International Journal of Control and Automation
Vol.9, No.2 (2016), pp.191-198
http://dx.doi.org/10.14257/ijca.2016.9.2.18

Robust H_∞ Guaranteed Cost Control for a Class of Uncertain Nonlinear Time-delay Systems

Cheng'ai Yuan,Wenhui Dong and Peishu Qu

College of Physics and Electronic engineering, Dezhou University, Dezhou 253023, China
eternitybeyong@163.com,dongwh_81@163.com,qupsh@163.com

Abstract

The design of robust guaranteed cost nonlinear controller is studied for a class of time-delay nonlinear systems with norm-bound time-varying uncertainties. By means of Lyapunov functional and linear matrix inequality (LMIs) technique, a sufficient condition of H_∞ robust stabilization which satisfies guaranteed cost index is obtained for the systems. Furthermore, the state feedback nonlinear controller of the systems is given in terms of the feasible solutions of LMIs. Finally, an illustrative example shows the effectiveness of the proposed method.

Keywords*: Nonlinear systems, Time-delay systems, H-infinity control, Guaranteed cost control*

1. Introduction

It is well known that uncertainty and time-delay widely exist in practical control systems, such as power systems, network systems, communication systems, and so on. They are often the main reasons for instability and performance decrease of systems. So it is necessary to study the uncertain time-delay systems and many results have been achieved [1-4].

H_∞ control [5-7] is an effective method for handling disturbance, and has attracted extensively concern because of its good application background. Guaranteed cost control [8-9] makes the systems have a certain upper bound of the quadratic performance index, and guarantees the closed-loop systems stable. In recent years, the research of multi-objective control has attracted more and more attention, and many results have been reported [10-11].

At present, the controllers are mostly linear. Because of the complexity of the actual control systems, sometimes, the linear controllers are not suitable for some systems. Until now, there are a few researches about nonlinear controllers, so the study of nonlinear controllers is very important and meaningful.

In this paper, based on the Lyapunov stability theory and linear matrix inequality (LMI) tools, we apply the advantage of H_∞ control and guaranteed cost control to the research of uncertain nonlinear time-delay system. The closed-loop system gets a good dynamic performance through designing a nonlinear state feedback controller. Finally, an illustrative example shows the effectiveness of the proposed method.

ISSN: 2005-4297 IJCA

2. Problem Description

Consider the following uncertain nonlinear time-delay systems:

$$\begin{cases} \dot{x}(t) = \bar{A}x(t) + \bar{A}_1 x(t-\tau) + b\phi(u(t)) + Dw(t) \\ z(t) = Cx(t) \\ x(t) = \varphi(t),\ t \in [-\tau, 0] \end{cases} \quad (1)$$

where $\bar{A} = A + \Delta A(t)$, $\bar{A}_1 = A_1 + \Delta A_1(t)$. $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}$ is the control input, $z(t) \in \mathbf{R}^p$ is the controlled output, $w(t) \in \mathbf{R}^q$ is the disturbance input that belongs to $L_2[0, \infty]$. A, A_1, b, C, D are known real constant matrices with appropriate dimensions. $\phi : \mathbf{R} \to \mathbf{R}$ is the sufficiently smooth function satisfying the conditions [12]:

$\phi(0) = 0$, $\phi'(0) = 0$, $\cdots$, $\phi^{(m-1)}(0) = 0$, and $\phi^{(m)}(0) \neq 0$ (m is a positive odd number). $\tau > 0$ is a constant time delay. $\Delta A(t)$, $\Delta A_1(t)$ are unknown matrices representing norm-bounded parametric uncertainties and are assumed to be of the form:

$$[\Delta A(t)\ \ \Delta A_1(t)] = HF(t)[N_a\ \ N_{a1}] \quad (2)$$

where H, N_a, N_{a1} are known real constant matrices with appropriate dimensions, and $F(t)$ is an unknown real matrix satisfying $F^{\mathrm{T}}(t)F(t) \leq I$.

For the nonlinear systems (1), we choose the following performance index:

$$J = \int_0^{+\infty} (x^{\mathrm{T}}(t) Q x(t) + \mathrm{r}\phi^2(u(t)))\mathrm{dt} \quad (3)$$

where Q is a symmetric positive-definite matrix, $\mathrm{r} > 0$ is a known constant.

The closed-loop system of the system (1) controlled by $u(t) = \alpha(x(t))$ can be written as:

$$\begin{cases} \dot{x} = \bar{A}x(t) + \bar{A}_1 x(t-\tau) + b\phi(\alpha(x(t))) + Dw(t) \\ z(t) = Cx(t) \\ x(t) = \varphi(t),\ t \in [-\tau, 0] \end{cases} \quad (4)$$

Definition 1 Given a constant $\gamma > 0$, matrix $Q > 0$, the nonlinear systems (1) and the performance index (3), if there exists a nonlinear controller $u(t) = \alpha(x(t))$, for all admissible uncertainties (2), the corresponding closed-loop system (4) has the following properties:

1) When the external disturbance $w(t) = 0$, the closed-loop system (4) is locally asymptotically stable.

2) Under condition $x(t) \equiv 0, t \in [-\tau, 0]$, when $w(t)$ belongs to $L_2[0, \infty]$, and $w(t) \neq 0$, the controlled output $z(t)$ satisfies $\|z(t)\|_2 \leq \gamma \|w(t)\|_2$.

3) When the external disturbance $w(t) = 0$, performance index (3) of the closed-loop system (4) is bounded.

Then, $u(t) = \alpha(x(t))$ is a robust H_∞ guaranteed cost controller of system (1).

Lemma 1[13] Given matrices Σ, T, and L with appropriate dimensions and with Σ symmetrical,

$\Sigma + T F^{T}(t) L + L^{T} F(t) T^{T} < 0$

for any $F(t)$ satisfying $F^{T}(t)F(t) \le I$, if and only if there exists a scalar $\varepsilon > 0$, such that $\Sigma + \varepsilon L^{T} L + \varepsilon^{-1} T T^{T} < 0$.

3. Main Results

Theorem 1. For uncertain nonlinear time-delay system (1), given $\gamma > 0$ and performance index (3), if there exist symmetric matrices $P > 0$, $R > 0$ and row vector K, such that

$$\begin{bmatrix} \Xi & P\bar{A}_1 & PD & C^{T} \\ * & -R & 0 & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \tag{5}$$

$$K^{T} K < G_0 \ (\text{given}\, G_0 > 0) \tag{6}$$

where, $\Xi = P(\bar{A} + bK) + (\bar{A} + bK)^{T} P + R + Pbb^{T} P + Q + [\mathrm{r}(1+\mathrm{s})^2 + \mathrm{s}^2] K^{T} K$,and $\mathrm{s} \ge 0$ satisfying:

$$\left| \frac{\phi^{(m+1)}(\xi)}{(m+1)!} \left(\frac{m!}{\phi^{(m)}(0)} \right)^{\frac{m+1}{m}} \left(\lambda_{max}(G_0 \mu^2) \right)^{\frac{1}{2m}} \right| \le \mathrm{s} \tag{7}$$

$\lambda_{\max}(.)$ denotes the maximum eigenvalue of matrix $(.)$, μ is the given positive constant.Then the nonlinear state feedback controller is a robust H_∞ guaranteed cost controller of system (1).

$$u(t) = \alpha(x(t)) = \left(\frac{m!(K x(t))}{\phi^{(m)}(0)} \right)^{\frac{1}{m}} \tag{8}$$

Proof: Consider the domain of $x(t) = 0$ as follows $\Omega = \{x(t): \|x\| \le \mu\}$ .From (6) and (8), we can obtain $u(t)$ is bounded on Ω, therefore $\phi^{(m+1)}(u(t))$ is bounded on Ω .From Taylor expansion formula, we have:

$$\phi(u) = \frac{\phi^{(m)}(0)}{m!} u^{m} + \frac{\phi^{(m+1)}(\xi)}{(m+1)!} u^{m+1}, \quad \text{where } \xi \in (0, u).$$

Lyapunov functional is constructed as follows:

$V(x(t)) = x^{T}(t) P x(t) + \int_{t-\tau}^{t} x^{T}(s) R x(s) \mathrm{d}s$.

Then the time-derivative of $V(x(t))$ along the solution of the system (4) gives:

$$\dot{V}(x(t)) = x^{\mathrm{T}}(t)[P(\bar{A}+bK)+(\bar{A}+bK)^{\mathrm{T}}P+R]x(t)+ \\ 2x^{\mathrm{T}}(t)P\bar{A}_1x(t-\tau)+2x^{\mathrm{T}}(t)PDw(t)-x^{\mathrm{T}}(t-\tau)Rx(t-\tau)+ \\ 2x^{\mathrm{T}}(t)Pb\frac{\phi^{(m+1)}(\xi)}{(m+1)!}\left(\frac{m!}{\phi^{(m)}(0)}\right)^{\frac{m+1}{m}}(Kx)^{1+\frac{1}{m}}.$$

For the last term of the above formula, from (6) and (7), we can obtain:

$$2x^{\mathrm{T}}(t)Pb\frac{\phi^{(m+1)}(\xi)}{(m+1)!}\left(\frac{m!}{\phi^{(m)}(0)}\right)^{\frac{m+1}{m}}(Kx)^{1+\frac{1}{m}} \le x^{\mathrm{T}}(t)Pbb^{\mathrm{T}}Px(t)+s^2x^{\mathrm{T}}(t)K^{\mathrm{T}}Kx(t).$$

Therefore, we have:

$$\dot{V}(x(t)) \le x^{\mathrm{T}}(t)[P(\bar{A}+bK)+(\bar{A}+bK)^{\mathrm{T}}P+R+Pbb^{\mathrm{T}}P+s^2K^{\mathrm{T}}K]x(t)+ \\ 2x^{\mathrm{T}}(t)P\bar{A}_1x(t-\tau)+2x^{\mathrm{T}}(t)PDw(t)-x^{\mathrm{T}}(t-\tau)Rx(t-\tau). \qquad (9)$$

First, prove the local asymptotic stability of the closed-loop system (4). When $w(t)=0$ and $x(t)\in\Omega$, from (9), we can obtain:

$$\dot{V}(x(t)) \le [x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)]\times\begin{bmatrix} P(\bar{A}+bK)+(\bar{A}+bK)^{\mathrm{T}}P+ \\ R+Pbb^{\mathrm{T}}P+s^2K^{\mathrm{T}}K & P\bar{A}_1 \\ * & -R\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}.$$

According to (5) , when $w(t)=0$ and $x(t)\in\Omega$,we have

$$\dot{V}(x(t)) < -x^{\mathrm{T}}(t)(Q+r(1+s)^2K^{\mathrm{T}}K)x(t) \le -\lambda_{min}(Q+r(1+s)^2K^{\mathrm{T}}K)\|\boldsymbol{x}(t)\|^2 \le 0 \quad (10)$$

where $\lambda_{\min}(.)$ denotes the minimum eigenvalue of $(.)$.

According to the Lyapunov stability theory, the closed-loop system (4) is locally asymptotically stable. Then, from (10), we can obtain

$$x^{\mathrm{T}}(t)(Q+r(1+s)^2K^{\mathrm{T}}K)x(t) < -\dot{V}.$$

By making integral on both sides of the above formula from 0 to $+\infty$ about t, and using the asymptotic stability of the system (4), we can obtain:

$$J=\int_0^{+\infty}(x^{\mathrm{T}}(t)Qx(t)+r\phi^2(u(t)))\mathrm{d}t \\ \le \int_0^{+\infty}x^{\mathrm{T}}(t)(Q+r(1+s)^2K^{\mathrm{T}}K)x(t)\mathrm{d}t < \varphi^{\mathrm{T}}(0)P\varphi(0)+\int_{-\tau}^{0}\varphi^{\mathrm{T}}(s)R\varphi(s)\mathrm{d}s$$

Therefore, $u(t)=\alpha(x(t))=\left(\frac{m!(Kx(t))}{\phi^{(m)}(0)}\right)^{\frac{1}{m}}$ is a guaranteed cost controller of the closed-loop system (4).

Finally, we will establish the H_∞ performance of the system (4). We introduce the following performance index

$$J_1=\int_0^{+\infty}(z^{\mathrm{T}}(t)z(t)-\gamma^2w^{\mathrm{T}}(t)w(t))\mathrm{d}t$$

Under the zero initial condition, for any $w(t) \neq 0$ and $w(t) \in L_2[0, +\infty]$, from (9), it can be shown that:

$$J_1 \leq \int_0^{+\infty} (z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) + \dot{V}(x(t))\mathrm{d}t \leq [x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)\ \ w^{\mathrm{T}}(t)] \times$$

$$\begin{bmatrix} \Xi - Q - \mathrm{r}(1+\mathrm{s})^2 K^{\mathrm{T}}K & P\bar{A}_1 & PD \\ * & -R & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \times [x^{\mathrm{T}}(t)\ \ x^{\mathrm{T}}(t-\tau)\ \ w^{\mathrm{T}}(t)]^{\mathrm{T}}.$$

According to the conditions of Theorem1, we have $J_1 < 0$.

Remark 1. In the system (1), if $\phi(u) = u$, we can get $u = Kx$ from formula (8), so the results of this paper include the situation of the linear H_∞ guaranteed cost controller.

Remark 2. The inequality of Theorem 1 has uncertain matrices , and can not be solved directly using the LMI toolbox. In order to solve the problem, we will deal with the uncertain matrices of theorem 1, and transform the inequality to LMI.

Theorem 2. For uncertain nonlinear time-delay system (1), given $\gamma > 0$ and performance index (3), if there exist symmetric matrices $0 < X < I$ and $\bar{R}$, row vector W and constant $\varepsilon > 0$, such that

$$\begin{bmatrix} \Theta & A_1 X & W^{\mathrm{T}} & XC^{\mathrm{T}} & D & XN_a^{\mathrm{T}} & XQ \\ * & -\bar{R} & 0 & 0 & 0 & XN_{a1}^{\mathrm{T}} & 0 \\ * & * & -\sigma I & 0 & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & -\varepsilon I & 0 \\ * & * & * & * & * & * & -Q \end{bmatrix} < 0 \qquad (11)$$

$$\begin{bmatrix} -G & W^{\mathrm{T}} \\ * & -I \end{bmatrix} < 0, \ (\text{given}\, G > 0) \qquad (12)$$

where $\Theta = AX + XA^{\mathrm{T}} + bW + W^{\mathrm{T}}b^{\mathrm{T}} + \bar{R} + bb^{\mathrm{T}} + \varepsilon HH$, $\sigma = [\mathrm{r}(1+\mathrm{s})^2 + \mathrm{s}^2]^{-1}$ and $\mathrm{s} \geq 0$ satisfying (7).

Furthermore, if the linear matrix inequality (11) and (12) have feasible solutions, and then the nonlinear state feedback controller:

$$u(t) = \alpha(x(t)) = \left(\frac{\mathrm{m}!(K\,x(t))}{\phi^{(\mathrm{m})}(0)} \right)^{\frac{1}{\mathrm{m}}}$$

is a robust H_∞ guaranteed cost controller of system (1). The corresponding performance index satisfying:

$$J < \varphi^{\mathrm{T}}(0)X^{-1}\varphi(0) + \int_{-\tau}^{0} \varphi^{\mathrm{T}}(s)X^{-1}\bar{R}X^{-1}\varphi(s)\mathrm{d}s\,.$$

Proof: According to the Schur complement lemma, matrix inequality (5) is equivalent to

$$\begin{bmatrix} \Xi_1 & P\bar{A}_1 & K^{\mathrm{T}} & C^{\mathrm{T}} & PD \\ * & -R & 0 & 0 & 0 \\ * & * & -\sigma I & 0 & 0 \\ * & * & * & -I & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix} < 0 \tag{13}$$

where $\Xi_1 = P(\bar{A}+bK)+(\bar{A}+bK)^{\mathrm{T}}P+R+Pbb^{\mathrm{T}}P+Q$, $\sigma=[\mathrm{r}(1+\mathrm{s})^2+\mathrm{s}^2]^{-1}$.

Using the lemma 1 and the Schur complement lemma, we can get that the inequality (13) is established if and only if there exists $\varepsilon > 0$, such that:

$$\begin{bmatrix} \Xi_2 & PA_1 & K^{\mathrm{T}} & C^{\mathrm{T}} & PD & N_a^{\mathrm{T}} \\ * & -R & 0 & 0 & 0 & N_{a1}^T \\ * & * & -\sigma I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \tag{14}$$

where $\Xi_2 = P(A+bK)+(A+bK)^{\mathrm{T}}P+R+Pbb^{\mathrm{T}}P+Q+\varepsilon PHH^{\mathrm{T}}P$.

Pre-and post-multiplying (14) by diag$\{P^{-1}\ \ P^{-1}\ \ I\ I\ \ I\ I\}$, respectively, and letting $X=P^{-1}, W=KP^{-1}, \bar{R}=P^{-1}RP^{-1}$, using the Schur complement lemma, we can get that (14) is equivalent to (11).

Letting $G=P^{-1}G_0P^{-1}$, it is obvious that (6) is equivalent to (12). Therefore, if the linear matrix inequality (11) and (12) have feasible solutions X, W, $\bar{R}$ and ε, then the nonlinear state feedback controller

$$u(t)=\alpha(x(t))=\left(\frac{\mathrm{m}!(K\,x(t))}{\phi^{(\mathrm{m})}(0)}\right)^{\frac{1}{\mathrm{m}}}$$

is a robust H_∞ guaranteed cost controller of system (1). The corresponding performance index satisfying:

$$J<\varphi^{\mathrm{T}}(0)X^{-1}\varphi(0)+\int_{-\tau}^{0}\varphi^{\mathrm{T}}(s)X^{-1}\bar{R}X^{-1}\varphi(s)\mathrm{d}s\,.$$

4. Simulation Example

Consider the uncertain nonlinear time-delay systems (1), parameters are as follows:

$$A=\begin{bmatrix} -5 & 1 \\ 1 & -4 \end{bmatrix}, A_1=\begin{bmatrix} 0 & 0 \\ -0.1 & -0.1 \end{bmatrix}, b=\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C=D=\begin{bmatrix} 0.4 & 0.1 \\ 0 & 0.4 \end{bmatrix}, \Delta A(t)=\begin{bmatrix} 0.3\sin t & 0 \\ 0 & 0.1\cos t \end{bmatrix} \Delta A_1(t)=\begin{bmatrix} 0.2\sin t & 0 \\ 0.3\cos t & 0 \end{bmatrix},$$

We choose

$$H=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ N_a=\begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix},\ N_{a1}=\begin{bmatrix}0.2 & 0\\ 0.3 & 0\end{bmatrix},\ F(t)=\begin{bmatrix}\sin t & 0\\ 0 & \cos t\end{bmatrix},$$

$\phi(u)=u^3$, $\varphi(t)=\begin{bmatrix}e^{-\frac{1}{2}t}\\ -e^{-\frac{1}{2}t}\end{bmatrix}$, $\tau=1$, $r=1$, $Q=I_2$, $\gamma=0.6$, $\mu=1$, $G=I_2$. We can get

$$m=3,\ s=0,\ \sigma=1,\ \varphi(0)=\begin{bmatrix}1\\ -1\end{bmatrix}.$$

Solve the inequality of linear matrix (11) and (12), and it is feasible with the following results:

$$X=\begin{bmatrix}0.7102 & 0.0479\\ 0.0479 & 0.7757\end{bmatrix}$$

$$W=\begin{bmatrix}-0.1195 & -0.3379\end{bmatrix}$$

Furthermore, the nonlinear robust H_∞ guaranteed cost controller of system (1) can be obtained as:

$$u(t)=-(0.1395x_1+0.4270x_2)^{\frac{1}{3}}.$$

5. Conclusion

Uncertainty, time-delay and nonlinear widely exist in most practical control systems. In this paper, the problem of guaranteed cost control for a class of uncertain nonlinear time-delay continuous systems is discussed. A sufficient condition is presented to ensure the system to be stable with performance condition and guaranteed cost performance index. Furthermore, the design method of the nonlinear state feedback controller is proposed. The control function discussed in this paper can be a sufficiently smooth odd function, which include the common linear function.

References

[1] H.S. Wu, "Adaptive robust control of uncertain nonlinear systems with nonlinear delayed state perturbations", Automatica, vol. 45, no. 8, **(2009)**, pp.1979-1807.

[2] H.Y. Shao, "Improved delay-dependent stability criteria for systems with a delay varying in a range", Automatica, vol. 44, no. 12, **(2008)**, pp. 3215-3218.

[3] S.P. Ma, C.H. Zhang and Z.L. Cheng, "Delay-dependent Robust stability and stabilization for uncertain discrete singular systems with delays", Asian journal of control, vol. 11, no. 3, **(2009)**, pp. 309-318.

[4] C. Lin, Q.G. Wang and T.H. Lee, " A less conservative robust stability test for linear uncertain time-delay systems", IEEE trans on automatic control, vol. 51, no. 1, **(2006)**, pp. 87-91.

[5] S.Y. Xu, L. James and Y. Zou, "New results on delay-dependent robust H∞control for systems with time-varying delays", Automatica, vol. 42, no. 2, **(2006)**, pp. 343-348.

[6] V. Suplin, E. Fridman and U.H. Shaked, "Infinity control of linear uncertain time-delay systems-A projection Approach", IEEE trans on automatic control, vol. 51,no. 4, **(2006)**, pp. 680-685.

[7] F. Yang, Z. Wang, "Robust H-infinity control with missing measurements and time delays", IEEE trans on automatic control, vol. 52, **(2007)**, pp. 1666-1672.

[8] H. Mukaindhani, "An LMI approach to guaranteed cost control for uncertain delay systems", IEEE Tran on circuits and systems: Fundamental Theory and Applications, vol. 50, no. 6, pp. 795-801.

[9] S.H. Esfakani, S.O.R. Moheimani, "LMI approach to suboptimal guaranteed cost for uncertain time-delay systems", IEEE proceedings control theory and applications, vol. 145, no. 6, **(1998)**, pp. 491- 498.

[10] X. Fu and C.N. Tong, "Robust H∞ guaranteed cost control for the uncertain systems with dependent time-delay", Journal of Central South University (Science and Technology), vol. 37, no.1, **(2006)**, pp.

3744-3752.
[11] W. Yuan, Q. Zhang and B. Du, "H∞ guaranteed cost control for uncertain discrete singular systems", Systems Engineering and Electronics, vol. 29, no. 1, pp. 121-124.
[12] Q. Fu, W. Qu, C. Yang, "Robust Control for a Class of Nonlinear Systems with Bounded Disturbance", Acta Automatica Sinica, vol. 33, no. 11, **(2007)**, pp. 1209-1210.
[13] L. Xie, "Output feedback $H\infty$ control of systems with parameter uncertainty", International Journal of control, vol.63, no.4, pp. 741-750.