A Multi-class Bernoulli Feedback Queue with Gate Mechanism

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Abstract

We consider a single server queue in which multi-class customers arrive according to Poisson process and service times are exponentially distributed. The server works following a gate mechanism in which arriving customers do not enter service immediately and wait to form a batch. This batch of customers get service after the completion of service to the previous batch. We also assume that after the service time of all customers, a customer of one class may join the next batch to get the service of another or the same class. The number of customers in a batch, duration of service time to a batch and the probability for the busy period to end in finite time are obtained. Elegant expressions are obtained when there are only two classes. These results are extended to multi-class customers in a varying environment. The connections between queues and branching processes is exploited to obtain these results. Numerical illustrations are presented.

Keywords: Branching processes; busy period; feedback; gated service.

1. Introduction

The phenomenon of feedback in queueing systems occurs in many practical situations. In telecommunication systems messages which turned out errors at destination are sent again.

In a single-operator repair facility where jobs arrive and proceed from one category to another according to what inspections, tests, and repairs are performed, a computer system where jobs change categories as execution ensues, and single-technician medical laboratory in which different samples arrive to be processed through random sequences of tests. Queueing models are the main quantitative tools in evaluating the operating performance of call centers. In a call center a customer may call again if his or her problems are not solved completely after service.

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In software product development processes, the feeding-back flow of software project represents re-work that must be done on projects as the result of problems discovered during independent outside testing or early use.

The assumption of Poisson traffic flow in queueing network is a common simplifying assumption in modeling practice. There have been many rigorous justifications of this type of phenomenon in different contexts [4].

In a cyclic priority queue, with several classes, after completing one service, a customer may enter the next with a prescribed probability or leave the system. The feedback probabilities may be different for different classes. In modeling such a system one can adjust the feedback probabilities to match job service requirements [3].

Feedback is introduced as a mechanism for scheduling the call service in which service consists of a random number of stages [1]. In telephony, a particular unit must perform more than one class of service; central control units may have to handle a variety of requests for service from calls in various stages of processing. For example, an originating call may require N distinct services, ranging from first obtaining dial tone to finally obtaining a connection to the called party. If more than one of these services is provided by the same control unit (or units), then this situation can be represented by a feedback queueing system. In such situations an incoming call will enter the queue and, after having obtained its first service, will immediately rejoin the queue until, finally, after the completion of its N-th service, the call will leave the system or no longer demand service from the control unit.

A single-server queueing network is equivalent to a multi-class system with feedback probabilities; one can view the customers at station i as the class i customers. With this interpretation, there is a single service facility with many classes of customers, and each time a customer's service is completed he either leaves the system, rejoins the same class, or proceeds to a new class.

Indeed, one can combine these two viewpoints and consider a single-server queueing network with several classes of customers. Then each index i $(i = 1, 2, \cdots, N)$ corresponds to a particular class/node pair, and the customers can change from class to class and/or node to node. Each time a customer's service at station i is completed, with some probability $p_{ij} (j = 1, 2, \cdots, N)$ he proceeds to station j or with probability $p_{i0} = 1 - \sum_{j=1}^{N} p_{ij}$ he leaves the system [9].

In semiconductor manufacturing and thin film lines, components visit some machine (or set of machines) more than once. Thus parts at different stages of their life may be in contention for service at the same machine. This gives rise to the problem of machine scheduling. The Polling system in computer network can be modeled as queue with gated vacation. In order to include transmission's error, the polling system is modeled as queue with gated vacation and feedback where old customers have different feedback parameter and different service time distribution compared to new customers [2].

Feedback queueing systems with single and multiple types of feedback have been widely investigated (eg. [6]). For queueing systems with single type of feedback, Takacs (1963) considered an M/G/1 Bernoulli feedback queue and obtained the queue size distribution and the total response time distribution of a customer. Lam and Shankar (1981) studied M/M/1 feedback queue with multi-class customers and FCFS policy where the number of feedbacks for a customer is generally distributed depending on the class to which the customer belongs, and they obtained the system distribution and the total response time distribution.

In this paper, we consider a single server queue with n classes of customers in which arrivals occur according to a Poisson process and service times of customers are independent exponential random variables. The server works according to a gated discipline. Upon arrival customers of all classes join a queue in front of a gate. Whenever all customers present at the service area have received service, the gate opens. The customers waiting for service enter the service area and the gate closes. That is, once a batch of customers begins service, customers who arrive during their service time receive service only when all members of the previous batch have completed their service. We also assume that after the service time of all customers, a customer of one class may join the next batch to get the service of the another or the same class.

We obtain exact results for the number served and the duration of service for $k-$th batch using branching process techniques. The finiteness of the busy period can be related to the extinction probability of a multi-type branching process.

The initial idea of using branching processes as a tool in queueing theory seems to be due to Borel as pointed out by Kendall [5]. For a detailed account of this technique, see [8].

2. Multi-class queue

We now consider a multi-class queue. We define the random times $0 = T_0, T_1, T_2, ...$ where T_{n+1} is the instant in which all customers, if any, present at T_n complete service. We denote $\mathbf{X_k} = (X_{1k}, X_{2k}, \cdots, X_{nk}),\quad X_{jk} =$ the number of class j customers in the k-th batch for a process. Let $\mathbf{X}(t)$ denote the number of customers in the system at t+0 and $\mathbf{X_k} = \mathbf{X}(T_k)$, number of customers in the k-th batch.

The bivariate sequence $\{\mathbf{X_k}, T_{k+1} - T_k\}$ is a Markov sequence because of the assumptions of Poisson input and exponential service times. Let the probability generating function of $\mathbf{X_k} = (X_{1k}, X_{2k}, \cdots, X_{nk}),\ k = 1, 2, 3, \ldots$ be

$$\mathbf{f_k}(\mathbf{z}) = [f_k^{(1)}(\mathbf{z}), f_k^{(2)}(\mathbf{z}), \cdots, f_k^{(n)}(\mathbf{z})] \quad ,$$

with the vector symbol $\mathbf{z} = (z_1, z_2, \cdots, z_n)$.

Assume initially there is one customer of class i. During its service, customers of class i arrive according to Poisson process with the rate $\lambda_i^{(1)}$, $i = 1, 2, \cdots, n$ and all customers together form the first batch and get exponential service with the same rate $\mu^{(1)}$. We denote $\rho_i^{(1)} = \frac{\lambda_i^{(1)}}{\mu^{(1)}}$. After completion of service a customer from the first batch will wait to get class i service with probability $p_i^{(1)}, i = 1, 2, \cdots, n$ in the next batch. We have assumed service times are independent of the class. During the service to the first batch customers of all classes arrive to form the next batch with rates $\lambda_i^{(2)}$, $i = 1, 2, \cdots, n$ and service rates $\mu^{(2)}$ and we denote $\rho_i^{(2)} = \frac{\lambda_i^{(2)}}{\mu^{(2)}}$ and so on. When customer of class i is getting service initially, arrivals to the system is governed by the following probability generating function:

$$\begin{aligned} & f_1^{(j)}(z_1, z_2, \cdots, z_n) = E[z_1^{X_{11}} z_2^{X_{21}} \cdots z_n^{X_{n1}} | X_{j\ 0} = 1, X_{k\ 0} = 0, k \neq j] \\ = & \frac{1 - p_{j1}^{(1)}(1 - z_1) - p_{j2}^{(1)}(1 - z_2) - \cdots - p_{jn}^{(1)}(1 - z_n)}{1 + \rho_1^{(1)}(1 - z_1) + \rho_2^{(1)}(1 - z_2) + \cdots + \rho_n^{(1)}(1 - z_n)}, \ j = 1, 2, \cdots, n, \\ & \text{and } \ p_{j1}^{(1)} + p_{j2}^{(1)} + \cdots + p_{jn}^{(1)} < 1. \end{aligned}$$

We follow the notation given below:

$$\mathbf{1} = \begin{pmatrix} 1, & 1, & \cdots, & 1 \end{pmatrix}', \quad \mathbf{z} = \begin{pmatrix} z_1, & z_2, & \cdots, & z_n \end{pmatrix}',$$
$$\mathbf{f_i}(\mathbf{z}) = \begin{pmatrix} f_i^{(1)}(\mathbf{z}), & f_i^{(2)}(\mathbf{z}), & \cdots, & f_i^{(n)}(\mathbf{z}) \end{pmatrix}'.$$
$$\boldsymbol{\rho}_i = \begin{pmatrix} \boldsymbol{\rho}_1^{(i)}, & \boldsymbol{\rho}_2^{(i)}, & \ldots & \boldsymbol{\rho}_n^{(i)} \end{pmatrix} \quad \text{and}$$
$$\mathbf{M}_i = \begin{pmatrix} \rho_1^{(i)} + p_{11}^{(i)} & \rho_2^{(i)} + p_{12}^{(i)} & \ldots & \rho_n^{(i)} + p_{1n}^{(i)} \\ \rho_1^{(i)} + p_{21}^{(i)} & \rho_2^{(i)} + p_{22}^{(i)} & \ldots & \rho_n^{(i)} + p_{2n}^{(i)} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_1^{(i)} + p_{n-11}^{(i)} & \rho_2^{(i)} + p_{n-12}^{(i)} & \ldots & \rho_n^{(i)} + p_{n-1n}^{(i)} \\ \rho_1^{(i)} + p_{n1}^{(i)} & \rho_2^{(i)} + p_{n2}^{(i)} & \ldots & \rho_n^{(i)} + p_{nn}^{(i)} \end{pmatrix}, \quad i = 1, 2, \cdots.$$

Now, for $j = 1, 2, \cdots, n$, we have

$$1 - f_1^{(j)(\mathbf{z})} = \frac{(\rho_1^{(1)} + p_{j1}^{(1)})(1 - z_1) + \cdots + (\rho_n^{(1)} + p_{jn}^{(1)})(1 - z_n)}{1 + \rho_1^{(1)}(1 - z_1) + \cdots + \rho_n^{(1)}(1 - z_n)}$$

Simple calculations yield,

$$\mathbf{f_i}(\mathbf{z}) \;\; = \;\; \mathbf{1} - \frac{1}{1 + \boldsymbol{\rho}_i(\mathbf{1} - \mathbf{z})} \mathbf{M}_i(\mathbf{1} - \mathbf{z}) \tag{1}$$

is the vector whose components represent the pgfs of the number of customers arrived in the i-th batch during the service of one customer of class j, $j = 1, 2, \cdots, n$ in the $(i-1)$-th batch for $i = 1, 2, \cdots$ initially.

The pgf $\mathbf{g}_n(\mathbf{z})$ of the number of customers in the n-th batch is given by

$$\mathbf{g}_n(\mathbf{z}) \;\; = \;\; \mathbf{f}_1(\mathbf{f}_2(\cdots(\mathbf{f}_n(\mathbf{z})))), \;\; n = 1, 2, \ldots .$$

$$\mathbf{g}_k(\mathbf{z}) = \left(g_k^{(1)}(\mathbf{z}), \quad g_k^{(2)}(\mathbf{z}), \quad \cdots, \quad g_k^{(n)}(\mathbf{z}) \right)' \tag{2}$$

$$= \mathbf{1} - \frac{\mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_k(\mathbf{1}-\mathbf{z})}{1+(\boldsymbol{\rho}_k+\boldsymbol{\rho}_{k-1}\mathbf{M}_k+\boldsymbol{\rho}_{k-2}\mathbf{M}_{k-1}\mathbf{M}_k+\cdots+\boldsymbol{\rho}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_k)(\mathbf{1}-\mathbf{z})} \tag{3}$$

We observe that the product $\mathbf{M}_i\mathbf{M}_{i+1}\cdots\mathbf{M}_k$, $i = 1, 2, \cdots, k$ are $n \times n$ matrices and ρ_i, $i = 1, 2, \cdots, k$ are $1 \times n$ matrices. Hence $\boldsymbol{\rho_k} + \boldsymbol{\rho_{k-1}}\mathbf{M_k} + \boldsymbol{\rho_{k-2}}\mathbf{M_{k-1}}\mathbf{M_k} + \cdots + \boldsymbol{\rho_1}\mathbf{M_2}\mathbf{M_3}\cdots\mathbf{M_k}$ is a $1 \times n$ matrix. Since $\mathbf{1}-\mathbf{z}$ is a $n \times 1$ matrix, $\mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_k(\mathbf{1}-\mathbf{z})$ is a $n\times 1$ matrix and $(\boldsymbol{\rho_k}+\boldsymbol{\rho_{k-1}}\mathbf{M_k}+\boldsymbol{\rho_{k-2}}\mathbf{M_{k-1}}\mathbf{M_k}+\cdots+\boldsymbol{\rho_1}\mathbf{M_2}\mathbf{M_3}\cdots\mathbf{M_k})(\mathbf{1}-\mathbf{z})$ is a scalar. The mean matrix of $\mathbf{g_k}(\mathbf{z})$ is

$$= \begin{pmatrix} \frac{\partial g_k^{(1)}}{\partial z_1} & \frac{\partial g_k^{(1)}}{\partial z_2} & \cdots & \frac{\partial g_k^{(1)}}{\partial z_n} \\ \frac{\partial g_k^{(2)}}{\partial z_1} & \frac{\partial g_k^{(2)}}{\partial z_2} & \cdots & \frac{\partial g_k^{(2)}}{\partial z_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_k^{(n)}}{\partial z_1} & \frac{\partial g_k^{(n)}}{\partial z_2} & \cdots & \frac{\partial g_k^{(n)}}{\partial z_n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix}$$

$$= \mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_k \tag{4}$$

Assume that $(\boldsymbol{\rho_k}+\boldsymbol{\rho_{k-1}}\mathbf{M_k}+\cdots+\boldsymbol{\rho_1}\mathbf{M_2}\mathbf{M_3}\cdots\mathbf{M_k})(\mathbf{1}-\mathbf{z}) = \boldsymbol{l_1}(\mathbf{1}-\boldsymbol{z_1})+\boldsymbol{l_2}(\mathbf{1}-\boldsymbol{z_2})+\cdots+\boldsymbol{l_n}(\mathbf{1}-\boldsymbol{z_n})$. Hence (2) implies that

$$g_k^{(j)}(\mathbf{z}) = 1 - \frac{a_{j1}(1-z_1)+a_{j2}(1-z_2)+\cdots+a_{jn}(1-z_n)}{l_1(1-z_1)+l_2(1-z_2)+\cdots+l_n(1-z_n)}, \quad (j = 1, 2, \cdots, n). \tag{5}$$

Let N be the number of batches served in a busy period starting with class i customer. Then

$$\mathbf{P}(N \le k) = \mathbf{g}_k(\mathbf{0}) = \mathbf{1} - \frac{\mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_k\mathbf{1}}{1+(\boldsymbol{\rho}_k+\boldsymbol{\rho}_{k-1}\mathbf{M}_k+\cdots+\boldsymbol{\rho}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_k)\mathbf{1}}.$$

Starting with class i customer probabilities for the busy period to end in the $k-$th batch using the relation $\mathbf{P}(N = k) = \mathbf{P}(N \le k) - \mathbf{P}(N \le k-1) = \mathbf{g}_k(\mathbf{0}) - \mathbf{g}_{k-1}(\mathbf{0})$ can be found.

Moments

Now, we find the second order moments. Starting with the class j customer, in the k-th generation, $E[X_pX_i] = \frac{\partial^2 g_k^{(j)}}{\partial z_p \partial z_i}(\mathbf{1}) = a_{ji}l_p + a_{jp}l_i$, where $\boldsymbol{\rho_k}\boldsymbol{I} + \boldsymbol{\rho_{k-1}}\boldsymbol{M_k}+$

$\cdots + \boldsymbol{\rho_1 M_2 M_3 \cdots M_k = (l_1, l_2, \cdots, l_n)}$, and a_{ji} and a_{jp} are elements of the mean matrix $\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_k$ given in (4).

Busy period

We present a theorem for the probabilities for the busy period to end in finite time.

Theorem 1 *Let q_j denote the probability that starting with class j, $j = 1, 2, \cdots, n$ customer busy period ends in finite time. Let π denote the largest eigenvalue of the mean matrix M of the process,*

$$M = \begin{pmatrix} \rho_1 + p_{11} & \rho_2 + p_{12} & \dots & \rho_n + p_{1n} \\ \rho_1 + p_{21} & \rho_2 + p_{22} & \dots & \rho_n + p_{2n} \\ \dots & \dots & \dots & \dots \\ \rho_1 + p_{n-11} & \rho_2 + p_{n-12} & \dots & \rho_n + p_{n-1n} \\ \rho_1 + p_{n1} & \rho_2 + p_{n2} & \dots & \rho_n + p_{nn} \end{pmatrix}.$$

Then
(i) For $\pi > 1$, $q_j < 1$, $\forall j$ and satisfy the equations

$$f^{(j)}(q_1, q_2, \cdots, q_n) = \frac{1 - p_{j1}(1-q_1) - p_{j2}(1-q_2) - \cdots - p_{jn}(1-q_n)}{1 + \rho_1(1-q_1) + \rho_2(1-q_2) + \cdots + \rho_n(1-q_n)} = q_j,$$

and
(ii) For $\pi \leq 1$, $q_j = 1$, $i = 1, 2, \cdots, n$.

We compute numerically the probabilities for the busy period to end in finite time when the system has five classes of customers.

We use the following parameters in Theorem (1):

$$p_{11} = .2, p_{12} = .13, p_{13} = .1, p_{14} = .12, p_{15} = .15;$$
$$p_{21} = .25, p_{22} = .1, p_{23} = .12, p_{24} = .16, p_{25} = .15;$$
$$p_{31} = .1, p_{32} = .21, p_{33} = .12, p_{34} = .12, p_{25} = .13;$$
$$p_{41} = .01, p_{42} = .1, p_{43} = .08, p_{44} = .01, p_{45} = .02;$$
$$p_{51} = .13, p_{52} = .14, p_{53} = .01, p_{54} = .21, p_{55} = .11;$$

and

$$\lambda_1 = .8, \lambda_2 = 1.4, \lambda_3 = .6, \lambda_4 = 1.3 \text{ and } \lambda_5 = 1.$$

Since $\rho_1 = \frac{\lambda_1}{\mu}$, $\rho_2 = \frac{\lambda_2}{\mu}$, $\rho_3 = \frac{\lambda_3}{\mu}$, $\rho_4 = \frac{\lambda_4}{\mu}$ and $\rho_5 = \frac{\lambda_5}{\mu}$, we get different sets of values for different values of the common service rate $\mu = 2, 5, 8, 11, 14$ and 17. With these values busy period probabilities are obtained and displayed in the following table.

Table-1. Probabilities for the busy period to end in finite time

μ	π	q_1	q_2	q_3	q_4	q_5
2	3.1311	0.1307	0.1104	0.1356	0.2584	0.1616
5	1.6044	0.3598	0.3329	0.3659	0.5358	0.4079
8	1.2241	0.6109	0.5905	0.6153	0.7474	0.6509
11	1.0518	0.8743	0.8669	0.8758	0.9247	0.8898
14	0.9536	1.0000	1.0000	1.0000	1.0000	1.0000
17	0.8902	1.0000	1.0000	1.0000	1.0000	1.0000

In Table-1, π represents the maximum eigenvalue for the mean matrix. Here $q_i, i = 1, 2, \cdots, 5$, represents the probability for the busy period to end in finite time starting with one customer of class i. We see that for $\pi < 1$, $q_i = 1, \forall i$.

In Table-2, we are depicting probabilities of the busy period to end before batch n starting with one class i customer initially, $i = 1, 2, 3, 4, 5$ with service rate $\mu = 2$. Here these probabilities monotonically increase to the limit q_i. Observe that the first row in Table-1 is same as the last row in Table-2.

Table-2. Probabilities for the busy period to end before batch n

n	class - 1	class - 2	class - 3	class - 4	class - 5
2	0.0845	0.0620	0.0901	0.2197	0.1127
3	0.1162	0.0955	0.1211	0.2460	0.1476
4	0.1261	0.1057	0.1310	0.2545	0.1571
5	0.1293	0.1089	0.1341	0.2571	0.1601
6	0.1303	0.1099	0.1351	0.2580	0.1611
7	0.1306	0.1102	0.1354	0.2583	0.1614
8	0.1307	0.1103	0.1355	0.2583	0.1615
9	0.1307	0.1103	0.1356	0.2584	0.1615
10	0.1307	0.1104	0.1356	0.2584	0.1615
11	0.1307	0.1104	0.1356	0.2584	0.1616

Remark: In the case of two-class queue, we discuss the probability that the busy period ends in finite time. When customer of class i, $(i = 1, 2)$ is getting service initially, arrivals to the system is governed by the following probability generating function:

$$f_1^{(1)}(z_1, z_2) = \frac{1 - a_1(1 - z_1) - a_2(1 - z_2)}{1 + \rho_1(1 - z_1) + \rho_2(1 - z_2)},$$

and

$$f_1^{(2)}(z_1, z_2) = \frac{1 - b_1(1 - z_1) - b_2(1 - z_2)}{1 + \rho_1(1 - z_1) + \rho_2(1 - z_2)},$$
$$i = 1, 2, 3, \cdots, \ a_1 + a_2 < 1 \ \text{ and } \ b_1 + b_2 < 1$$

Mean matrix is $\mathbf{M} = \begin{pmatrix} \rho_1 + a_1 & \rho_2 + a_2 \\ \rho_1 + b_1 & \rho_2 + b_2 \end{pmatrix}$
with the largest eigenvalue

$$\pi = \frac{a_1 + b_2 + \rho_1 + \rho_2 + \sqrt{(a_1 + \rho_1 - b_2 - \rho_2)^2 + 4(a_2 + \rho_2)(b_1 + \rho_1)}}{2}.$$

Theorem 2 *Let q_1 and q_2 respectively denote the probabilities for the busy period to end in finite when service start with class 1 and class 2 customer initially.*
i) If $\pi > 1$, then $q_1 < 1$, $q_2 < 1$ and

$$\begin{aligned} q_1 &= \frac{(1 - b_1 - b_2)a_2 - (1 - a_1 - a_2)(b_2 - \pi)}{(a_1 - \pi)(b_2 - \pi) - a_2 b_1} \\ q_2 &= \frac{(1 - a_1 - a_2)b_1 - (1 - b_1 - b_2)(a_1 - \pi)}{(a_1 - \pi)(b_2 - \pi) - a_2 b_1}. \end{aligned}$$

ii) If $\pi \leq 1$, then $q_1 = 1$, $q_2 = 1$.

Proof:
Using the results from [7], q_1 and q_2 satisfy $f^{(1)}(q_1, q_2) = q_1$ and $f^{(2)}(q_1, q_2) = q_2$.
That is,

$$\frac{1 - a_1(1 - q_1) - a_2(1 - q_2)}{1 + \rho_1(1 - q_1) + \rho_2(1 - q_2)} = q_1 \tag{6}$$

$$\text{and} \quad \frac{1 - b_1(1 - q_1) - b_2(1 - q_2)}{1 + \rho_1(1 - q_1) + \rho_2(1 - q_2)} = q_2 \tag{7}$$

Solving these two equations, we get

$$q_1 = \frac{(1 - b_1 - b_2)a_2 - (1 - a_1 - a_2)(b_2 - \pi)}{(a_1 - \pi)(b_2 - \pi) - a_2 b_1}$$

and

$$q_2 = \frac{(1 - a_1 - a_2)b_1 - (1 - b_1 - b_2)(a_1 - \pi)}{(a_1 - \pi)(b_2 - \pi) - a_2 b_1}.$$

Now,

$$\begin{aligned} \pi > 1 \quad &\Leftrightarrow \quad \pi > 1 > b_2 - a_2 \\ &\Leftrightarrow \quad (\pi - 1)(\pi - (b_2 - a_2)) > 0 \\ &\Leftrightarrow \quad a_2 - b_2 + \pi - a_2\pi + b_2\pi - \pi^2 < 0 \\ &\Leftrightarrow \quad a_2 - a_2 b_2 - (b_2 - \pi) + a_1(b_2 - \pi) + a_2(b_2 - \pi) \\ &\qquad\qquad < a_1(b_2 - \pi) - \pi(b_2 - \pi) \\ &\Leftrightarrow \quad (1 - b_1 - b_2)a_2 - (1 - a_1 - a_2)(b_2 - \pi) < (a_1 - \pi)(b_2 - \pi) - a_2 b_1 \\ &\Leftrightarrow \quad q_1 < 1 \end{aligned}$$

Similarly,

$$\pi > 1 \quad \Leftrightarrow \quad q_2 < 1$$

When $\pi \leq 1$ both $q_1 = 1$ and $q_2 = 1$.

3. Duration of service time

In this section we obtain the duration of service time of any given batch. Since all the service rates are equal replacing the variables by $\frac{\mu^{(k)}}{s+\mu^{(k)}}$ we get the Laplace transform of service time of k-th batch. Now,

$$\mathbf{z} = \begin{pmatrix} z_1, & z_2, & \cdots, & z_n \end{pmatrix}' = \begin{pmatrix} \frac{\mu^{(k)}}{s+\mu^{(k)}}, & \frac{\mu^{(k)}}{s+\mu^{(k)}}, & \cdots, & \frac{\mu^{(k)}}{s+\mu^{(k)}} \end{pmatrix}'$$

$$\mathbf{1} = \begin{pmatrix} 1, & 1, & \cdots, & 1 \end{pmatrix}' \text{ and } \mathbf{1} - \mathbf{z} = \frac{s}{\mu^{(k)} + s}\mathbf{1}.$$

Therefore,

$$\hat{\mathbf{g}}_{\mathbf{k}}(\mathbf{s}) = \begin{pmatrix} \hat{g}_k^{(1)}(\mathbf{s}), & \hat{g}_k^{(2)}(\mathbf{s}), & \cdots, & \hat{g}_k^{(n)}(\mathbf{s}) \end{pmatrix}'$$

$$= \mathbf{1} - \frac{s\mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_k\mathbf{1}}{\mu^{(k)} + s + s(\boldsymbol{\rho}_k\mathbf{I} + \boldsymbol{\rho}_{k-1}\mathbf{M}_k + \boldsymbol{\rho}_{k-2}\mathbf{M}_{k-1}M_k + \cdots + \boldsymbol{\rho}_1\mathbf{M}_2\mathbf{M}_3\cdots\mathbf{M}_k)\mathbf{1}}.$$

Here $\hat{g}_k^{(j)}(\mathbf{s}), j = 1, 2, \cdots, n$ represent the Laplace transforms of the distributions of the duration of the service time of $k-$th batch starting initially with $j-$th class customer.

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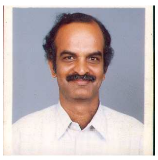

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