boldsymbol{z}\|^2 = \sum_{i=1}^{n} z_i^2$ can be approximated to a Gaussian distribution.

Moreover, for Turbo-like codes, the distribution of squared radius R^2 in the decision region can also be approximated to a Gaussian distribution [7]. Equation (3) can also be written as

$$
\begin{aligned}
P_e &= \Pr\left(\|\boldsymbol{Z}\| > R\right) = \Pr\left(\|\boldsymbol{Z}\|^2 > R^2\right) \\
&= \Pr\left(\|\boldsymbol{Z}\|^2 - R^2 > 0\right)
\end{aligned}
\tag{5}
$$

Since $\|\boldsymbol{Z}\|^2$ and R^2 are approximated to Gaussian distributions, $\|\boldsymbol{Z}\|^2 - R^2$ can also be viewed as a Gaussian random variable with mean $c_1 - n/\beta$ and variance $c_2 + 2n/\beta^2$. So Equation (3) becomes [13]

$$
\begin{aligned}
P_e &= \Pr\left(\|\boldsymbol{Z}\|^2 - R^2 > 0\right) \\
&\approx \mathrm{Q}\left(\frac{c_1 - n/\beta}{\sqrt{c_2 + 2n/\beta^2}}\right)
\end{aligned}
\tag{6}
$$

While n is larger, the signal space is bigger. So the squared radius R^2 is increased, correspondingly, its mean and variance are larger.

3. Simulation Results

Next we consider two error correcting codes commonly used in wireless communications, including Turbo codes and LDPC codes. In Figure 1 and Figure 2, we adopt a Turbo code. The code length of the Turbo code is 1152, the code rate is 1/2, the generator is [13, 15, 17]. Accordingly, for the LDPC code, the parameters are 1152 code length with 3/4 code rate and 576 code length with 1/2 code rate in Figures 3 and 4. Figure 1 is the comparison of WER between the Monte Carlo simulation and the evaluated by Equation (6). In this figure, the Turbo code adopting Log-Map decoding algorithms and Max-Log-Map algorithm with a random interleave, respectively. Figure 2 describes the WER for the Turbo code of Log-Map algorithm with CDMA2000 interleave and a random interleave. From these two figures, we see the maximum error of WER between the radius and Monte Carlo method is about 0.03dB.

In Figure 3, the LDPC code adopts SPA and MSA with 25 maximum iteration [14], respectively. Figure 4 describes the WER of LDPC adopting SPA algorithm with 25 maximum iteration and 50 maximum iteration, respectively. It can be seen from the figures that the radius method fits very well with Monte Carlo simulation, while the maximum error between the simulation and the evaluated is less than 0.05dB. So by using the radius of decision region, we can relatively accurately evaluate the WER of error correcting codes with different decoding algorithms.

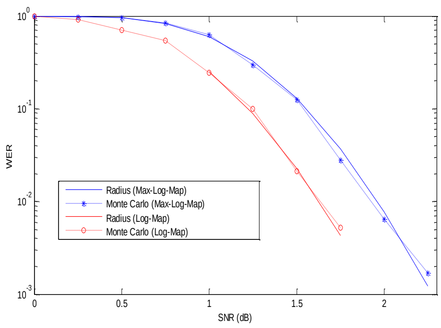

Figure 1. The WER of Turbo Codes with Log-Map Algorithm and Max-Log-Map Algorithm, Respectively

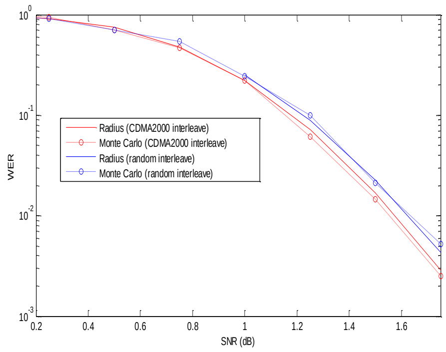

Figure 2. The WER of Turbo Codes Adopting Log-Map Decoding Algorithms with the CDMA2000 Interleave and a Random Interleave, Respectively

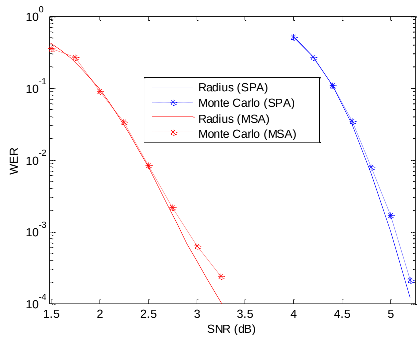

Figure 3. The WER of LDPC Codes with SPA Decoding Algorithms and MSA Decoding Algorithm, Respectively

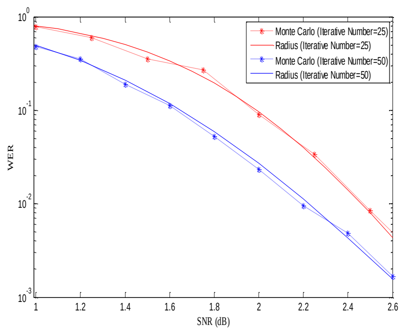

Figure 4. The WER of Turbo Codes Adopting SPA Algorithms with 25 Maximum Iteration and 50 Maximum Iteration, Respectively

4. Conclusion

Since the code word error rate is completely determined by the decision region, the radius of decision region introduces an effective method to evaluate the performance of error correcting codes with different decoding algorithms. Simulation results show that the maximum WER between Monte Carlo and the radius method is about 0.05dB. Despite that the method is demonstrated with only several codes with different decoders in this paper, it is straightforward to generalize this method to any situations where the error rate is characterized by the decision region, such as MIMO detection, equalization, etc. In these situations, the decision region may not have the same shape for different transmitted signals. Nevertheless, the average error rate can still be evaluated by the average radius of decision region in a similar way.

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