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Probability Fuzzy Attribute Implications for Interval-Valued Fuzzy Sets

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Abstract

Recently Burusco introduced interval-valued fuzzy formal contexts into fuzzy formal concept analysis. The most interesting work mainly including fuzzy attribute implications from fuzzy formal context, however, were presented under the framework of residuated lattice. In this paper, we first show that the study of interval-valued fuzzy set can be fitted into the framework of residuated lattice. Secondly, considering that the definition of fuzzy attribute implication in fact implies a minimal *degree and thus may be impractical in some applications, we introduce probability information to fuzzy inclusion degree and then to fuzzy attribute implication, and discuss some properties of this definition. The result verifies the correctness of probability fuzzy attribute implication in some illustrations.*
Keywords : *probability fuzzy attribute implication; probability inclusion degree; fuzzy attribute implication; interval-valued fuzzy formal context; residuated lattice*

1: Introduction

As an extension of Zadeh-style fuzzy set [36], interval-valued fuzzy set introduced by [32] exhibits a subinterval of $[0, 1]$, instead of a fixed value, for an object as its membership degree. This approach indeed introduces uncertainty into fuzziness and shows more flexibility than classical fuzzy set. Subsequently, interval-valued fuzzy set have been studied from various viewpoints [28, 29, 18] and also on the relationships with other extensions of fuzzy set [13, 14, 12, 3, 24, 25, 23].

In Formal Concept Analysis (FCA) [22], interval-valued fuzzy set was first introduced by Burusco in [11] as interval-valued formal contexts (IVFF contexts). Due to its versatility on handling uncertainty and incomplete data, this model is getting more and more attention [11, 2, 34, 1, 16]. For example, Djouadi etc. [16] studied the condition under which an fuzzy implication based on interval values satisfies the fuzzy closure properties with respect to

Galois connection, and then the algorithm for building concept lattice for interval-valued formal concepts under a generalized Gödel implication. On the other hand, in some cases, there may be absent values in IVFF contexts, called incomplete IVFF contexts, which may be caused by omission during the transfer process, or by no proper values for the cells. In order to handle the cases, [1, 2] discussed the completion of such contexts by applying association rules and fuzzy proposition respectively. With a different viewpoint, Wu etc. [34] proposed real formal concept analysis (real FCA) based on grey-rough set theory [35]. The basic idea behind real FCA is to view interval data as grey numbers so as to define operators for interval data by exploiting the operators on grey number from grey-rough set theory [35].

All the investigations on IVFF contexts are indeed on the basis of the idea that IVFF contexts are a novel model for handling uncertainty. However, the most useful framework developed by Belohlavek [10] is based on complete residuated lattice [26, 5, 8]. In this framework, Belohlavek presented the semantical and syntactical characteristics of fuzzy attribute implications in two style, i.e., crisp style and Pavelka-style [30], and in two settings, i.e., within a data table with fuzzy attributes [9] and in the logic way [7]. Besides, [6] presented a system of pseudo-intents, which was shown to be complete, non-redundant, and furthermore under a special setting, minimal in size. For the sake of applying these results into IVFF contexts, in this paper, in the light of [23] we will show that the study of interval-valued fuzzy set can be fitted into the framework.

Moreover, we state that the definition of fuzzy attribute implication (FAI) in fact implies a *minimal* degree and thus may be impractical in some applications. To produce a more reasonable result, this paper introduces probability information to fuzzy attribute implication, and then discusses some properties of this definition. As a matter of fact, in order to introduce probability information to FCA, various approaches have been investigated [15, 19, 20, 17]. [19, 17] considered four derivation operators, called sufficiency, possibility, necessity and dual sufficiency, aiming at providing set approximation [21, 33] for fuzzy sets, while [21, 33] obtained inspiration from Rough set theory [31] and defined some set approximations for fuzzy set. [15] presented an explicit formula for intent-extent mapping when only individual descriptions were given. Under the formula, the so-called stochastic Galois lattice was then introduced and their properties were discussed hereafter. Afterwards, [20] evaluated the mean and variance of the size of the random Galois lattice built from a sample of binary random vectors with i.i.d. Bernoulli(p) components, along with the similar results for the mean and the variance of the number of closed α-frequent itemsets. The distinction between the approaches and our study lies in that we just pay our attention to adding probability information to FAIs, while others may concern how to model fuzzy concept lattice based on probability information.

The paper will be organized as follows. Section 2 gives a viewpoint of interval-valued fuzzy set as L-fuzzy set, where L is a complete residuated lattice. Next in Section 3 we describe an illustration in which the original definition of fuzzy inclusion degree is impractical, and furthermore we present probability inclusion degree to combine probability information into fuzzy inclusion degree. Another viewpoint of probability inclusion degree will be presented in Section 4 based on weighting. In Section 5, by applying probability inclusion degree we construct some variations of FAI including probability FAI and give some properties of them. During the process, some illustrations are also provided to verify the results obtained.

Objects	Specific gravity	Freezing point	Iodine value	Saponification value
Linseed oil	[0.930,0.935]	[-27,-18]	[170,204]	[118,196]
Perilla oil	[0.930,0.937]	[-5,-4]	[192,208]	[188,197]
Cottonseed oil	[0.916,0.918]	[-6,-1]	[99,113]	[189,198]
Sesame oil	[0.920,0.926]	[-6,-4]	[104,116]	[187,193]
Camellia oil	[0.916,0.917]	[-21,-15]	[80,82]	[189,193]
Olive oil	[0.914,0.919]	[0,6]	[79,90]	[187,196]
Beef tallow	[0.860,0.870]	[30,38]	[40,48]	[190,199]
Hog fat	[0.858,0.864]	[22,32]	[53,77]	[190,202]

Table 1. Fats and oil data

2: Interval-valued Fuzzy Sets: A Perspective of Residuated Lattice

An interval-valued fuzzy set A in a universe U is given by $A(u) = [\underline{\mu}(u), \overline{\mu}(u)]$, where $\underline{\mu}$, $\overline{\mu} : U \rightarrow [0, 1]$ are the so-called lower and upper membership functions such that for any $u \in U$, $0 \leq \underline{\mu}(u) \leq \overline{\mu}(u) \leq 1$. That is, the degree to which an element necessarily belongs to A equals to $\underline{\mu}$, while the degree to which the element possibly belongs to A equals to $\overline{\mu}$. The constrained condition of $\underline{\mu}(u) \leq \overline{\mu}(u)$ means that necessity implies possibility. It is easy to see that Zadah-style fuzzy set is a special case of interval-valued fuzzy set when $\underline{\mu} = \overline{\mu}$. In this paper, we also call the values of $\underline{\mu}(u)$and $\overline{\mu}(u)$ lower and upper approximations of μ respectively.

Interval-valued fuzzy set can also be constructed based on a complete lattice L, just by defining $\underline{\mu}$, $\overline{\mu} : U \rightarrow L$. Moreover, if L is a complete residuated lattice [26, 5, 8] we can take interval-valued fuzzy set as a special L-fuzzy set as shown below.

Lemma 1. *[B. V. Gasse, etc.[23]] For a residuated lattice L, define the sublattice $T(L)$ of $L \times L$ as $T(L) = \{(x_1, x_2) \in L \times L | x_1 \leq x_2\}$, and then we can generate a residuated lattice $\mathcal{L} = (T(L), \wedge, \vee, \otimes, \rightarrow, \mathbf{0}, \mathbf{1})$ such that*

- $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$,
- $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$,
- $(a_1, b_1) \otimes (a_2, b_2) = (a_1 \otimes a_2, b_1 \otimes b_2)$,
- $(a_1, b_1) \rightarrow (a_2, b_2) = ((a_1 \rightarrow a_2) \wedge (b_1 \rightarrow b_2), b_1 \rightarrow b_2)$,
- $\mathbf{0} = (0, 0)$ *and* $\mathbf{1} = (1, 1)$

for $(a_1, b_1), (a_2, b_2) \in T(L)$.

Using the above lemma, one can observe that each interval-valued fuzzy set A can then be viewed as a mapping $A : U \rightarrow T(L)$, and thus is a special case of L-fuzzy set.

3: Probability Inclusion Degree

For data table, entries can also be interval values, called interval-valued fuzzy formal contexts (IVFF contexts for short) [11, 16]. Formally, an IVFF contexts is a triple $\langle X, Y, I \rangle$, where X is a set of objects, Y is a set of attributes, and I is a mapping $X \times Y \rightarrow T(L)$.

An example concerning interval values is taken from [27, 13] and shown in Figure 1.

From the table, one can see that not all intervals are contained in $[0, 1]$, for example,

Objects	Specific gravity	Freezing point	Iodine value	Saponification value
Linseed oil	[0.930,0.935]	[0.000,0.138]	[0.774,0.976]	[0.000,0.929]
Perilla oil	[0.930,0.937]	[0.338,0.354]	[0.905,1.000]	[0.833,0.940]
Cottonseed oil	[0.916,0.918]	[0.323,0.400]	[0.351,0.435]	[0.845,0.952]
Sesame oil	[0.920,0.926]	[0.323,0.354]	[0.381,0.452]	[0.821,0.893]
Camellia oil	[0.916,0.917]	[0.092,0.185]	[0.238,0.250]	[0.845,0.893]
Olive oil	[0.914,0.919]	[0.415,0.508]	[0.232,0.298]	[0.821,0.929]
Beef tallow	[0.860,0.870]	[0.877,1.000]	[0.000,0.048]	[0.857,0.964]
Hog fat	[0.858,0.864]	[0.754,0.908]	[0.077,0.220]	[0.857,1.000]

Table 2. Fats and oil data after transformed

in the attributes “Freezing point”, “Iodine value” and “Saponification value”. Although this inconsistency does not matter for the following process, one would like to convert these intervals into $[0,1]$, just for convenience and intuition, by any interesting methods. It should be noted that in order to avoid causing damage to the information behind the intervals, the methods of converting should at least keep the interval-valued structures consistent with original ones. That is, we need to assure that the obtained residuated lattice is isomorphic to the original one. In this example, since any intervals are contained in real line, any scaling will be valid. Then Table 1 is, after scaled into $[0,1]$, transformed into Table 2.

So far we have considered how to apply residuated lattice structure to interval-valued fuzzy set, and thus the study of IVFF contexts can be subsumed into the framework of Belohlavek [6]. However, we note here that in some cases this framework may be impractical as shown below.

To start with, in this paper we restrict our discussion to the special residuated lattice $[0,1]$ and denote by L^U the collection of all $[0,1]$-fuzzy sets. Then, recall that the study of FAIs developed by Belohlavek in [6] depended basically upon the inclusion degree (i.e., subsethood degree) between fuzzy sets (or L-sets):

$$S(A,B) = \bigwedge_{u\in U} (A(u) \to B(u)),$$

for two fuzzy sets A and B on the universe U. In other words, $S(A,B)$ equals to the minimal degree among all the degrees to which elements u belong to B whenever they belong to A. In another sense of approximation [21, 33], this degree is actually only a lower approximation of the subsethood relation, since here we obtain the *minimal* degree.

In some applications, this lower approximation may be correct and practical, when we want to rank fuzzy sets strictly. However, in others (for example, data tables may contain noise and the minimal degree is obtained accidentally on some noise object), this would be impractical. For example, let A, B be two fuzzy sets on interval $[0,1]$ with $A(u) \leq B(u)$ for any $u \in U$ except $w \in U$ for which $A(u) = 1$ and $B(u) = 0$ (see Figure 1). In this case, $S(A,B) = 0$. This result would be unacceptable, since A is almost everywhere contained in B; with the terms from Measure Theory, A is not contained in B merely in some subsets of $[0,1]$ with measure 0. Therefore, to make it more practical, it is necessary to add the measure information to the definition of S. Then, an intuitive method is to add probability information to the definition; for example, we can reformulate the inclusion degree and define probability inclusion degree as

$$S_P(A,B) = \int_U (A(u) \to B(u)) P(du)$$

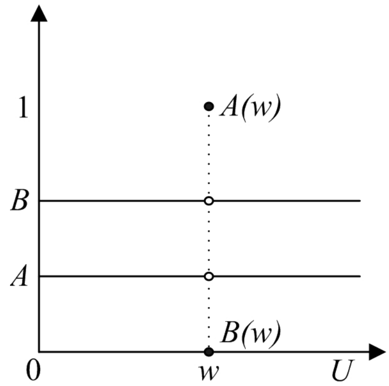

Figure 1. A **is contained in** B **except at the point of** w

where $P(du)$ is a probability measure on U, and note that we replace S by S_P to distinguish from the original one. Since any point w of U will be of measure 0, we then obtain the expected result $S_P(A, B) = 1$, provided that U is an infinite set. For finite universe, the inclusion degree turns out to be

$$S_P(A, B) = \sum_{u \in U} ((A(u) \to B(u))P(u)).$$

Obviously, S_P is more acceptable than S in this case.

Defining $S_M(A, B) = \bigvee_{u \in U}(A(u) \to B(u))$, we have the following boundaries for probability inclusion degree.

Theorem 1. *For any A, $B \in L^U$, we have* $\bigwedge_{u \in U}(A(u) \to B(u)) = S(A, B) \leq S_P(A, B) \leq S_M(A, B)$.

Proof. It is straightforward. □

In order to make an intuitive impression on the differences of those inclusion degrees, we compute the inclusion degrees with uniform distribution for probability measure P for all pairs of distinct objects (up to $C_8^2 = 56$ pairs) of Table 2, and show the results in Figure 2 and Figure 3. Since inclusion degrees are interval values, for clarification we plot the lower approximations of these inclusion degrees in Figure 2 and upper approximations in Figure 3. In the figures, we plot the inclusion degrees on y-axis against the pairs of attributes on x-axis, where circle-markers stand for values of $S(A, B)$, square-marker for $S_P(A, B)$ and star-marker for $S_M(A, B)$. From the figures, it is easy to see that the three degrees have explicit order for each pair of attributes, and moreover the probability inclusion degree $S_P(A, B)$ is nearly an average of $S(A, B)$ and $S_M(A, B)$. Actually, in both cases, lower and upper approximations of S_M are up close to 1 for almost all pairs, which means that for any pair of objects, there must be at least one attribute that makes the two objects similar. However, we observe that the original inclusion degree S is beyond the upper boundary S_M and thus can not give a better approximation for the inclusion degree, because they always select the minimal degree as their inclusion degree. Moreover, by varying the probability measure P, the probability inclusion degree is capable of ranging from $S(A, B)$, the possible minimal inclusion degree, to $S_M(A, B)$, the possible maximal inclusion degree as Section 4 shows.

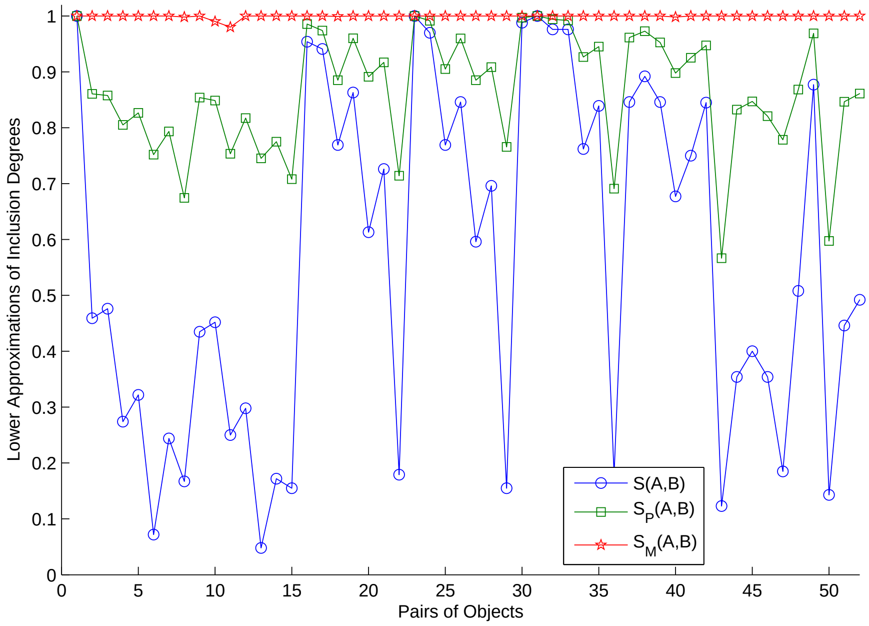

Figure 2. Lower approximation of inclusion degrees for all pairs of distinct objects in Table 2

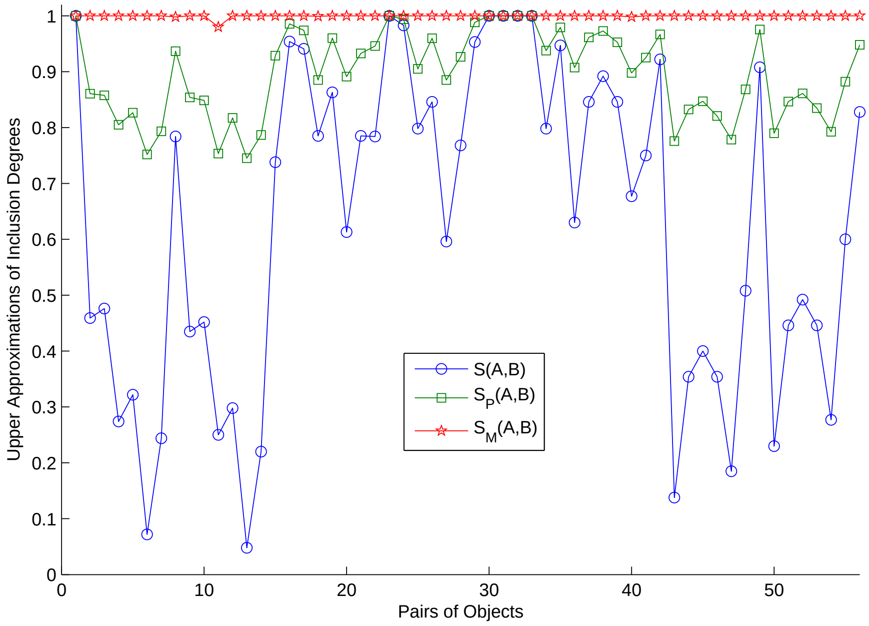

Figure 3. Upper approximation of inclusion degrees for all pairs of distinct objects in Table 2

4: Weighting Related Viewpoint on Probability Inclusion Degree

Another viewpoint of S_P is to regard $P(du)$ as weights such that $\sum_U P(du) = 1$, and thus S_P is the weighted average. For finite objects U such as Table 2, to compute the inclusion degree of objects fuzzy sets, for example, the object fuzzy set

$$\text{Perilla oil} = ([0.930, 0.937]/\text{Specific gravity}, [0.338, 0.354]/\text{Freezing point}, \\ [0.905, 1.000]/\text{Iodine value}, [0.833, 0.940]/\text{Saponification value}), \quad (1)$$

we can give all the attributes the same weights in case they share the same importance, while we can give some attributes higher weights if they are regarded as more important than others. For example, setting same weights to all attributes and using Łukasiewicz implication, one can calculate that

$$\begin{aligned} S_P(\text{Perilla oil}, \text{Linseed oil}) =& \frac{1}{4}([0.930, 0.937] \to [0.930, 0.935] + [0.338, 0.354] \to [0.000, 0.138 \\ &+ [0.927, 1.024] \to [0.793, 1.000] + [0.833, 0.940] \to [0.000, 0.929 \\ =& [0.675, 0.937] \end{aligned}$$

while

$$\begin{aligned} S(\text{Perilla oil}, \text{Linseed oil}) =& ([0.930, 0.937] \to [0.930, 0.935] \wedge [0.338, 0.354] \to [0.000, 0.138] \\ &\wedge [0.927, 1.024] \to [0.793, 1.000] \wedge [0.833, 0.940] \to [0.000, 0.929]) \\ =& [0.167, 0.784] \end{aligned}$$

To demonstrate the difference between S_P and S more clearly, we give the Euclidean distance between the two objects, Perilla oil and Linseed oil: $d(\text{Perilla oil}, \text{Linseed oil}) = 0.873$. Note that the Euclidean distance is computed with 8 components, so the average distance for each component is only 0.109, which means that the two objects are very close, taking into account that the interval is $[0, 1]$. Since $S(\text{Linseed oil}, \text{Perilla oil}) = S_P(\text{Linseed oil}, \text{Perilla oil}) = 1$, our result $S_P(\text{Perilla oil}, \text{Linseed oil}) = [0.675, 0.937]$ thus implies that Perilla oil is not only contained in but is closer to Linseed oil than S(Perilla oil,Li which equals to $= [0.167, 0.784]$, at least in probability sense.

Now if it is thought that the attributes “Specific gravity” and “Iodine value” are of more importance than the others, one can assign the weights 0.5 and 0.5 to them respectively and assign 0 to the others. In this case, we have

$$\begin{aligned} S_P(\text{Perilla oil}, \text{Linseed oil}) =& (\frac{1}{2} \times [0.930, 0.937] \to [0.930, 0.935] + 0 \times [0.338, 0.354] \to [0.00 \\ &+ \frac{1}{2} \times [0.927, 1.024] \to [0.793, 1.000] + 0 \times [0.833, 0.940] \to [0.0 \\ =& [0.935, 0.987] \end{aligned}$$

which shows that the two objects can be considered as two quite similar ones under the attributes “Specific gravity” and “Iodine value”, by ignoring the effect of the other attributes.

To illustrate the effect of probability measure on probability inclusion degree, we generate randomly 1000 probability distributions and show the effect on the lower and upper approximations of $S_P(\text{Perilla oil}, \text{Linseed oil})$ in Figure 4 and Figure 5 respectively. From the figures, it is easy to see that the probability inclusion degree can vary from the minimum $S(A, B)$ to the maximum $S_M(A, B)$.

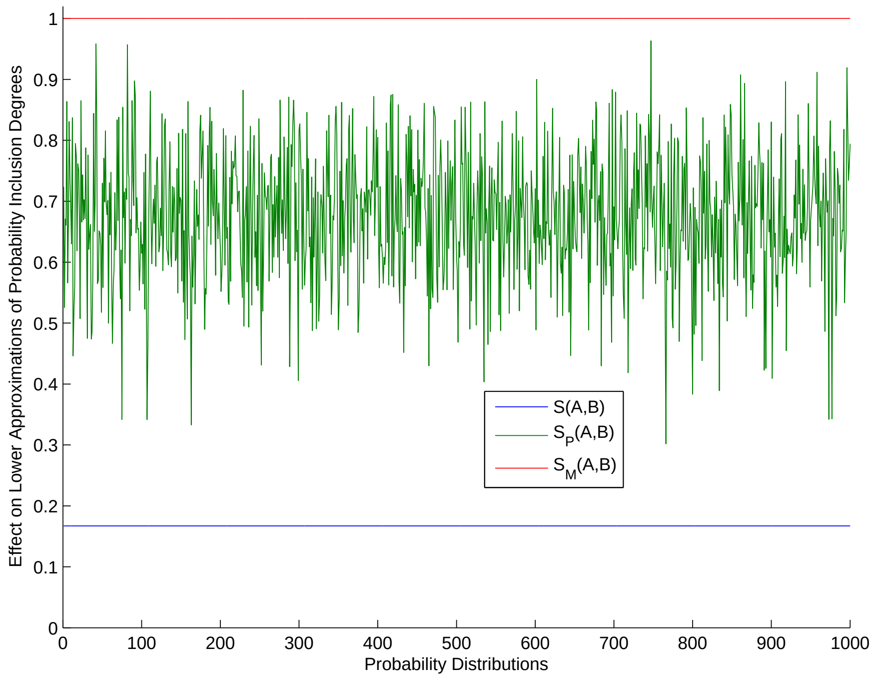

Figure 4. Effect of probability distributions on lower approximations of S_P

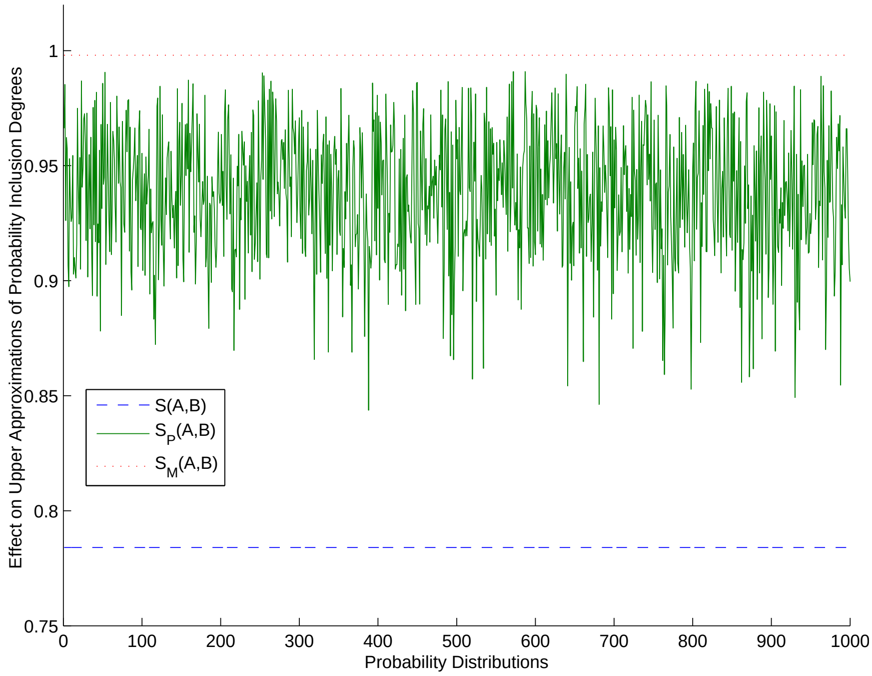

Figure 5. Effect of probability distributions on upper approximations of S_P

5: Probability Fuzzy Attribute Implications

In this section, we consider how to introduce probability information to FAIs. First, recall the definition of FAI [10]. An FAI over a finite set Y of attributes is an expression

$A \Rightarrow B$, where $A,\ B \in L^Y$. Note that FAI at the time is just a formula without any meaning. To specify meaning to FAI, Belohlavek [10] defined a degree $\|A \Rightarrow B\|_M \in L$ for a fuzzy set $M \in L^Y$ by

$$\|A \Rightarrow B\|_M = S(A, M)^* \to S(B, M).$$

Obviously, the degree $\|A \Rightarrow B\|_M \in L$ expresses the possibility that if A is contained in M, then B should also be contained in M. Using probability inclusion degree, we can define the probability FAI degree as

$$\|A \Rightarrow B\|_M^P = S_P(A, M)^* \to S_P(B, M).$$

Generally speaking, there is no explicit order between $\|A \Rightarrow B\|_M^P$ and $\|A \Rightarrow B\|_M$. This is due to varying probability measure P for one hand. For another, the value of $\|A \Rightarrow B\|_M^P$ depends on two independent fuzzy sets A and B, so taking different pairs of fuzzy sets will come to different orders. In order to deal with the case, we define the lower and upper probability FAI degrees as the boundaries for the approximation as follows:

$$\|A \Rightarrow B\|_M^{\downarrow P} = S_P(A, M)^* \to S(B, M),$$

and

$$\|A \Rightarrow B\|_M^{\uparrow P} = S(A, M)^* \to S_P(B, M).$$

Thus we have six possible FAI degrees for any $A \Rightarrow B$ and $M \in L^Y$, namely,

$$\begin{aligned}
\|A \Rightarrow B\|_M^L &= (\bigvee_{y\in Y} A(y) \to M(y))^* \to (\bigwedge_{y\in Y} A(y) \to M(y))\\
\|A \Rightarrow B\|_M^{\downarrow P} &= S_P(A, M)^* \to S(B, M)\\
\|A \Rightarrow B\|_M &= S(A, M)^* \to S(B, M)\\
\|A \Rightarrow B\|_M^P &= S_P(A, M)^* \to S_P(B, M)\\
\|A \Rightarrow B\|_M^{\uparrow P} &= S(A, M)^* \to S_P(B, M)\\
\|A \Rightarrow B\|_M^U &= (\bigwedge_{y\in Y} A(y) \to M(y))^* \to (\bigvee_{y\in Y} A(y) \to M(y))
\end{aligned}$$

These FAI degrees have the following order.

Theorem 2. *For any $A,\ B,\ M \in L^Y$, we have the following order:*

1. $\|A \Rightarrow B\|_M^L \le \|A \Rightarrow B\|_M^{\downarrow P}$
2. $\|A \Rightarrow B\|_M^{\downarrow P} \le \|A \Rightarrow B\|_M \le \|A \Rightarrow B\|_M^{\uparrow P}$
3. $\|A \Rightarrow B\|_M^{\downarrow P} \le \|A \Rightarrow B\|_M^P \le \|A \Rightarrow B\|_M^{\uparrow P}$
4. $\|A \Rightarrow B\|_M^{\uparrow P} \le \|A \Rightarrow B\|_M^U$

Proof. The proof follows from Theorem 1 and the fact that the implication operator in residuated lattice is antitone in the first variable and isotone in the second variable. □

There exist natural meanings behind these degrees. First, we argue that only these degrees lying between the intervals $\|A \Rightarrow B\|_M^L$ and $\|A \Rightarrow B\|_M^U$ are valid degrees. Second, for practical purpose, only the degrees ranging from $\|A \Rightarrow B\|_M^{\downarrow P}$ to $\|A \Rightarrow B\|_M^{\uparrow P}$ are considered meaningful for a fixed probability measure P, since one can not reach at one time the limits $\|A \Rightarrow B\|_M^L$ and $\|A \Rightarrow B\|_M^U$ that obviously adopt different probability measures at the premise and the consequence. Third, we note that there does not exist explicit order between $\|A \Rightarrow B\|_M^P$ and $\|A \Rightarrow B\|_M$.

For a given interval-valued data table $\langle X, Y, I\rangle$, the probability valid degree to which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ is given by

$$\|A \Rightarrow B\|^P_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|^P_{\{I_x|x\in X\}},$$

where I_x represents an object fuzzy set (e.g. the "Perilla oil fuzzy set", see Eq. (1)), while Belohlavek defined it by

$$\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = \|A \Rightarrow B\|_{\{I_x|x\in X\}}.$$

Furthermore we define the lower and upper probability valid degrees (with respect to $\langle X, Y, I\rangle$) by:

$$\|A \Rightarrow B\|^{\downarrow P}_{\langle X,Y,I\rangle} = \bigwedge_{\{I_x|x\in X\}} \|A \Rightarrow B\|^{\downarrow P}_{I_x},$$

and

$$\|A \Rightarrow B\|^{\uparrow P}_{\langle X,Y,I\rangle} = \bigwedge_{\{I_x|x\in X\}} \|A \Rightarrow B\|^{\uparrow P}_{I_x}.$$

Using Theorem 2, we can show the following results.

Theorem 3. *For an IVFF context and any A, $B \in L^Y$, we have the following properties:*

1. $\|A \Rightarrow B\|^L = \bigwedge_{\{I_x|x\in X\}} \|A \Rightarrow B\|^L_{I_x} \leq \|A \Rightarrow B\|^{\downarrow P}_{\langle X,Y,I\rangle}$
2. $\|A \Rightarrow B\|^{\downarrow P}_{\langle X,Y,I\rangle} \leq \|A \Rightarrow B\|_{\langle X,Y,I\rangle} \leq \|A \Rightarrow B\|^{\uparrow P}_{\langle X,Y,I\rangle}$
3. $\|A \Rightarrow B\|^{\downarrow P}_{\langle X,Y,I\rangle} \leq \|A \Rightarrow B\|^{P}_{\langle X,Y,I\rangle} \leq \|A \Rightarrow B\|^{\uparrow P}_{\langle X,Y,I\rangle}$
4. $\|A \Rightarrow B\|^{\uparrow P}_{\langle X,Y,I\rangle} \leq \bigwedge_{\{I_x|x\in X\}} \|A \Rightarrow B\|^U_{I_x} = \|A \Rightarrow B\|^U$.

Proof. Following from Theorem 2.

6: Rationality of Probability FAI

To see the differences between FAI valid degrees and probability FAI valid degrees, we randomly generate 1000 FAIs of Table 2, and obtain the most different pairs of $\|A \Rightarrow B\|_{\langle X,Y,I\rangle} = [0.1466, 0.1466]$ and $\|A \Rightarrow B\|^P_{\langle X,Y,I\rangle} = [0.7866, 0.7866]$ according to lower and upper approximations. We list the premise and consequence of the FAI in Table 3, and then we compute the FAI degrees and probability FAI degrees as shown in Table 4.

According to Table 4, both of values of $\|A \Rightarrow B\|_{\langle X,Y,I\rangle}$ and $\|A \Rightarrow B\|^P_{\langle X,Y,I\rangle}$ are determined by object "Beef tallow", i.e., the fuzzy set

$$M = \text{Beef tallow} = \{[0.860, 0.870]/\text{Specific gravity}, [0.877, 1.000]/\text{Freezing point}, [0.000, 0.048]/\text{Iodine value}, [0.857, 0.964]/\text{Saponification value}\}.$$

	A			
FAI No.	Specific gravity	Freezing point	Iodine value	Saponification value
1	[0.1540,0.3134]	[0.6745,0.9910]	[0.0428,0.1305]	[0.5269,0.7404]
2	...	...	...	...

	B			
FAI No.	Specific gravity	Freezing point	Iodine value	Saponification value
1	[0.3365,0.7288]	[0.6526,0.9344]	[0.7605,0.9839]	[0.3159,0.5028]
2	...	...	...	...

Table 3. Premise and Consequence of FAI

Objects	$\|A \Rightarrow B\|_{I_x}$	$\|A \Rightarrow B\|_{I_x}^P$
Linseed oil	[1.0000,1.0000]	[1.0000,1.0000]
Perilla oil	[1.0000,1.0000]	[1.0000,1.0000]
Cottonseed oil	[0.9420,1.0000]	[0.8769,0.8769]
Sesame oil	[0.9720,1.0000]	[0.8812,0.8812]
Camellia oil	[1.0000,1.0000]	[0.8307,0.8307]
Olive oil	[0.7301,0.7970]	[0.8427,0.8427]
Beef tallow	[0.1466,0.1466]	[0.7866,0.7866]
Hog fat	[0.3165,0.3190]	[0.8232,0.8232]

Table 4. FAI degrees and probability FAI degrees

In fact, we have

$$\|A \Rightarrow B\|_M = [0.9572 \rightarrow 0.2395 \wedge 0.9175 \rightarrow 0.0641, 0.9175 \rightarrow 0.0641] = [0.2823 \wedge 0.1466, 0.$$

$$\|A \Rightarrow B\|_M^P = [0.9893 \rightarrow 0.8099 \wedge 0.9794 \rightarrow 0.7660, 0.9794 \rightarrow 0.7660] = [0.8206 \wedge 0.7866, 0.$$

Then, the second components decide the values of $\|A \Rightarrow B\|_M$ and $\|A \Rightarrow B\|_M^P$, and the most distinct parts are $S(B, M) = [0.0641, 0.0641]$ and $S_P(B, M) = [0.7660, 0.7660]$. Thus the analysis of FAI valid degrees is then reduced to the differences of $S(B, M)$ and $S_P(B, M)$. As shown before, the former is assigned to the minimal degree and the later is a weighted average. Accordingly, if one do not want the FAI valid degree determined by some objects that perhaps contain noise, probability FAI valid degree will be proper and required.

7: Conclusion and Further Work

In this paper, we considered probability inclusion degree and probability FAI. We first show that the study of interval-valued fuzzy set can be fitted into the framework developed by Belohlavek. Second, we state that the definition of FAI by Belohlavek in fact implies a *minimal* degree and thus may be impractical in some applications. Thus this paper introduces probability information to fuzzy inclusion degree and then to fuzzy attribute implication, and discuss some properties of this definition. The result shows their differences and verifies the correctness of probability fuzzy attribute implication in some illustrations.

Note that the results obtained in the paper are useful not only for IVFF contexts, but also for fuzzy formal contexts. In this sense, probability FAIs are in fact pretty general contribution to fuzzy concept lattice. Thus it gives rise to the further work including how to develop general semantical and syntactical aspects for probability FAI, to apply them into practical ends, and to study the relationship between probability FAI and fuzzy concept lattice.

One of shortcomings of the paper is that we limit our discussion only to the interval $[0, 1]$ and fail to give further results on general residuated lattice, due to the fact that one can not directly give meaningful operators between probability measure and residuated lattice. One approach to solve this problem is to consider a probability measure $P : U \to L$, which is defined on residuated lattice L and such that $\bigvee_{u \in U} P(u) = 1$. And then define the probability inclusion degree as $S_P(A, B) = \bigwedge_{u \in U}(P(u) \to (A(u) \to B(u)))$, which however is still under investigation.

Here we also want to mention intuitionistic fuzzy set introduced by [4]. An intuitionistic fuzzy set A in a universe is defined by $A(u) = \{\mu(u), \nu(u)\}$, where μ, $\nu : U \to [0, 1]$ are the so-called truth- and falsity-membership functions such that $0 \le \mu(u) + \nu(u) \le 1$. Here, μ takes the same meaning as $\underline{\mu}$ in interval-valued fuzzy set, while ν assigns each element the degree to which the element does not belong to A. It was shown [12] that intuitionistic fuzzy set is a model equivalent to interval-valued fuzzy set, if we set $\mu = \underline{\mu}$ and $\nu = 1 - \overline{\mu}$. Thus the framework of this paper can be applied to intuitionistic fuzzy set without any difficulty.

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