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Nonlinear Integrable Couplings of the Kaup-Newell Hierarchy

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Abstract

In this paper, a six-dimensional Lie algebra is first introduced, whose corresponding loop algebra is constructed, for which an isospectral problem is established. By zero curvature equations, we obtain the nonlinear integrable couplings of the Kaup-Newell (KN) hierarchy.

***Keywords**: power line communication,Lie algebra, soliton equation, zero curvature equations,KN hierarchy,nonlinear integrable couplings*

1. Introduction

In 1989, Tu [1-2] presented a simple method to obtain the integrable soliton hierarchy and its Hamiltonian structures. This approach is called Tu's scheme and it has been applied to find a lot of interesting Hamiltonian integrable systems of soliton equations [3-17]. In References [18, 19], we further found that some soliton hierarchies also can be obtained by the vector-form Lie algebra and their Hamiltonian structures can be constructed by the quadratic-form identity. In 2006, the trace identity was been generalized to zero curvature equations associated with non-semi-simple Lie algebra. The study of integrable couplings has attracted much attention [20-31]. It originated from the investigations into the symmetry problems and associated centerless Virasoro algebras. Integrable couplings have much richer mathematical structure than scalar integrable equations. The related theory generalizes the symmetry and helps us work towards completely classifying integrable equations from an algebraic point of view. A few ways to construct integrable couplings of soliton equations are presented by perturbation, enlarging spectral problems, constructing new loop Lie algebra and creating semidirect sums of Lie algebra [22-26]. Recently, Ma and Zhu [20] presented a scheme for constructing nonlinear continuous and discrete integrable couplings using the block type matrix algebra.

The notion on integrable couplings was proposed when Ma and Fuchsstiener studied of Virasoro symmetric algebras. In detail, for a given integrable system of evolution type

$$u_t = K(u) \tag{1}$$

the following bigger integrable system:

$$\begin{cases} u_t = K(u), \\ v_t = S(u,v). \end{cases} \tag{2}$$

is called an integrable couplings of the system (1). Especially, if the second equation in

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the system (2) is nonlinear for v, then the system (2) is called a nonlinear integrable couplings of the system (1). Constructing nonlinear integrable couplings is one of the pretty interesting topics in the soliton theory. There are much richer mathematical structures behind integrable couplings than scalar integrable equations. Moreover, the study of integrable couplings generalizes the symmetry problem and provides clues toward complete classification of integrable equations.

In thibns paper, we hope to construct nonlinear integrable couplings of soliton equations through vector-form Lie algebras. Specifically, we would like to construct vector-form Lie algebra G to obtain nonlinear integrable couplings of the *Kaup-Newell (*KN) hierarchy.

2. The KN Hierarchy

We present a brief description of the zero-curvatrue representation for the Kaup-Newell hierarchy associated with the following eigenvalue problem.

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U(u,\lambda)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3}$$

$$U(u,\lambda)=\begin{pmatrix} -\lambda^2 & \lambda q \\ \lambda r & \lambda^2 \end{pmatrix} \tag{4}$$

Where $u=(u_1,u_2)^T=(q,r)^T$. First, we solve the adjoint representation of (4).

$$V_x=[U,V]\equiv UV-VU, \tag{5}$$

with

$$V=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}=\sum_{m=0}^{\infty} V_m\lambda^{-m}, \quad V_m(u)=\begin{pmatrix} a_m(u) & b_m(u) \\ c_m(u) & -a_m(u) \end{pmatrix}. \tag{6}$$

Equations (5) and (6) lead to

$$b_{m+2}=-qa_{m+1}-\frac{1}{2}b_{m,x},$$

$$c_{m+2}=-ra_{m+1}+\frac{1}{2}c_{m,x},$$

$$a_{m,x}=qc_{m+1}-rb_{m+1}=\frac{1}{2}(qc_{m-1,x}+rb_{m-1,x}), \tag{7}$$

$$a_{2m+1}=b_{2m}=c_{2m}=0, m=0,1,\cdots,$$

$$a_0 = 1, a_2 = -\frac{1}{2}qr,$$

$$b_1 = -q, c_1 = -r, b_3 = \frac{1}{2}(q^2 r + q_x)$$

$$c_3 = \frac{1}{2}(qr^2 - r_x),$$

$$a_4 = \frac{3}{8}q^2 r^2 + \frac{1}{4}(rq_x - qr_x),$$

which gives rise to

$$\begin{pmatrix} c_{2m+1} \\ b_{2m+1} \end{pmatrix} = L \begin{pmatrix} c_{2m-1} \\ b_{2m-1} \end{pmatrix}, \tag{8}$$

$$L = \frac{1}{2}\begin{pmatrix} D - rD^{-1}qD & -rD^{-1}rD \\ -qD^{-1}qD & -D - qD^{-1}rD \end{pmatrix} \tag{9}$$

with

$$D = \frac{\partial}{\partial x}, D^{-1}D = DD^{-1} = 1.$$

Set

$$V^{(n)}(u,\lambda) = \sum_{i=0}^{n-1} \begin{pmatrix} a_{2i}\lambda^{2n-2i} & b_{2i+1}\lambda^{2n-2i-1} \\ c_{2i+1}\lambda^{2n-2i-1} & -a_{2i}\lambda^{2n-2i} \end{pmatrix} \tag{10}$$

and

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{t_n} = V^{(n)}(u,\lambda)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{11}$$

Then the compatibility condition of (3) and (11) gives rise to a zero-curvature representation for the KN hierarchy,

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, n = 1, 2, \cdots \tag{12}$$

We have

$$V_x^{(n)} - [U, V^{(n)}] = \begin{pmatrix} 0 & \lambda b_{2n-1,x} \\ \lambda c_{2n-1,x} & 0 \end{pmatrix}, n = 1, 2, \cdots \tag{13}$$

So (12) and (13) yield the KN hierarchy,

$$u_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \begin{pmatrix} c_{2n-1} \\ b_{2n-1} \end{pmatrix} = -JL^{n-1} \begin{pmatrix} r \\ q \end{pmatrix}, \tag{14}$$

where

$$J = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}.$$

that (14) can be cast in the Hamiltonian form

$$\begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = J \begin{pmatrix} c_{2n-1} \\ b_{2n-1} \end{pmatrix} = J \frac{\delta H_{2n-2}}{\delta u}, \tag{15}$$

where

$$\begin{pmatrix} c_{2m+1} \\ b_{2m+1} \end{pmatrix} = \frac{\delta H_{2m}}{\delta u} = L \frac{\delta H_{2m-2}}{\delta u} = L \begin{pmatrix} c_{2m-1} \\ b_{2m-1} \end{pmatrix} \tag{16}$$

$$H_{2m} = \frac{1}{2m}(4a_{2m+2} - rb_{2m+1} - qc_{2m+1}), \quad H_0 = -qr, H_{2m+1} = 0. \tag{17}$$

Here $\frac{\delta}{\delta u} = (\frac{\delta}{\delta q}, \frac{\delta}{\delta r})^T$ stands for the variational derivative

$$\frac{\delta}{\delta u_i} = \sum_{k \geq 0} (-\partial)^k \frac{\partial}{\partial u_i^{(k)}}, \quad u_i^{(k)} = \frac{\partial^k u_i}{\partial x^k}, \partial = \frac{\partial}{\partial x}. \tag{18}$$

3. A Vector-form Lie Algebra

Let us consider a vector space,

$$R^6 = \left\{ a = (a_1, a_2, a_3, a_4, a_5, a_6)^T, a_i \in R \right\}.$$

For $a, b \in R^6$, define an operation as follows:

$$[a,b]=\begin{pmatrix} a_2b_3-a_3b_2 \\ 2(a_1b_2-a_2b_1) \\ 2(a_3b_1-a_1b_3) \\ 2(a_1b_4-a_4b_1+a_6b_2-a_2b_6+a_6b_4-a_4b_6) \\ 2(a_3b_6-a_6b_3+a_5b_1-a_1b_5+a_5b_6-a_6b_5) \\ a_2b_5-a_5b_2+a_4b_3-a_3b_4+a_4b_5-a_5b_4 \end{pmatrix} \quad (19)$$

It is easy to verify that R^6 is a Lie algebra if equipped with (19).

Denoted by

$$G=span\{e_1,e_2,e_3,e_4,e_5,e_6\} \quad (20)$$

where

$$e_i=(e_{i1},e_{i2},e_{i3},e_{i4},e_{i5},e_{i6})^T,$$

$$e_{ij}=\begin{cases}1, i=j, \\ 0, i\neq j,\end{cases} \qquad i=1,2,3,4,5,6.$$

Define operation relations among $e_i(i = 1, 2, 3, 4, 5, 6)$ as follows:

$$\begin{aligned} &[e_1,e_2]=2e_2, \quad [e_1,e_3]=-2e_3, \quad [e_2,e_3]=e_1, \\ &[e_1,e_4]=2e_4, \quad [e_3,e_4]=-e_6, \quad [e_1,e_5]=-2e_5, \\ &[e_4,e_5]=e_6, \quad [e_2,e_5]=e_6, \quad [e_2,e_6]=-2e_4, \\ &[e_3,e_6]=2e_5, \quad [e_4,e_6]=-2e_4, \quad [e_5,e_6]=2e_6. \end{aligned} \quad (21)$$

It is easy to see that the Lie algebra G is isomorphic to the Lie algebra R^6.

Set

$$G_1=span\ \{e_1,e_2,e_3\},$$

$$G_2=span\ \{e_4,e_5,e_6\},$$

then we find that

$$G=G_1\oplus G_2,[G_1,G_2]\subset G_2,$$

which satisfy the sufficient condition on generating integrable couplings.

Define a loop algebra corresponding to the Lie algebra G, denoted by

$$\tilde{G}=span\{e_1(n),e_2(n),e_3(n),e_4(n),e_5(n),e_6(n)\} \quad (22)$$

where

$$e_1(n)=e_1\lambda^{2n}, e_2(n)=e_2\lambda^{2n+1}, e_3(n)=e_3\lambda^{2n+1},$$

$$e_4(n)=e_4\lambda^{2n+1}, e_5(n)=e_5\lambda^{2n+1}, e_6(n)=e_6\lambda^{2n}, n\in Z.$$

The corresponding commutative relations are given as

$$\begin{aligned}
&[e_1(n),e_2(n)]=2e_2(m+n), && [e_1(n),e_3(n)]=-2e_3(m+n),\\
&[e_2(n),e_3(n)]=e_1(m+n+1), && [e_1(n),e_4(n)]=2e_4(m+n),\\
&[e_3(n),e_4(n)]=-e_6(m+n+1), && [e_1(n),e_5(n)]=-2e_5(m+n),\\
&[e_4(n),e_5(n)]=e_6(m+n+1), && [e_2(n),e_5(n)]=e_6(m+n+1),\\
&[e_2(n),e_6(n)]=-2e_4(m+n), && [e_3(n),e_6(n)]=2e_5(m+n),\\
&[e_4(n),e_6(n)]=-2e_4(m+n), && [e_5(n),e_6(n)]=2e_6(m+n)
\end{aligned} \tag{23}$$

4. Nonlinear Integrable Couplings of the KN Hierarchy

By means of loop algebra $\tilde{G}$, let us consider the following isospectral problem

$$\varphi_x=[U,\varphi] \tag{24}$$

$$U=e_1(1)+qe_2(0)+re_3(0)+pe_4(0)+se_5(0)$$

Set

$$\varphi_t=[V,\varphi] \tag{25}$$

$$V=\sum_{m\geq 0}(A_me_1(-m)+B_me_2(-m)+C_me_3(-m)+D_me_4(-m)+E_me_5(-m)+F_me_6(-m))$$

then the stationary zero curvature equation

$$V_x=[U,V] \tag{26}$$

gives

$$\begin{aligned}
A_{mx} &= qC_{m+1} - rB_{m+1},\\
B_{mx} &= 2B_{m+1} - 2qA_m,\\
C_{mx} &= -2C_{m+1} + 2rA_m,\\
D_{mx} &= 2D_{m+1} - 2qF_m - 2pA_m - 2pF_m,\\
E_{mx} &= -2E_{m+1} + 2rF_m + 2sA_m + 2sF_m,\\
F_{mx} &= (q+p)E_{m+1} - (r+s)D_{m+1} + pC_{m+1} - sB_{m+A}.
\end{aligned} \tag{27}$$

Taking

$$A_0 = \beta_1, B_0 = C_0 = D_0 = E_0, F_0 = \beta_2,$$

then we can obtain

$$A_1 = -\frac{1}{2}\beta_1 qr, B_1 = \beta_1 q, C_1 = \beta_1 r,$$

$$D_1 = \beta_2 q + (\beta_1 + \beta_2)p, E_1 = \beta_2 r + (\beta_1 + \beta_2)s$$

$$F_1 = -\frac{1}{2}\beta_2 qr - \frac{1}{2}(\beta_2 + \beta_1)(rp + ps + qs), \cdots\cdots$$

Noting that

$$V_+^{(n)} = \sum_{m=0}^{n}(A_m e_1(n-m) + \lambda B_m e_2(n-m) + \lambda C_m e_3(n-m)$$

$$+\lambda D_m e_4(n-m) + \lambda E_m e_5(n-m) + F_m e_6(n-m))$$

$$V_-^{(n)} = \lambda^{2n}V - V_+^{(n)},$$

a direct calculation gives

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -A_{nx}e_1(0) - 2B_{n+1}e_2(0) + 2C_{n+1}e_3(0)$$

$$-2D_{n+1}e_4(0) + 2E_{n+1}e_5(0) - F_{nx}e_6(0)$$

Taking

$$V^{(n)} = V_+^{(n)} - A_n e_1(0) - F_n e_6(0),$$

we find that

$$-V_x^{(n)} + [U, V^{(n)}] = (2qA_n - 2B_{n+1})e_2(0) + (2C_{n+1} - 2rA_n)e_3$$

$$+\left(2pA_n + 2(q+p)F_n - 2D_{n+1}\right)e_4(0) + \left(2E_{n+1} - 2sA_n - 2(r+s)F_n\right)e_5(0).$$

Therefore, the zero curvature equation

$$U_t - V^{(n)} + [U, V^{(n)}] = 0 \tag{28}$$

admits the Lax integrable system

$$u_t = \begin{pmatrix} q \\ r \\ p \\ p \end{pmatrix}_t = \begin{pmatrix} 2B_{n+1} - 2qA_n \\ -2C_{n+1} + 2rA_n \\ 2D_{n+1} - 2pA_n - 2(q+p)F_n \\ -2E_{n+1} + 2sA_n + 2(r+s)F_n \end{pmatrix} = \begin{pmatrix} B_{mx} \\ C_{mx} \\ D_{mx} \\ E_{mx} \end{pmatrix} = J \begin{pmatrix} 2C_n + E_n \\ 2B_n + D_n \\ C_n + E_n \\ B_n + D_n \end{pmatrix}, \tag{29}$$

where

$$J = \begin{pmatrix} 0 & \partial & 0 & -\partial \\ \partial & 0 & -\partial & 0 \\ 0 & -\partial & 0 & 2\partial \\ -\partial & 0 & 2\partial & 0 \end{pmatrix}$$

is a Hamiltonian operation.

From Equation (27), we obtain the recurrence operator L which satisfied that

$$\begin{pmatrix} 2C_{n+1} + E_{n+1} \\ 2B_{n+1} + D_{n+1} \\ C_{n+1} + E_{n+1} \\ B_{n+1} + D_{n+1} \end{pmatrix} = L \begin{pmatrix} 2C_n + E_n \\ 2B_n + D_n \\ C_n + E_n \\ B_n + D_n \end{pmatrix}, \tag{30}$$

where

$$L = (L_1, L_2),$$

$$L_1 = \begin{pmatrix} -\frac{1}{2}\partial - \frac{1}{2}r\partial^{-1}q\partial & \frac{1}{2}r\partial^{-1}r\partial \\ -\frac{1}{2}q\partial^{-1}q\partial & \frac{1}{2}\partial - \frac{1}{2}q\partial^{-1}r\partial \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$L_2=\begin{pmatrix} -\frac{1}{2}r\partial^{-1}p\partial-\frac{1}{2}s\partial^{-1}q\partial-\frac{1}{2}s\partial^{-1}p\partial & -\frac{1}{2}s\partial^{-1}r\partial-\frac{1}{2}r\partial^{-1}s\partial-\frac{1}{2}s\partial^{-1}s\partial \\ -\frac{1}{2}p\partial^{-1}q\partial-\frac{1}{2}q\partial^{-1}p\partial-\frac{1}{2}p\partial^{-1}p\partial & -\frac{1}{2}p\partial^{-1}r\partial-\frac{1}{2}q\partial^{-1}s\partial-\frac{1}{2}p\partial^{-1}s\partial \\ -\frac{1}{2}\partial-\frac{1}{2}(r+s)\partial^{-1}(q+p)\partial & -\frac{1}{2}(r+s)\partial^{-1}(r+s)\partial \\ -\frac{1}{2}(q+p)\partial^{-1}(q+p)\partial & -\frac{1}{2}\partial-\frac{1}{2}(q+p)\partial^{-1}(r+s)\partial \end{pmatrix}$$

As a result, the system (29) can be written as

$$u_t=\begin{pmatrix} q \\ r \\ p \\ p \end{pmatrix}_t = JL^{n-1}\begin{pmatrix} (2\beta_1+\beta_2)r+(\beta_1+\beta_2)s \\ (2\beta_1+\beta_2)q+(\beta_1+\beta_2)q \\ (\beta_1+\beta_2)(r+s) \\ (\beta_1+\beta_2)(q+p) \end{pmatrix} \tag{31}$$

When taking p=s=0, (31) reduces to the KN hierarchy. Therefore, the system (31) is the integrable couplings of the KN hierarchy according to the definition of integrable couplings. It is a nonlinear integrable couplings because the commutators [e_4, e_5], [e_4, e_6] and [e_5, e_6] can generate nonlinear terms.

5. Conclusion

In this paper, we introduced a kind of Lie algebra which allows us to construct nonlinear integrable couplings of the KN hierarchy. The loop algebra presented in this paper can be used to other known integrable hierarchies of soliton equations for generating the nonlinear integrable couplings.

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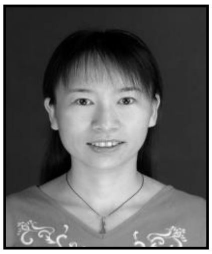

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