International Journal of Future Generation Communication and Networking
Vol. 6, No. 4, August, 2013

Min-Sum Algorithm with Maximum Average Mutual Information Quantization

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Abstract

The specific methods of the maximum average mutual information quantization are analyzed and deduced. With this method to quantify the initial message, the variable messages and check messages are converted into integers. Then the min-sum decoding algorithm based on integer arithmetic is implemented. Simulation results demonstrate that the decoding performance of min-sum algorithm based on integer with 7-bit maximum average mutual information quantization is almost the same as that of min-sum algorithm based on integer with 8-bit uniform quantization. Meanwhile, all the variables in the algorithm are fixed-length integers, so it is convenient for hardware implementation. Although the decoding performance is slightly worse than that of the sum-product decoding algorithm, the decoding time is greatly reduced. So it is very convenient for future generation communication.

Keywords: *Low Density Parity Check Codes (LDPC); maximum average mutual information quantization; uniform quantization; min-sum algorithm; integer operation*

1. Introduction

LDPC codes enjoy the good properties of approaching the Shannon limit, and its practical application is of great significance. In [1] Sae-Young Chung designed the code whose performance comes within 0.0045dB of Shannon capacity limit under the binary input Gaussian channel. It's common to use high-precision floating-point operations on theoretical study of the LDPC codes, but it is difficult to implement high-precision floating point numbers operation in practical application, especially in FPGA. Therefore, there is a great need to design effective quantitative method which enables its implementation in hardware [2].

At present, more and more people research LDPC at home and abroad. Reference [3] also quantifies the initial message under 3 dB SNR and 1/2 code rate. Then a good performance is achieved, and the resulting quantization tables are applicable to different SNR and any rate of LDPC decoding. Reference [4, 5] quantifies not only the received signal in the sum product decoding algorithm, but also intermediate variables, and the maximum number of iterations is set to 200. Reference [6] studied the complexity of quantization of min-sum decoding algorithm and sum-product decoding under three different measurements and studied their respective quantization decoding algorithm accuracy in detail. It is concluded that min-sum decoding algorithm is suitable for integer quantization. After quantization all messages of the min-sum decoding algorithm have become integral, so it is run faster. Based on the detailed studies of hyperbolic curve, reference [7] designs a correction factor to make the decoding performance of the rule codes and irregular codes improved significantly under different SNR. The maximum average mutual information quantization is proposed in [8]. But it is too complex and the details of its solution are not provided. In addition, the initial message, variable messages and check messages are separately quantified. So the average of 15 times iterations in each decoding totally requires 2+4*15=62 times look-up.

With a method of maximum average mutual information quantization to quantify the received signal, this paper designs a quantized min-sum decoding algorithm whose variable is integer actually represented in 7-bit. When the maximum times of iterations is 20, its performance approaches that of high-precision floating-point sum product decoding algorithm. And this quantitative method only requires once table look-up. So it is convenient for hardware implementation, and the decoding speed is faster.

The rest of this paper is arranged as follows: Section 2 describes the principle of min-sum algorithm for LDPC codes; Section 3 analyzes the principle of maximum average mutual information quantization; Section 4 describes how to implemented in LDPC decoding using maximum average mutual information quantization; Section 5 simulates and analyzes sum-product decoding algorithm of the LLR measure and min-sum algorithm based on integer operation; Section 6 draws some conclusions.

2. The Principle of Min-Sum Algorithm for LDPC Codes

Min-sum algorithm is implemented on the basis of sum-product decoding algorithm under the LLR measure [9]. The formulas [10] of sum-product decoding algorithm under the LLR measure are

$$u_j = 2y_j / \sigma_n^2 \tag{1}$$

$$v_{ij} = u_j + \sum_{k \in M(j)\backslash i} u_{kj} \tag{2}$$

$$\tanh\left(u_{ij} / 2\right) = \prod_{k \in N(i)\backslash j} \tanh\left(v_{ik} / 2\right) \tag{3}$$

Where y_i is channel output messages; σ_n^2 is the variance of Gaussian noise; u_j is the initial message; v_{ij} indicates the variable message; u_{ij} is the check message and its initial value is 0; *i, j* are the serial number of row and column in check matrix H.

Differencing from sum-product decoding algorithm under the LLR measure, Min-sum algorithm modifies formula (3). To illustrate the modification, we first introduce the feature of the hyperbolic tangent function that is given by

$$\tanh\left(x\right) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \tag{4}$$

Figure 1 is the hyperbolic tangent curve. We can see that symbols of the arguments and the dependent variable keep consistent.

According to min-sum algorithm, we multiply the variable message whose absolute value of confidence is lowest by signs of each variable message in the check equation. This result is the new check information, so the equation (3) can be transformed into

$$u_{ij} = \min\left(\left|v_{ik}\right|_{k \in N(i)\backslash j}\right) \prod_{k \in N(i)\backslash j} \mathrm{sgn}(v_{ik}) \tag{5}$$

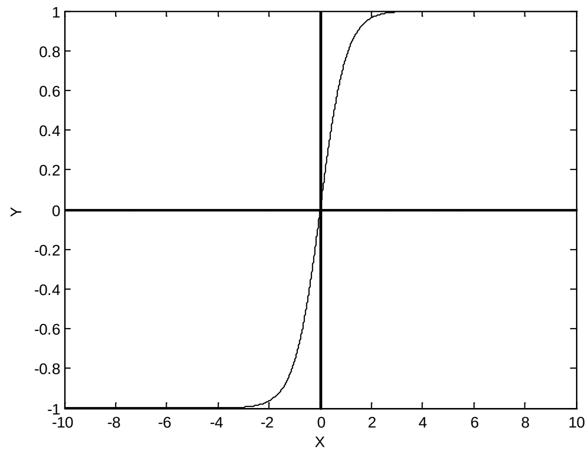

Figure 1. Hyperbolic Tangent Curve

Thus equations (1), (2), (5) constitute min-sum algorithm iteration formulas where equation (1) is the initial message; equation (2) is the variable message; and equation (5) is the check message. The initial value of u_{ij} is 0, the decision rule is $\hat{x}_j = \left[v_j \le 0 \right]$,

$$v_j = u_j + \sum_{k \in M(j)} u_{kj} \tag{6}$$

We have the variable message corrected by the correction factor K, then the formula (2) is updated to [7]

$$v_{ij} = K(u_j + \sum_{k \in M(j) \backslash i} u_{kj}) \tag{7}$$

Reference [7] pointed out that a straight line with a slope of 0.8 fits hyperbolic curve better. Here, the correction factor K is set to 0.8 and it is used to correct check messages u_{ij}. The min-sum decoding algorithm can be divided into three steps:

1) Initialization: send initial message to v_{ij}, and initialize u_{ij} to be 0;
2) Update variable messages and check messages referring to (7) and (5);
3) Try to decode referring to the value of equation (6).

Supposing that the decoding codeword sequence is $\hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_N)$ and the LDPC code parity check matrix is $H_{M \times N}$, if $H_{M \times N}\hat{x} = 0$, the decoding ends, otherwise return to step 2 to execute a new round of calculation.

3. Maximum Average Mutual Information Quantization

In general, for a given quantization length, uniform quantization performance is worse than that of the non-uniform quantization [11]. That is to say, to achieve a certain performance, non-uniform quantization needed less quantization bits. In order to improve computing speed in FPGA, it requires that the quantization bits be as little as possible [12], so we can use non-uniform quantization. maximum average mutual information quantization is a non-uniform quantization. The average mutual information represents the average information provided by the received signal. In order to get optimized error correction performance after quantization without known the statistical distribution of the source, average mutual information between the received signal and the quantized symbols must be maximum. The maximum average mutual information quantization maximizes the average mutual information between the quantized symbols and the received signal. So the quantized symbols contain greatest information of the received signal. Consequently, this quantitative approach provides as much source information as possible for LDPC decoding.

For a $N = 2^n$ layer quantization, assuming quantization interval is $X_1 < X_2 < \cdots < X_{N+1}$, let the representative element for each quantization interval be Y_i $(1 \le i \le N, i \in Z)$. The average mutual information between the input and output is

$$I(X;Y) = H(Y) - H(Y|X) \tag{8}$$

As quantified equipment is predetermined, each given source X after quantization corresponds to a certain output Y, i.e, the uncertainty of the output Y is zero on condition that X be known.

$$H(Y|X) = 0 \tag{9}$$

Thus

$$I(X;Y) = H(Y) - H(Y|X) = H(Y) \tag{10}$$

So maximizing the input and output average mutual information is equivalent to maximizing the entropy $H(Y)$ of the output signal Y. Because the output Y after quantization is discrete, it can be regarded as discrete information source. The maximum entropy theorem of discrete information sources tells us: When the probability of each Y_i is equal, $H(Y)$ takes the maximum value. Thus, assuming $p(Y_i)$ represents the probability of Y_i, so as to maximize the $H(Y)$,

$$p(Y_i) = 1/N\ . \tag{11}$$

For $p(Y_i) = 1/N$, the probability that source X falls in the quantization interval of representative element Y_i is 1/N. If the source X is a continuous signal, using

$$\int_{-\infty}^{X_i} f(x)dx = i/N \qquad (1 \le i \le N) \tag{12}$$

We can determine the region boundaries for each quantization interval. Here, we expressed them as $X_1, X_2, \cdots, X_{N+1}$. Where $f(x)$ is the probability density function of source X.

The representatives of each quantization interval element Y_i can be quantified based on minimum mean square (MSE) criterion. That is[13][14]

$$Y_i = \frac{\int_{X_i}^{X_{i+1}} xf(x)dx}{\int_{X_i}^{X_{i+1}} f(x)dx} \tag{13}$$

4. Maximum Average Mutual Information Quantization in the Implementation of LDPC Decoding

To facilitate hardware implementation, we quantify the initial message in min-sum decoding algorithm with the method of maximum average mutual information quantization. Then the original information is directly mapped to the corresponding integer value via a table. Finally, variable message and check message will become integer owing to the fact that the initial message is integer.

For the binary input channels, the output bit sequence of channel encoder entered channel by the bipolar mapping. Suppose $u \in \{0,1\}$, bipolar map is $x = 1-2u$, thus $x = \{+1,-1\}$.

In the stationary memoryless AWGN channel, BPSK modulation, the transmission signal $a \in \{+1,-1\}$ is the map of transmitted bits $\{0,1\}$. "-1","+1" is basically the same number in actual communication system, so you can assume that the probability of "-1"or"+1" is 1 / 2. Assuming the source information is X; the received initial message is Y; and the quantized output is Z. Then the likelihood functions as:

$$f(y|x=a) = \frac{1}{\sqrt{2\pi}\sigma_n}\exp\left\{-\frac{(y-a)^2}{2\sigma_n^2}\right\} \tag{14}$$

Where σ_n^2 is Gaussian noise variance, $a \in \{+1,\ -1\}$. By the assumption we know the probability of a is equal to ± 1 , so

$$\begin{aligned} f(y) &= \sum_a f(a)\times f(y|a) = f(a=1)\times f(y|a=1) \\ &+ f(a=-1)\times f(y|a=-1) = \frac{1}{2\sqrt{2\pi}\sigma_n} \\ &\times\left\{\exp\left[-\frac{(y-1)^2}{2\sigma_n^2}\right] + \frac{1}{2\sqrt{2\pi}\sigma_n}\exp\left[-\frac{(y+1)^2}{2\sigma_n^2}\right]\right\} \end{aligned} \tag{15}$$

By the above equation we know: after the signal transmitted through AWGN channel, the received information y_j obey mathematical expectation for $\mu = \pm 1$ normal distribution, as shown in Figure 2.

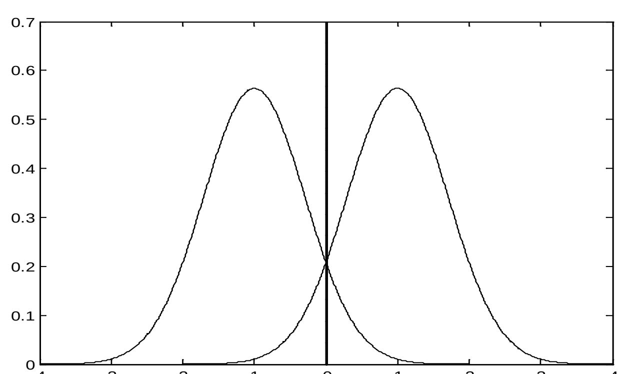

Figure 2. Distribution Map of the Initial Information

Let R : LDPC code rate, E_b : the energy per information bit before modulated, E_s : a bit of the signal energy, N_0 : noise power spectral density.

Because

$$SNR|_{\frac{E_b}{N_0}} = 10\lg(\frac{E_b}{N_0}) = 10\lg(\frac{E_s}{N_0} \times \frac{1}{R}) \tag{16}$$

We can get $N_0 = \frac{E_s}{R} \times 10^{-\frac{SNR}{10}}$, Usually take $E_s = 1$, Then

$$N_0 = 10^{-\frac{SNR}{10}} \times \frac{1}{R} \tag{17}$$

$$\sigma_n^2 = N_0 / 2 \tag{18}$$

By means of above formula the Gaussian noise variance σ_n^2 can be obtained.

We quantify the initial message of Figure 2 when the SNR is 3dB, the code rate is $R = 1/2$. Therefore,

$$N_0 \approx 1.00238 \tag{19}$$

$$\sigma_n^2 = N_0 / 2 = 0.50119 \tag{20}$$

The initial message y_j is thought as the source X to be quantified in section 3. Let $\int_{-\infty}^{\hat{y}_1} f(y)dy = 1/\mathrm{N} = 1/2^n$, we can get the first quantization interval, say $\hat{y}_1$, for the n bit quantization. And let $\int_{-\infty}^{\hat{y}_2} f(y)dy = 2/\mathrm{N} = 2/2^n$ we can get $\hat{y}_2$ .Similarly, let $\int_{-\infty}^{\hat{y}_i} f(y)dy = i/N = i/2^n$, $\hat{y}_i$ $(1 \le i \le N, i \in Z)$ is obtained.

We intend to get the integer quantified value for the initial message u_j , and it seems unnecessary to use the minimum mean square error criteria (Refer to (13)) to quantify. In effect, they can be accomplished as follows. Let u_j take the values below.

$$u_j = \begin{cases} \frac{N}{2}, & y_j > \hat{y}_{N-1} \\ -\frac{N}{2} + k, & \hat{y}_{k-1} < y_j \le \hat{y}_k, (\frac{N}{2} < k < N, k \in Z) \\ 0, & y_j = 0 \\ -\frac{N}{2} + i, & \hat{y}_i \le y_j < \hat{y}_{i+1}, (1 \le i < \frac{N}{2}, i \in Z) \\ -\frac{N}{2}, & \hat{y}_1 < y_j \end{cases} \tag{21}$$

Thus, through table look-up the original message y_j can be directly converted into the corresponding initial message integer u_j, and then decoding can be operated in accordance with the following steps:

1) Initialization: send initial message to v_{ij}, and initialize u_{ij} to be 0;
2) Update variable messages and check messages referring to (7) and (5);
3) Try to decode referring to the value of equation (6).

Suppose that the decoding codeword sequence is $\hat{x} =$ ($\hat{x}_1$, $\hat{x}_2$, $\cdots$, $\hat{x}_N$), and the LDPC code parity check matrix is $H_{M\times N}$. If $H_{M\times N}\hat{x} = 0$, we consider the decoding successful, otherwise return to step 2 to calculate a new round of updates.

In this article it can be seen that the initial information is quantified with a new method, that is, we initialize the received raw information y_j not via formula (1) but mapping the original information y_j to the corresponding integer value directly through a table. Having the initial message been converted into the integer values, we start to decode. Actually, the table look-up realizes the quantization operation.

5. Performance Simulation and Analysis

The quasi-cyclic LDPC codes [15] whose rate is $R = 1/2$ and code length is $N = 2048$ are studied here. We simulate sum-product decoding algorithm under the LLR measure and the min-sum algorithm respectively in stationary memoryless AWGN channel. The maximum number of iterations is 20 times, and we count 1 million groups at each SNR. Figure 3 is the BER simulation results.

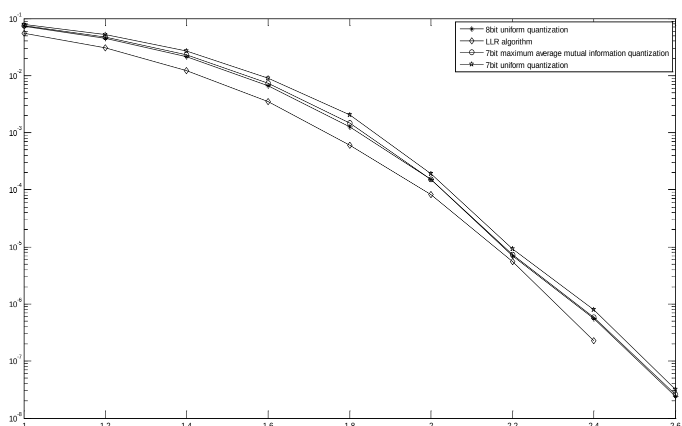

Figure 3. Performance of the Maximum Average Mutual Information Quantization and Uniform Quantizer

We can see from the Figure 3 when the length of quasi-cyclic LDPC codes is $N = 2048$, the decoding performance of min-sum algorithm based on integer with 7-bit maximum average mutual information quantization is better than that of min-sum algorithm based on integer with 7-bit uniform quantization [7], and its performance is almost the same as the decoding performance of min-sum algorithm based on integer with 8-bit uniform quantization [7] , and its performance is very close to that of sum-product decoding algorithm based on precision floating point operation. Meanwhile, all the variables in this algorithm are fixed-length integers, so it is convenient for hardware implementation.

We separately adopt sum-product decoding algorithm under LLR measure and min-sum algorithm based on integer with the 7-bit maximum average mutual information quantization and uniform quantization to decode this quasi-cyclic LDPC codes. The 200M-bit symbols decoding time are illustrated in Table I. The maximum iteration times are 20 and the SNR is 1 dB or 2 dB. At C simulation platform (the computer processor is Pentium (R) Dual-core E5500@2.80GHz, the memory storage size is 2 GB).

Table I. Comparison of Decoding Speed of Different Decoding Algorithm (Unit: Seconds)

Algorit-hm / Time / SNR	LLR algorithm	8bit uniform quantizati-on	7bit uniform quantizati-on	7bit maximum average mutual information quantization
1dB	5610.44	614.19	613.79	613.790
2dB	2731.96	358.89	354.86	354.864

The above table shows that: The decoding speed of min-sum algorithm based on integer is much faster than the that of sum-product algorithm under the LLR measure because a high-precision floating-point arithmetic is unnecessary; The decoding speed of min-sum algorithm based on integer with 7-bit maximum average mutual information is as fast as that with 7-bit uniform quantization, and its speed is slightly faster than that with the 8-bit uniform quantization; And the size of 8-bit uniform quantization table is as 2 times as 7-bit maximum average mutual information quantization table.

6. Conclusions

In this paper, we quantify the initial message in min-sum decoding algorithm with the method of maximum average mutual information quantization. Then the original information is directly mapped to the corresponding integer value via a table. This initial message is converted into integer value and then participates in the decoding operation. According to equation (2) and (5), variable message and check message will become integer owing to the fact that the initial message is integer (the variable message corrected by the correction factor in (7) just need to be rounded). Finally, the improved min-sum decoding algorithm is obtained. This algorithm performance is slightly worse than that of the high-precision floating-point LLR decoding algorithm, however, it enjoys the advantages of faster decoding speed and easy of hardware implementation. As soft decoding is used in LDPC decoding, which can maximize the use of information, the decoder enjoys a greater probability of deciding the correct code word. The maximum average mutual information quantization maximizes the average mutual information between the quantized symbols and the received signal. So the quantized symbols contain greatest information of the received signal, this is to say, the quantitative approach provide as much information as possible for LDPC decoding. In general, its quantitative results are much better. Therefore, this quantitative approach is a best quantitative one in the situation that the statistical distribution of the source is even unknown and only the received signal can be obtained. Decoding algorithm proposed in this article improves the decoding speed and is easy of hardware implementation, though its performance is slightly worse than the sum-product decoding algorithm based on high-precision floating-point operation.

Acknowledgements

I would like to acknowledge the Doctorate Foundation of Northwestern Polytechnical University and Natural Science Foundation of Shaanxi Province. This work was supported by them.

References

[1] S.-Y. Chung, G. D. Forney Jr., T. J. Richardson and R. Urbanke, “On the design of low-density parity-check codes within 0.0045dB of the Shannon limit”, IEEE Communications Letters, vol. 2, no. 5, **(2001)**.

[2] D. Oh and K. K. Parhi, “Optimally quantized offset min-sum algorithm for flexible LDPC decoder”, 2008 42nd Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, United States, **(2008)** October 26-29.

[3] Y. Xiaodong, “LDPC codes integer algorithm and its application”, Master's Thesis, Air Force Engineering University , Xi’an, **(2008)**.

[4] H. Yucheng, S. Shaohui and W. Xinmei, “Fast Decoding of LDPC Codes Using Quantization”, IEE Electronics Letters, vol. 4, no. 38, **(2002)**.

[5] S.-H. Sun, R. Sun and X.-M. Wang, “How to quantify BP decoding algorithm of LDPC codes”, Electronics, vol. 2, no. 31, **(2003)**.

[6] J. Heo and K. M. Chugg, “Optimization of Scaling Soft Information in Iterative Decoding Via Density Evolution Methods”, IEEE Transactions on Communications, vol. 6, **(2005)**, pp. 53.

[7] X. D. Ye, L. H. Ma and W. M. Wang, “The min-sum algorithm of LDPC codes based on integer operation”, Communications, vol. 6, no. 31, **(2010)**.

[8] J. Lee and J. Thorpe, “Memory-Efficient Decoding of LDPC Codes”, Information Theory, ISIT 2005, Proceedings, International Symposium on, Piscataway, NJ, **(2005)** September 4-9.

[9] D. Oh and K. K. Parhi, “Min-sum decoder architectures with reduced word length for LDPC codes”, Circuits and Systems I: Regular Papers, IEEE Transactions, vol. 1, no. 57, **(2010)**.

[10] T. J. Richardson and R. L. Urbanke, “The capacity of low-density parity-check codes under message-passing decoding”, IEEE Trans. Inf. Theory, vol. 2, no. 47, **(2001)**.

[11] H. Guangshu, “Digital Signal Processing (2nd Edition)”, Tsinghua University Publishing, Beijing, **(2003)**.

[12] X. Chen, J. Kang and L. S. Akell, “Accelerating FPGA-based emulation of quasi-cyclic LDPC codes with vector processing”, 2009 Design, Automation and Test in Europe Conference and Exhibition, Nice, France, **(2009)** April 20-24.

[13] H. Mao, K. Toru and I. Hiroyasu, “Delta-Sigma Modulator with Non-Uniform Quantization”, 2011 IEEE Radio and Wireless Week, RWW 2011-2011 IEEE Radio and Wireless Symposium, RWS, Phoenix, AZ, United States, **(2011)** January 16-19.

[14] S. P. Lloyd, “Least Squares Quantization in PCM”, IEEE Transactions on Information Theory, vol. 2, no. 28, **(1982)**.

[15] CCSDS, Low-Density Parity-Check Codes for Use in Near-Earth and Deep Space [S]. Consultative Committee for Space Data Systems Standard, 131.1-O-2, Orange Book, **(2007)** September.

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