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An Improved Affine Algorithm for the Non-Probabilistic Reliability Index of Interval Model

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Abstract

Aimed at that standard interval arithmetic is easy to cause error explosion, as well as traditional affine arithmetic is easy to cause calculation error due to not considering the correlation among the first degree item, quadratic term, even the multiple-item and the nonlinear term expressed by the same noise, this paper proposed an improved algorithm which took affine algorithm operation as the main body. When there are those correlation items, the algorithm calculates the upper and lower bound of their total value by interval arithmetic, then uses the bound information and introduces the new noise affine form to instead of those terms. The method improved the conventional affine operation by combining interval arithmetic. When it applied to calculate the non-probabilistic reliability index, the calculation results has higher accuracy than the interval arithmetic and the conventional affine algorithm, and it has strong practical significance.

Keywords*: Non-probabilistic reliability; Interval arithmetic; Correlation; New noise; Affine arithmetic*

1. Introduction

Assuming $x = (x_1, x_2, \cdots, x_n)$ $(x_i \in x_i^I$ $(i = 1, 2, \cdots, n)$ as an interval input vector which will affect the structural response. Considering the structural response to the input parameter, the allowable interval of known structural response was given as follow：

$$y \in y^I \in [y^l, y^u] \tag{1}$$

Then supposes the structural response function as：

$$f(x) = f(x_1, x_2, \cdots, x_n) \tag{2}$$

So the structural performance function is：

$$M = g(x, y) = y - f(x) = y - f(x_1, x_2, \cdots, x_n) \tag{3}$$

When $f(x_1, x_2, \cdots, x_n)$ is continuous for x_i $(i = 1, 2, \cdots, n)$, the values of M, $f(x)$ must be Intervals. Setting M^u, M^l and f^u, f^l as the upper and lower bound for M and $f(x)$ respectively，then the non-probabilistic reliability index of interval model is defined as:

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$$\eta = \frac{M^c}{M^r} = \frac{y^c - f^c}{y^r + f^r} \tag{4}$$

In the formula (4), $y^c = \frac{y^u + y^l}{2}$, $y^r = \frac{y^u - y^l}{2}$, $f^c = \frac{f^u + f^l}{2}$, $f^r = \frac{f^u - f^l}{2}$, $M^c = \frac{M^u + M^l}{2} = y^c - f^c$, $M^r = \frac{M^u - M^l}{2} = y^r + f^r$, they are respectively the centers and radius of corresponding intervals.

At present, the General method for solving the non-probabilistic reliability index η of interval model is to use the interval arithmetic by interval variables instead of point variables. This method is widely used in computer science, engineering, finance and other fields due to its convenience for handling uncertain data, automatically recording the truncation errors and round-off errors in the computer floating point arithmetic, estimating the range of function value in a certain interval, and so on.

However, the irrationality of the interval arithmetic rules [1] and the thorough neglect of the correlation among variables, so its computation results often are expanded [2] and overflowed [3]. When nonlinear degree of performance function is high or has a deep nesting structure, the error explosion will appear, and as a result that the calculated value of η will lose its practical significance. To overcome these disadvantages, many solutions such as subdividing interval, changing expression form of the function, redefining interval arithmetic, combining monotonous interval, interval finite element method [4], Interval truncation method and Sub interval perturbation method [5] came out. But they could hardly be generalized because of complicated process, extensive calculation and low stability.

As a modified form of interval arithmetic, the conventional affine arithmetic is different from interval arithmetic in realizing the relativity of computation and input data, then recording the relativity automatically and applying to independent calculation. In addition, the calculation rules of affine algorithm are optimized [1] and more stable. Thus its operation interval result can be narrower and more accurate, especially in long chain calculation. So the affine algorithm has been widely used in artificial intelligence systems analysis [6], system stability analysis [7], circuit response boundary analysis [8] and computer graphics [9].

Yet errors will still exist when using affine algorithm to calculate η . For example, the multiply and division operation of affine algorithm only approximate the quadratic term and nonlinear term by introducing the new noise affine form. Besides, the traditional affine algorithm may produce errors because of its neglect of the correlation among the first degree item, quadratic term, even the multiple-item and nonlinear term expressed by the same noise element during the computing process.

To further overcome these limitations, since affine form and interval form of the uncertain variable have a mutual transformation, this paper proposed an improved algorithm which takes affine algorithm operation as the main body, with the help of h interval arithmetic: considering the correlation among the first degree item, quadratic term, even the multiple-item and the nonlinear term expressed by the same noise element, the algorithm calculates the upper and lower bound of their total value by interval arithmetic, then introduces the new noise affine form to instead of those terms for operation according to the rules. Obviously the improved arithmetic can not only consider the correlation among uncertain variables, but also realize the relativity among the first degree item, quadratic term, even the multiple-item and nonlinear term expressed by the same noise element during the computing process. Examples show, as compared with interval arithmetic or traditional affine algorithm, this improved method which handles related items expressed by the same noise element by combining interval arithmetic doesn't only operate easily but can also make the interval results narrower and more accurate when is applied to calculate the non-probabilistic reliability index of interval model which contains complicated nonlinear explicit performance function. The method has better practical significance.

2. Affine Forms and Affine Arithmetic

2.1. Affine Forms

As considering the correlation among variables, the affine arithmetic can conclude more accurate results. Its basic idea is to express interval variable x as a kind of affine form. Because of some self-reasons or environment, assuming that the truth value of uncertain variable x can be effected by t kinds of noises, then the affine form(written as $\hat{x}$)can be expressed as a first-order polynomials of these t kinds of noise symbols:

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_t\varepsilon_t \tag{5}$$

In formula(5), x_0 is the center value of $\hat{x}$, $\varepsilon_j \in [-1,1]\ (j = 1, 2, \cdots, t)$ represent the j th noise symbol, the real coefficient $x_j \in \mathrm{R}\ (j = 1, 2, \cdots, t)$ is called the j th partial increment of $\hat{x}$.

Since $\varepsilon_j \in [-1,1]\ (j = 1, 2, \cdots, t)$ which represent different kinds of noises are all independent of each other, when $|\varepsilon_1| = |\varepsilon_2| = \cdots = |\varepsilon_t| = 1$, the value of $\hat{x}$ will reach the upper or lower bound, in other words, $x_0 + \sum_{j=1}^{t} |x_j|$ or $x_0 - \sum_{j=1}^{t} |x_j|$.

Now, $\hat{x}$ can be transferred to interval form $\overline{x}$:

$$\overline{x} = [x_0 - \sum_{j=1}^{t} |x_j|, x_0 + \sum_{j=1}^{t} |x_j|] \tag{6}$$

The transformation process taking affine form into interval form represented by formula (6) will lose some original information of uncertain variable; however loosed information cannot be recovered through inverse process. If the same noise element ε_j appear in two or more affine forms such as $\hat{x}$ and $\hat{y}$, it means that the uncertainties of $\hat{x}$ and $\hat{y}$ have some connection and interdependence.

2.2. Affine Arithmetic

The returning value of any affine arithmetic that takes affine form or real number as independent variable must be an affine form (first-order polynomials of the noise symbol). Only the four fundamental operations of affine form are specified here.

Assuming $\beta \in \mathrm{R}, \xi \in \mathrm{R}$ and affine form: $\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_t\varepsilon_t$, $\hat{y} = y_0 + y_1\varepsilon_1 + y_2\varepsilon_2 + \cdots + y_t\varepsilon_t$, then,

$$\hat{x} \pm \hat{y} = (x_0 \pm y_0) + (x_1 \pm y_1)\varepsilon_1 + \cdots + (x_t \pm y_t)\varepsilon_t \tag{7}$$

$$\beta\hat{x} = (\beta x_0) + (\beta x_1)\varepsilon_1 + (\beta x_2)\varepsilon_2 + \cdots + (\beta x_t)\varepsilon_t \tag{8}$$

$$\hat{x} \pm \xi = (x_0 \pm \xi) + x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_t\varepsilon_t \tag{9}$$

$$\hat{x} \cdot \hat{y} = (x_0 \cdot y_0) + \sum_{j=1}^{t} (x_0 y_j + x_j y_0)\varepsilon_j + \left(\sum_{j=1}^{t} x_j \varepsilon_j \right) \cdot \left(\sum_{j=1}^{t} y_j \varepsilon_j \right) \tag{10}$$

$$\hat{x} / \hat{y} = \hat{x} \cdot (1 / \hat{y}) = \hat{x} \cdot \left(1 / (y_0 + y_1\varepsilon_1 + y_2\varepsilon_2 + \cdots + y_t\varepsilon_t)\right) \tag{11}$$

That quadratic terms and nonlinear terms of noise symbol appear in multiplication and division formula respectively does not conform to affine form definition formula (5), so it needs to introduce an affine form of the new (the $t+1$ th) noise symbol ε_{t+1} to replace them.

According to Chebyshev affine approximation theory, set respectively b and a as the maximum and minimum in the domain of $\left(\sum_{j=1}^{t} x_j \varepsilon_j \right) \cdot \left(\sum_{j=1}^{t} y_j \varepsilon_j \right)$, then the optimal affine form of $\left(\sum_{j=1}^{t} x_j \varepsilon_j \right) \cdot \left(\sum_{j=1}^{t} y_j \varepsilon_j \right)$ is: $\left(\sum_{j=1}^{t} x_j \varepsilon_j \right) \cdot \left(\sum_{j=1}^{t} y_j \varepsilon_j \right) = \frac{a+b}{2} + \frac{b-a}{2}\varepsilon_{t+1}$

So the returning value of $\hat{x} \cdot \hat{y}$ will be

$$\hat{x} \cdot \hat{y} = (x_0 \cdot y_0 + \frac{a+b}{2}) + \sum_{j=1}^{t} (x_0 y_j + x_j y_0)\varepsilon_j + \frac{b-a}{2}\varepsilon_{t+1} \tag{12}$$

Similarly, according to Chebyshev affine approximation theory and Min-range affine theory, the returning value of $1/\hat{y}$ can be gotten:

$$\begin{aligned} 1/\hat{y} = &-(1/b^2)y_0 + (a+b)^2/(2ab^2) - (1/b^2)(y_1\varepsilon_1 + y_2\varepsilon_2 + \cdots + y_t\varepsilon_t) - \\ &(a-b)^2/(2ab^2)\varepsilon_{t+1} \end{aligned} \tag{13}$$

In the equation (13), $a = y_0 - |y_1| - |y_2| - \cdots - |y_t|$, $b = y_0 + |y_1| + |y_2| + \cdots + |y_t|$.

3. Non-Probabilistic Reliability Affine Arithmetic

Calculate the center point of interval input variables $x_1 \sim x_n$: $(x_i^l + x_i^u)/2 \quad (i = 1, 2, \cdots, n)$, taken as the center value of corresponding affine form, that is to set:

$$x_{i_0} = (x_i^l + x_i^u)/2 \quad (i = 1, 2, \cdots, n) \tag{14}$$

Take the interval radius of $x_1 \sim x_n$ as the coefficients of single noise symbol $\varepsilon_1 \quad (\varepsilon_1 \in [-1,1])$, namely:

$$x_{i_1} = (x_i^u - x_i^l)/2 \quad (i = 1, 2, \cdots, n) \tag{15}$$

With this, $x_1 \sim x_n$ can be transformed into n affine forms which possess single noise symbol:

$$\hat{x}_i = x_{i_0} + x_{i_1}\varepsilon_1 = \frac{x_i^u + x_i^l}{2} + \frac{x_i^u - x_i^l}{2}\varepsilon_1 \quad (i = 1, 2, \cdots, n) \tag{16}$$

Replace the corresponding interval variable x_i $(i = 1, 2, \cdots, n)$ in formula (2) with the equation above, the response function (also interval function) $f(x) = f(x_1, x_2, \cdots, x_n)$ will be transformed into affine function. Applied the rules of affine operation to compute the final returning value $f(\hat{x})$, it can be expressed as follows：

$$f(\hat{x}) = f_0 + f_1\varepsilon_1 + f_2\varepsilon_2 + \cdots + f_{t+1}\varepsilon_{t+1} \tag{17}$$

In the formula (17), $\varepsilon_2 \sim \varepsilon_{t+1}$ are the t new noise symbols introduced as they are needed.

Work out the upper and lower bound of $f(\hat{x})$:

$$\begin{aligned} f^u &= f_0 + |f_1| + |f_2| + \cdots + |f_{t+1}| \\ f^l &= f_0 - |f_1| - |f_2| - \cdots - |f_{t+1}| \end{aligned} \tag{18}$$

Similarly, worked out y^u, y^l as the upper and lower bound of allowable interval of structural response, and put those into formula (4), the non-probabilistic reliability index will be obtained:

$$\eta = \frac{M^c}{M^r} = \frac{y^c - f^c}{y^r + f^r} \tag{19}$$

In the formula (19), $y^c = \dfrac{y^u + y^l}{2}, x_n^r = \dfrac{y^u - y^l}{2}$, $f^c = \dfrac{f^u + f^l}{2}, f^r = \dfrac{f^u - f^l}{2}$.

4. The Improving of Affine Arithmetic

The use of approximations inevitably induced error in calculating the non-linear function by affine arithmetic, and the errors will increase with the growth of nonlinearity. Besides, calculating errors are further produced as general affine arithmetic just simply introduce the affine form of new noise symbols to respectively replace the first degree item, quadratic term and the multiple-item expressed by same noise symbol without considering the relativity among these terms.

In order to reduce the calculating errors from related items, In view of the fact that affine form and interval form can transform into each other, this paper proposes a modified affine arithmetic algorithm with the aid of interval arithmetic to determine the upper and lower bound. When there are the correlation terms expressed by the same noise symbol in operation, considering the correlation among them, the algorithm calculates the upper and lower bound of their total value by interval arithmetic. Since the values of noise symbol all range in [-1, 1], it’s easier to apply interval arithmetic to work out the limits of total correlation terms.

Based on this, introduces the new noise symbol affine form to instead of correlation terms for affine operation. Apparently the improved arithmetic can not only realize the correlation of interval input variables of structural response function and the correlation among allowable interval variable of structural response, but also give full consideration to the relativity among the first degree item, quadratic term and even the multiple-item term of noise during the computing process.

With the aid of interval arithmetic to handle correlation items during the process of affine algorithm, this method applies to calculate the non-probabilistic reliability index in interval model with complicated nonlinear explicit performance function, can get interval results more compact, and more close to the real value. The improved method has better practical significance.

Take example for calculating the multiple quadratic terms:

$$h = \sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij} \varepsilon_i \varepsilon_j, \quad \varepsilon_i, \varepsilon_j \in [-1,1] \ (i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m) \tag{20}$$

If $\max(\sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij} \varepsilon_i \varepsilon_j) = h^u$, $\min(\sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij} \varepsilon_i \varepsilon_j) = h^l$, then:

$$h = \sum_{i=1}^{n} \sum_{j=1}^{m} h_{ij} \varepsilon_i \varepsilon_j = h^c + h^r \varepsilon_k \tag{21}$$

In this formula (21), $h^c = (h^u + h^l)/2$, $h^r = (h^u - h^l)/2$, ε_k is the new introduced noise element.

This method handled quadratic correlation term has been also generalized to deal with the multiple-item and nonlinear term.

5. Calculating Example

5.1. Calculating Example 1

The cantilever beam is diagramed as in Figure 1. There are two concentrated forces p_1 and p_2 load at point C and B, where respectively distant fixed edge A from b_1 and b_2. The failure will occur if $|m_{max}| \geq |m_{cr}|$. Here, m_{cr} is the critical ultimate bending m_{max} is the maximum bending of the beam suffering. Assuming the basic interval variable: $p_1 \in [4.4, 5.6]\text{kN}$, $p_2 \in [1.7, 2.3]\text{kN}$, $b_1 \in [1.8, 2.2]\text{m}$, $b_2 \in [4.5, 5.5]\text{m}$. To apply the improved affine algorithm in this paper to calculate the non-probabilistic reliability index of this structure and compare it's results with which of the interval operation, conventional affine arithmetic in other documents and the exact value.

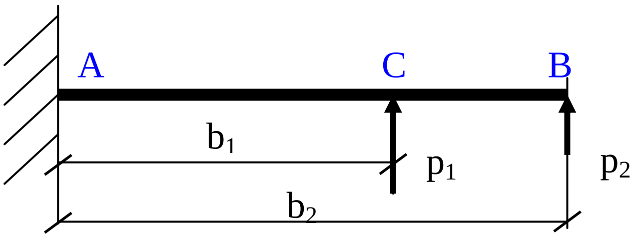

Figure 1. Cantilever Beam's Free-Body Diagram

The performance equations as follows:

$$M = m_{cr} - p_1 b_1 - p_2 b_2 = 0 \tag{22}$$

As the critical ultimate bending m_{cr}, force p_1, force p_2 and their locations are independent of each other, use the improved affine algorithm of this paper to transform respectively m_{cr} 、 p_1 、 b_1 、 p_2 and b_2 into affine forms with independent noise symbol: i.e. $m_{cr} = 36 + 4\varepsilon_1$, $p_1 = 5 + 0.6\varepsilon_2$, $b_1 = 2 + 0.2\varepsilon_3$, $p_2 = 2 + 0.3\varepsilon_4$, $b_2 = 5 + 0.5\varepsilon_5$, then,

$$\begin{aligned} M &= (36+4\varepsilon_1)-(5+0.6\varepsilon_2)(2+0.2\varepsilon_3)-(2+0.3\varepsilon_4)(5+0.5\varepsilon_5) \\ &= 16+4\varepsilon_1-1.2\varepsilon_2-\varepsilon_3-1.5\varepsilon_4-\varepsilon_5-0.12\varepsilon_2\varepsilon_3-0.15\varepsilon_4\varepsilon_5 \\ &= 16+4\varepsilon_1+(-1.2\varepsilon_2-\varepsilon_3-0.12\varepsilon_2\varepsilon_3)+(-1.5\varepsilon_4-\varepsilon_5-0.15\varepsilon_4\varepsilon_5) \end{aligned} \tag{23}$$

Considering that ε_2 and ε_3 are independent, $\varepsilon_2 \in [-1,1]$, $\varepsilon_3 \in [-1,1]$, while they are related to $\varepsilon_2\varepsilon_3$, for avoiding the expansion of interval, the related items of ε_2、 ε_3 and $\varepsilon_2\varepsilon_3$ are transformed together into new affine form and calculate their upper and lower bounds with the help of interval algorithm, so:

$$-1.2-1-0.12 \le -1.2\varepsilon_2-\varepsilon_3-0.12\varepsilon_2\varepsilon_3 \le 1.2+1-0.12 \tag{24}$$

i.e. $-2.32 \le -1.2\varepsilon_2-\varepsilon_3-0.12\varepsilon_2\varepsilon_3 \le 2.08$. Therefore, it can be introduced that the affine form $-0.12+2.2\varepsilon_6$ of new noise symbol ε_6 to replace $-1.2\varepsilon_2-\varepsilon_3-0.12\varepsilon_2\varepsilon_3$.

Similarly, ε_4、 ε_5 and $\varepsilon_4\varepsilon_5$ together introduce new affine form:

$$-1.5-1-0.15 \le -1.5\varepsilon_4-\varepsilon_5-0.15\varepsilon_4\varepsilon_5 \le 1.5+1-0.15 \tag{25}$$

i.e. $-2.65 \le -1.5\varepsilon_4-\varepsilon_5-0.15\varepsilon_4\varepsilon_5 \le 2.35$ .Therefore, the affine form $-0.15+2.5\varepsilon_7$ of new noise symbol ε_7 is introduced to replace $-1.5\varepsilon_4-\varepsilon_5-0.15\varepsilon_4\varepsilon_5$, then,

$$\begin{aligned} M &= 16+4\varepsilon_1+(-0.12+2.2\varepsilon_6)+(-0.15+2.5\varepsilon_7) \\ &= 15.73+4\varepsilon_1+2.2\varepsilon_6+2.5\varepsilon_7 \end{aligned} \tag{26}$$

Considering that ε_1 , ε_6 and ε_7 are independent of each other, so:

$$15.73-4-2.2-2.5 \le 15.73+4\varepsilon_1+2.2\varepsilon_6+2.5\varepsilon_7 \le 15.73-4-2.2-2.5 \tag{27}$$

i.e. $7.03 \le 15.73+4\varepsilon_1+2.2\varepsilon_6+2.5\varepsilon_7 \le 24.43$, then, $M \in [7.03, 24.43]$,
$M^c = (24.43+7.03)/2 = 15.73$, $M^r = (24.43-7.03)/2 = 8.7$.

Substitute them into formula (4); the non-probabilistic reliability index of affine arithmetic can be gotten:

$$\eta = M^c / M^r = \frac{15.73}{8.7} = 1.8080 \tag{28}$$

Based on estimation, the actual response interval of $M = m_{cr} - p_1 b_1 - p_2 b_2$ is $M \in [7.03, 24.43]$. To substitute it into formula (4), then the real value of non-probabilistic reliability index is $\eta_{True} = 1.8080$. Table 1 shows the calculated results through methods of this paper or the others.

Table 1. The Results Comparison of Different Method

	M	η	Error rate to η_{True}
True value[10]	[7.03, 24.43]	1.8080	0%
Interval algorithm[11]	[7.03, 24.43]	1.8080	0%
Conventional affine algorithm[12]	[7.03, 24.97]	1.7837	1.3452%
Method of this paper	[7.03, 24.43]	1.8080	0%

It can be seen from Table 1 that the exact result is obtained through both interval algorithm and the method of this paper. Because the values of m_{cr}, p_1b_1 and p_2b_2 are independent in performance function expressing formula $M = m_{cr} - p_1b_1 - p_2b_2$, the interval algorithm will not cause the expansion and overflow of result even without considering the correlation among variables.

Conventional affine algorithm causes some errors because Chebyshev approximation theory was used in handling noise quadratic term. It will inevitably produce errors in the multiplication and division operations. In addition, it ignores the relativity among the first degree item, quadratic term, and even the multiple-item during calculating, so the result also includes this part of calculating errors.

The method of this paper fully takes these correlations into consideration, identifies the upper and lower bound of their total value with help of interval arithmetic, then uses the bound information and introduces the new noise affine form to instead of those terms, and reach high accuracy.

5.2. Calculating Example 2

Shown in Figure 2, the length of the cantilever beam $l = 400mm$, its elastic modulus $E = 200GN/m^2$, and section moment of inertia $I = 200mm^4$. There is a concentrated force $P \in [200, 400]N$ act at $a \in [200, 300]mm$ away from fixed edge A. It's known that the allowable deflection along the opposite direction of y-axis at the cantilever end B is $f \in [80, 100]mm$, try to calculate the non-probabilistic reliability index.

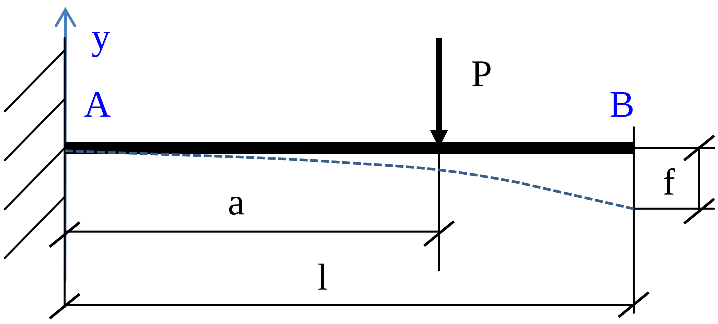

Figure 2. Cantilever Beam's Free-body Diagram

The performance function is:

$$M = y - f \tag{29}$$

The functional relationship between the deflection f along the opposite direction of y-axis at the cantilever end B and all other parameter above is:

$$f = \frac{Pa^2}{6EI}(3l - a) \tag{30}$$

5.2.1. Standard Interval Algorithm: Substitute interval variable P and a into formula (30) directly, to make sure the units of f and y are same as mm, it must be kept the units of force same as N and the units of length as mm. So the interval expressing formula of f is:

$$f = \frac{[200,400][200,300][200,300]}{6 \times 200 \times 10^9 \times 10^{-6} \times 200}(3 \times 400 - [200,300]) \tag{31}$$

Calculate the formula based on standard interval algorithm rules:

$$\begin{aligned} f &= \frac{[200,400][200,300][200,300]}{6 \times 200 \times 10^9 \times 10^{-6} \times 200}(3 \times 400 - [200,300]) \\ &= \frac{[200,400][200,300][200,300]}{6 \times 200 \times 10^9 \times 10^{-6} \times 200}(1200 - [200,300]) \\ &= \frac{[200,400][200,300][200,300]}{6 \times 200 \times 10^9 \times 10^{-6} \times 200}[900,1000]) \\ &= \frac{[200,400][200,300]}{6 \times 200 \times 10^9 \times 10^{-6} \times 200}([180000,300000]) \\ &= \frac{[200,400][36000000,90000000]}{2.4 \times 10^8} \\ &= \frac{[720000000000,36000000000]}{2.4 \times 10^8} = [30,150]mm \end{aligned} \tag{32}$$

Then, $f^c = 90$, $f^r = 60$. $y^c = (80 + 100)/2 = 90$, $y^r = (100 - 80)/2 = 10$. Substitute them into formula (4), the standard interval value of non-probabilistic reliability index η is:

$$\eta = \frac{y^c - f^c}{y^r + f^r} = \frac{90 - 90}{10 + 60} = 0 \tag{33}$$

5.2.2. The Improved Affine Arithmetic of this Paper: As the couple is independent from its application point, P and a can be transformed into affine forms with independent noise symbol respectively. $\hat{P} = 300 + 100\varepsilon_1$ and $\hat{a} = 250 + 50\varepsilon_2$. Substitute them into equation (28), the affine function is:

$$
\begin{aligned}
f &= \frac{(300+100\varepsilon_1)(250+50\varepsilon_2)(250+50\varepsilon_2)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\left(3\times 400-(250+50\varepsilon_2)\right)\\
&= \frac{(300+100\varepsilon_1)(250+50\varepsilon_2)(250+50\varepsilon_2)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}(950-50\varepsilon_2)\\
&= \frac{(300+100\varepsilon_1)(250+50\varepsilon_2)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}(250+50\varepsilon_2)(950-50\varepsilon_2)\\
&= \frac{(300+100\varepsilon_1)(250+50\varepsilon_2)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\left(250\times 950+(950\times 50-250\times 50)\varepsilon_2-50\times 50\varepsilon_2{}^2\right)\\
&= \frac{(300+100\varepsilon_1)(250+50\varepsilon_2)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}(237500+35000\varepsilon_2-2500\varepsilon_2{}^2)\\
&= \frac{(300+100\varepsilon_1)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\times\begin{pmatrix}250\times 237500+(250\times 35000+50\times 237500)\varepsilon_2\\ +(50\times 35000-250\times 2500)\varepsilon_2{}^2-50\times 2500\varepsilon_2{}^3\end{pmatrix}\\
&= \frac{(300+100\varepsilon_1)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\times(59375000+20625000\varepsilon_2+1125000\varepsilon_2{}^2-502500\varepsilon_2{}^3)
\end{aligned}
\tag{34}
$$

Since $\varepsilon_2\in[-1,1]$ is completely correlated with itself, the item $59375000+20625000\varepsilon_2+1125000\varepsilon_2{}^2-502500\varepsilon_2{}^3$ meets:

$$
\begin{aligned}
&59375000-20625000+1125000+502500\le\\
&59375000+20625000\varepsilon_2+1125000\varepsilon_2{}^2-502500\varepsilon_2{}^3\le\\
&59375000+20625000+1125000-502500
\end{aligned}
\tag{35}
$$

i.e. $40377500\le 158437500+19093750\varepsilon_2+328125\varepsilon_2{}^2-203125\varepsilon_2{}^3\le 80622500$.

So the affine form $60500000+20122500\varepsilon_3$ of new noise symbol $\varepsilon_3\in[-1,1]$ is introduced. And to substitute into formula (34):

$$
\begin{aligned}
f &= \frac{(300+100\varepsilon_1)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\times(59375000+20625000\varepsilon_2+1125000\varepsilon_2{}^2-502500\varepsilon_2{}^3)\\
&= \frac{(300+100\varepsilon_1)}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\times(60500000+20122500\varepsilon_3)\\
&= \frac{300\times 60500000+100\times 60500000\varepsilon_1+300\times 20122500\varepsilon_3+100\times 20122500\varepsilon_1\varepsilon_3}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\\
&= \frac{18150000000+6050000000\varepsilon_1+6036750000\varepsilon_3+2012250000\varepsilon_1\varepsilon_3}{6\times 200\times 10^{9}\times 10^{-6}\times 200}\\
&= 75.625+25.20833\varepsilon_1+25.15313\varepsilon_3+8.384375\varepsilon_1\varepsilon_3
\end{aligned}
\tag{36}
$$

Because of the independence between ε_1 and ε_3, then:

$$
\begin{aligned}
&75.625-25.20833-25.15313+8.384375\le\\
&75.625+25.20833\varepsilon_1+25.15313\varepsilon_3+8.384375\varepsilon_1\varepsilon_3\le\\
&75.625+25.20833+25.15313+8.384375
\end{aligned}
\tag{37}
$$

i.e. $33.64792\le 75.625+25.20833\varepsilon_1+25.15313\varepsilon_3+8.384375\varepsilon_1\varepsilon_3\le 134.3708$

Therefore, the affine form $84.00938+50.36146\varepsilon_4$ of new noise symbol ε_4 can be introduced to replace the formula above, working out the final returning value of structural response function:

$$f = 84.00938 + 50.36146\varepsilon_4 \tag{38}$$

So, $f \in [f^l, f^u] = [33.64792, 134.3708]$, the unit is mm, then, $f^c = 84.00938$, $f^r = 50.36146$, and substitute them into formula (19), the interval value of non-probabilistic reliability index η by the improved affine arithmetic of this paper is obtained:

$$\eta = \frac{y^c - f^c}{y^r + f^r} = \frac{90 - 84.00938}{10 + 50.36146} = 0.099246 \tag{39}$$

5.2.3. Traditional Affine Algorithm: The first half processes of traditional affine algorithm and arithmetic of this paper are same. Therefore it also obtains formula (30) by substituting affine form transformed from interval variable into formula (34), *i.e.*:

$$f = \frac{(300 + 100\varepsilon_1)}{6 \times 200 \times 10^9 \times 10^{-6} \times 200} \times (59375000 + 20625000\varepsilon_2 + 1125000\varepsilon_2^{\ 2} - 502500\varepsilon_2^{\ 3}) \tag{40}$$

The formula (40) contains quadratic term $1125000\varepsilon_2^{\ 2}$ and cubic term $-502500\varepsilon_2^{\ 3}$. Because $\varepsilon_2 \in [-1,1]$, $0 \le 1125000\varepsilon_2^{\ 2} \le 1125000$, the affine form $-562500 + 562500\varepsilon_3$ of the new noise symbol $\varepsilon_3 \in [-1,1]$ is introduced to replace them.

Similarly, $-502500 \le -502500\varepsilon_2^{\ 3} \le 502500$, the affine form $502500\varepsilon_4$ of the new noise symbol $\varepsilon_4 \in [-1,1]$ is introduced to replace it. So:

$$\begin{aligned} f &= \frac{(300 + 100\varepsilon_1)}{6 \times 200 \times 10^9 \times 10^{-6} \times 200} \times \begin{pmatrix} 59375000 + 20625000\varepsilon_2 - \\ 562500 + 562500\varepsilon_3 + 502500\varepsilon_4 \end{pmatrix} \\ &= \frac{(300 + 100\varepsilon_1)}{6 \times 200 \times 10^9 \times 10^{-6} \times 200} \times (58812500 + 20625000\varepsilon_2 + 562500\varepsilon_3 + 502500\varepsilon_4) \\ &= \frac{\begin{pmatrix} 17643750000 + 5881250000\varepsilon_1 + 6187500000\varepsilon_2 + 168750000\varepsilon_3 + \\ 150750000\varepsilon_4 + 2062500000\varepsilon_1\varepsilon_2 + 56250000\varepsilon_1\varepsilon_3 + 50250000\varepsilon_1\varepsilon_4 \end{pmatrix}}{6 \times 200 \times 10^9 \times 10^{-6} \times 200} \end{aligned} \tag{41}$$

As $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in [-1,1]$ and the numerator contain quadratic term $20625000\varepsilon_1\varepsilon_2$ which meets the relation $-20625000 \le 20625000\varepsilon_1\varepsilon_2 \le 20625000$, the affine form $20625000\varepsilon_5$ of the new noise symbol $\varepsilon_5 \in [-1,1]$ is introduced to replace it. Similarly, the quadratic term $56250000\varepsilon_1\varepsilon_3$ can be replaced by the affine form $56250000\varepsilon_6$ of the new noise symbol $\varepsilon_6 \in [-1,1]$, $50250000\varepsilon_1\varepsilon_4$ can be replaced by the affine form $50250000\varepsilon_7$ of $\varepsilon_7 \in [-1,1]$, then:

$$f = \frac{\begin{pmatrix} 17643750000 + 5881250000\varepsilon_1 + 6187500000\varepsilon_2 + 168750000\varepsilon_3 + \\ 150750000\varepsilon_4 + 20625000\varepsilon_5 + 56250000\varepsilon_6 + 50250000\varepsilon_7 \end{pmatrix}}{6 \times 200 \times 10^9 \times 10^{-6} \times 200} \tag{42}$$

The value range of the numerator is: $[-32201000000, 32201000000]$.

Therefore, $f \in [12.86042, 134.1708]$, $f^c = (12.86042 + 134.1708)/2 = 73.51563$,

$f^r = (134.1708 - 12.86042)/2 = 60.65521$.

Substitute them into formula (19), the non-probabilistic reliability index $\eta = \dfrac{90 - 73.51563}{10 + 60.65521} = 0.233307$.

Based on estimation, the actual response interval of f is $[33.33333, 135]mm$, *i.e.* $f^c = (135 + 33.33333)/2 = 84.16667$, $f^r = (135 - 33.33333)/2 = 50.83333$.

Substitute it into formula (19), the true value of non-probabilistic reliability index $\eta_{TRUE} = \dfrac{90 - 84.16667}{10 + 50.83333} = 0.09589$. Table 2 shows the calculated results through methods of this paper or the others.

Table 2. The Results Comparison of Different Method

	M	η	Error rate to η_{True}
True value	$[5.8333, 60.8333]$	0.09589	0%
Interval algorithm	$[-70, 70]$	0	100%
Conventional affine algorithm	$[-54.1708, 87.13958]$	0.233307	143.3061%
Method of this paper	$[5.9906, 60.3614]$	0.099246	3.4995%

It's obvious in Table 2 that the interval algorithm results include considerable errors and the reliability index is smaller than its true value η_{TRUE} , *i.e.* the result of the reliability index of interval algorithm is conservative. The reason is that the standard interval algorithm neglects the auto-correlation of a and $a^2(3l - a)$ during the process of computing and causes the error explosion. The standard interval algorithm can get $a^2(3l - a) \in [36000000, 90000000]$, while its true value interval is $[40000000,\ 81000000]$.

Moreover the reliability index value of the conventional affine algorithm also contains considerable errors but is higher than η_{TRUE} , in other words, its result η is overly optimistic without enough security. The reason is although the conventional affine algorithm is able to handle the relativity among the interval input variables in structural response function and allowable interval variables of structural response, it cannot take the correlation among the first degree item, quadratic term, even the multiple-item and nonlinear term which expressed by same noise symbol into effective treatment. In addition, the method is approximate optimization for multiplication computation. At the same time, the first degree item, quadratic term and related item appear frequency in this example, some approximate errors and considerable calculating errors are produced.

Only minor errors exist in the results of this paper's method because of inevitable rounding errors and accumulated errors.

6. Conclusions

Aimed at that standard interval arithmetic is easy to cause error explosion, as well as traditional affine arithmetic is easy to cause calculation error due to not considering the correlation among the first degree item, quadratic term, even the multiple-item and the nonlinear term expressed by the same noise, this paper proposed an improved algorithm which took affine algorithm operation as the main body. When there are those correlation items, the algorithm calculates the upper and lower bound of their total value with help of interval arithmetic. The method improved the conventional affine operation by combining interval arithmetic. When it applied to calculate the non-probabilistic reliability index, the results has much higher accuracy than the interval arithmetic and the conventional affine algorithm.

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References

[1] H. Shou, R. Martin, I. Voiculescu, A. Bowyer and G. Wang, “Affine arithmetic in matrix form for polynomial evaluation and algebraic curve drawing”, Progress in Natural Science, vol. 12, **(2002)**, pp. 77-80.

[2] D. Wang and J. Li, “A reliable approach to compute the static response of uncertain structural system”, Chinese Journal of Computations Mechanics, (In Chinese), vol. 20, **(2003)**, pp.662- 668.

[3] C. Liu and O. Chen, “Analysis of the random structure with the rough set theory”, Journal of Mechanical Strength, (In Chinese), vol. 26, **(2004)**, pp. 274 -277.

[4] S. Guo and Z. LV, “Interval arithmetic and static interval finite element method”, Application Mathematics and Mechanics, vol. 22, **(2001)**, pp. 1249-1254.

[5] X. Wang and Z. Qiu, “Robust reliability of structural vibration”, Journal of Beijing University of Aeronautics and Astronautics, (In Chinese), vol. 29, **(2003)**, pp. 1006- 1010.

[6] K. S. Goldenstein, C. Vogler and D. Metaxas, “Statistical cue integration in DAG deformable models”, IEEE Transactions on Pattern Analysis and Machine Intelligence, vol. 25, **(2003)**, pp. 801-813.

[7] Y. Xie and J. Chen, “Stability analysis of uncertain system using affine inequality”, Chinese High Technology Letters, (In Chinese), vol. 4, **(2008)**, pp. 867-871.

[8] N. Femia and G. Spagnuolo, “True worst-case circuit tolerance analysis using genetic algorithms and affine arithmetic”, IEEE Transactions on Circuits and Systems, vol. 47, **(2000)**, pp. 1285-1296.

[9] R. Martin, H. Shou and I. Voiculescu, “Comparison of interval methods for plotting algebraic curves”, Computer Aided Geometric Design, vol. 19, **(2002)**, pp553-587.

[10] J. G. Zhang, J. J. Chen, T. Jiang and X. S. Ma, “Research on the calculation of the non-probabilistic reliability of uncertain structures”, Journal of Mechanical Strength, (In Chinese), vol. 29, **(2007)**, pp. 58-62.

[11] J. H. Ma and F. X. Zhu, “Reliability measures of structures based on interval”, Journal of Mechanical Strength, (In Chinese), vol. 34, **(2012)**, pp. 48-52.

[12] T. Jiang, J. J. Chen and C. J. Zhang, “Non-probabilistic reliability index and affine arithmetic”, Journal of Mechanical Strength, (In Chinese), vol. 29, **(2007)**, pp. 251-255.

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