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Observer-Based Finite-Time Control for Robotic Manipulators with Uncertainties

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Abstract

In this paper, an observer-based finite-time control approach for robotic manipulators with uncertainties is proposed. First, a nonlinear observer is introduced to guarantee the finite-time convergence of the unmeasurable joint velocities, and a finite-time controller is designed based on the estimated velocities. Second, in the presence of uncertainties, a nonlinear sliding mode observer is further developed to guarantee the robustness to uncertainties. The proposed observers and the controllers are all finite-time convergent. The validity of the proposed methods are demonstrated by the simulations on a two-link robotic manipulator.

***Keywords**: robotic manipulator; observer; finite-time control; uncertainty*

1. Introduction

Robotic manipulator, one kind of the most important machinery in the field of industrial automation, is a coupled complex dynamic system with high linearity and strong time variation. In robotic manipulators, there always exist uncertainties and external disturbances, which may result in performance degradation. Due to its potential to achieve high tracking precision and good transient performance with high speed, continuous finite-time control has been paid more and more attention to in recently years. Different from asymptotic stability, finite-time stability [1] implies that system states can be stabilized to equilibrium in finite time and may give rise to fast transient and high-precision performances, so that it is especially useful for the high-quality control of industrial robots. Until now, several finite-time control approaches have been proposed, such as the finite-time stability approach of a homogeneous system [2, 3], the finite-time Lyapunov stability approach [4-6] and the terminal sliding mode approach[7-11] . Hong *et al*. firstly proposed a finite-time controller for robot manipulators based on state feedback and dynamic output feedback control [12]. Su proposed a global finite-time tracking controller by replacing the linear errors in linear proportional-derivative plus scheme with nonsmooth but continuous exponential ones[13]. Zhao *et al*. presented a robust finite-time stability control approach for robot manipulators via the backstepping method [6]. Su *et al*. addressed the global finite-time tracking of robotic manipulators by replacing the linear errors with nonlinear exponential-like ones [14]. However, most of the above-mentioned finite-time control approaches present two drawbacks: one is that the robustness of finite-time control is not considered, the other is that the states of robotic manipulators are all assumed to be available, which are different from many practical cases in which joint velocities are unmeasurable.

As we know, in high-performance robotic manipulators, tachometers are generally needed to obtain adequate velocity feedback signals even though high-precision measurements of joint displacements are typically available. However, due to the high cost of tachometers that may equal the cost of the motor, an observer that can generate clean velocity signals from joint displacement measurements, reduce the system cost and furthermore potentially improve the system performance is needed. Since linear observers

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cannot achieve adequate performance for robot systems, nonlinear observers have been paid more attemtion to [15]. Much of the effort has resulted in the extensions of sliding-mode observers [10, 16-18]. Different from linear observers that only achieve asymptotic convergence, sliding mode observers enable the output estimation error to converge to zero in finite time and are of strong robustness to uncertainties, so that widely applied to mechanical systems. Davila *et al*. developed a super-twisting second-order sliding-mode observer with finite-time convergence for uncertain mechanical systems [17]. Tan *et al*. proposed a terminal sliding mode observer for a class of nonlinear systems to achieve finite-time convergence for all error states [10]，Zhang *et al*. proposed an adaptive observer and a robust observer for a class of MIMO nonlinear systems with unmodeled dynamics, unknown parameters and external disturbance [19], which can make the observation error arbitrarily small.

In this paper, we develop an observer-based finite-time control approach for robotic manipulators based on the work of Davila *et al*. [17], in which the sliding mode observer and the controller are both finite-time convergent, so that high-performance robotic manipulator even without tachometers can be successfully designed. The main advantages of the proposed approach are as follows: (i) An observer-based finite-time control approach for robotic manipulator is proposed; and (ii) a nonlinear sliding mode observer is further developed to guarantee the robustness to uncertainties. It is proved that the proposed observers and the controllers are all finite-time convergent.

The paper is organized as follows: (1) in Section 2, the preliminaries about the finite-time stability and the related lemmas are introduced; in Section 3, based on full state information assumption, the robust finite-time control is discussed and the corresponding stability is analyzed; a nonlinear observer is designed to estimate joint velocities and guarantee the finite-time stability; and, in Section 4, an observer-based finite-time control method considering system uncertainties is developed. Simulation results are presented in Section 5 to demonstrate the effectiveness of the proposed methods. Finally, some conclusions are drawn in Section 6.

2. Preliminaries

2.1. Related Lemmas

Definition 1 [2, 20]. Considering a time-invariant nonlinear system in the form of

$$\dot{\zeta}(t) = f(\zeta), \quad f(0) = 0, \zeta \in \mathbf{R}^n, \zeta(0) = \zeta_0 \tag{1}$$

where $f : \hat{U}_0 \to R^n$ is continuous in an open neighborhood $\hat{U}_0$ of the origin. The equilibrium $\zeta = 0$ of the system is (locally) finite-time stable if:

(i) It is asymptotically stable in $\hat{U}$, an open neighborhood of the origin, with $\hat{U} \subseteq \hat{U}_0$;

(ii) It is finite-time convergent in $\hat{U}$, that is, for any initial condition $\zeta_0 \in \hat{U} \setminus \{0\}$, there is a settling time T>0 such that each solution $\psi(t, \zeta_0)$ of system (1) is defined with $\psi(t, \zeta_0) \in \hat{U} \setminus \{0\}$ for $t \in [0, T)$ and satisfies

$$\lim_{t \to T(\zeta_0)} \psi(t, \zeta_0) = 0 \tag{2}$$

And, if $t \geq T$, $\zeta(t, \zeta_0)$=0. Moreover, if $\hat{U} = R^n$, the origin $\zeta = 0$ is globally finite-time stable.

A scalar function $V(\zeta)$ is said to be homogeneous of degree $\sigma > 0$ with respect to (r_1, …, r_n), $r_i > 0$, i=1, …, n, if for any given $\varepsilon > 0$,

$$V\left(\varepsilon^{r_1}\zeta_1, \cdots, \varepsilon^{r_n}\zeta_n\right) = \varepsilon^{\sigma} V(\zeta) \tag{3}$$

A continuous vector field $f(\zeta)=\left[f_1(\zeta),\cdots,f_n(\zeta)\right]^{\mathrm{T}}$ is said to be homogeneous of degree with respect to $r=(r_1, \ldots, r_n)$ if, for any given $\varepsilon > 0$, there exists

$$f\left(\varepsilon^{r_1}\zeta_1,\cdots,\varepsilon^{r_n}\zeta_n\right)=\varepsilon^{\sigma}f\left(\zeta\right), i=1,\cdots,n\ \forall\zeta\in\mathbf{R}^n \tag{4}$$

System (1) is said to be homogeneous if $f(\zeta)$ is homogenous.

The finite-time stability lemmas are summarized in the following.

Lemma 1 [21]. Supposing that there exists a continuously differentiable function $V(x)$ defined in neighborhood $U\subset\mathbf{R}^n$ of the origin, and that real numbers $c > 0$ and $0 < \alpha < 1$, such that

(1) $V(x)$ is positive definite in U;

(2) $\dot{V}(x)+cV^{\alpha}(x)\le 0, \forall x\in U\setminus\{0\}$;

Then, there exists an area $U_0\subset\mathbf{R}^n$ such that any $V(x)$ starting from U_0 can reach $V(x)=0$ in finite time. The settling time T_{reach}, namely the time interval to reach $V(x)=0$, satisfies

$$T_{reach}(x_0)\le\frac{V(x_0)^{1-\alpha}}{c(1-\alpha)} \tag{5}$$

where $V(x_0)$ is the initial value of $V(x)$.

Lemma 2 [12]. Considering the following system

$$\dot{\xi}=f(\xi)+\hat{f}(\xi),\quad f(0)=0, \xi\in R^n \tag{6}$$

where $f(\xi)$ is a continuous and homogeneous vector field of degree $k<0$ with respect to $(r_1,\ldots, r_2)$ and $\hat{f}(\xi)$ satisfies $\hat{f}(0)=0$, and, assuming that $\xi=0$ is an asymptotically stable equilibrium point of the system $\dot{\xi}=f(\xi)$. Then, $\xi=0$ is a locally finite-time stable equilibrium point of the system if

$$\lim_{\varepsilon\to 0}\frac{\hat{f}_i(\varepsilon^{r_i}\xi_1,\cdots,\varepsilon^{r_i}\xi_n)}{\varepsilon^{k+r_i}}=0,\ i=1,\cdots,n, \forall\xi\ne 0. \tag{7}$$

Lemma 3 [12]. Global asymptotic stability and local finite-time stability of the closed-loop system imply global finite-time stability.

Lemma 4 [22]. For any real numbers l_i, $i = 1,\ldots, n$ and $0< \lambda < 2$, the following inequality holds:

$$\left(|l_1|^2+\cdots+|l_n|^2\right)^{\lambda}\le\left(|l_1|^{\lambda}+\cdots+|l_n|^{\lambda}\right)^2 \tag{8}$$

For the convenience of control design and analysis, we define the vector $Sig(\cdot)^{\alpha}\in R^n$ as follows:

$$Sig(\xi)^{\alpha}=[|\xi_1|^{\alpha}\operatorname{sgn}(\xi_1),\cdots,|\xi_n|^{\alpha}\operatorname{sgn}(\xi_n)]^T \tag{9}$$

where $\xi=[\xi_1,\cdots,\xi_n]^T\in\mathbf{R}^n$, $0 < \alpha < 1$, and $\operatorname{sgn}(\cdot)$ is the sign function.

2.2. Robotic Manipulator Dynamics

In this section, we consider the robotic manipulator described by

$$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+G(q)=\tau \tag{10}$$

where $q,\dot{q},\ddot{q}\in\mathbf{R}^n$ are the vectors of joint positions, velocities and accelerations, respectively; $M(q)\in\mathbf{R}^{n\times n}$ is a symmetric positive definite inertia matrix; $C(q,\dot{q})\in\mathbf{R}^{n\times n}$ is the centrifugal Coriolis matrix; $G(q)\in\mathbf{R}^n$ is the gravity term; and $\tau\in\mathbf{R}^n$ denotes the torque input vector on joints.

The following dynamic properties of the robotic manipulator will be used:

(1) Matrices $M(q)$ and $C(q,\dot{q})$ are respectively bounded by

$$M_m\le\|M(q)\|\le M_M\quad\forall q\in\mathbf{R}^n \tag{11}$$

and

$$0 < C_m \|y\|\|z\| \le \|C(x,y)z\| \le C_M \|y\|\|z\| \quad \forall x, y, z \in \mathbf{R}^n \tag{12}$$

where M_m, M_M, C_m and C_M are some unknown positive constants.

3. Observer-based Robust Finite-Time Control

3.1. Robust Finite-Time Control

The robust finite-time control problem for trajectory tracking is to design a control law τ such that the tracking error $e = 0$ can be achieved in finite time. Here, tracking errors $e(t)$ and $\dot{e}(t)$ are defined as follows:

$$\begin{cases} e = q - q_d \\ \dot{e} = \dot{q} - \dot{q}_d \end{cases} \tag{13}$$

where q_d is a twice differentiable desired trajectory and q is an actual trajectory. Letting $x_1 = e,\ x_2 = \dot{x}_1 = \dot{e}$, equation (10) can be expressed as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = M^{-1}(q)\left[\tau - C(q,\dot{q})(\dot{q}) - G(q)\right] - \ddot{q}_d \end{cases} \tag{14}$$

Introducing an auxiliary controller $\varphi(x_1) \in R^n$ with $\varphi_i(0) = 0$ and defining an error variable

$$z = x_2 - \varphi(x_1) \tag{15}$$

where $\varphi(x_1) = [\varphi_1(x_{11}), \varphi_2(x_{12}), \cdots, \varphi_n(x_{1n})]^T$.

we design the following auxiliary function to obtain the finite-time stability

$$\varphi(x_1) = -K_1 Sig(x_1)^{\alpha} \tag{16}$$

where K_1=diag(k_{11}, k_{12}, …,k_{1n}) is a positive definite diagonal matrix and $0 < \alpha < 1$.

And we have the following theorem.

Theorem 3.1. *Considering the robotic manipulator system* (14), *the robust finite-time control law is designed as follows:*

$$\tau = M(q)\left[\ddot{q}_d - x_1 - K_2 Sig(z)^{\alpha} + \dot{\varphi}(x_1)\right] + C(q,\dot{q})\dot{q} + G(q) \tag{17}$$

where z and $\varphi(x_1)$ are selected as (15) *and* (16), *respectively. K_2 = diag(k_{21}, k_{22}, …,k_{2n}) is a positive definite diagonal matrix. The closed-loop system of robotic manipulator is finite-time stable*.

In the following, the finite-time stability of robot system is proved.

Proof:

By substituting the control law (17) into (14), we can obtain the closed-loop system

$$\begin{cases} \dot{x}_1 = -K_1 Sig(x_1)^{\alpha} + z \\ \dot{z} = -K_2 Sig(z)^{\alpha} - x_1 \end{cases} \tag{18}$$

Selecting the Lyapunov function

$$V = \frac{1}{2}x_1^{\mathrm{T}} x_1 + \frac{1}{2} z^{\mathrm{T}} z \tag{19}$$

and differentiating V with respect to time, by combining (18), there comes

$$\begin{aligned}\dot{V} &= x_1^{\mathrm{T}}\dot{x}_1 + z^{\mathrm{T}}\dot{z} \\ &= x_1^{\mathrm{T}}\left(\varphi(x_1)+z\right) + z^{\mathrm{T}}[-K_2 Sig(z)^{\alpha} - x_1] \\ &= -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} + x_1^{\mathrm{T}}z + z^{\mathrm{T}}[-K_2 Sig(z)^{\alpha} - x_1] \\ &= -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} - z^{\mathrm{T}}K_2 Sig(z)^{\alpha} \\ &\le -\tilde{k}_1\left(\frac{1}{2}\sum_{i=1}^{n} x_{1i}^2\right)^{\mu} - \tilde{k}_2\left(\frac{1}{2}\sum_{i=1}^{n} z_i^2\right)^{\mu} \\ &= -\tilde{k}V^{\mu}\end{aligned} \tag{20}$$

where $\mu = (1+\alpha)/2$, $1/2 < \mu < 1$, $k_{1\min}=\min\{k_{1i}\}$, $k_{2\min}=\min\{k_{2i}\}$, $\tilde{k}_1 = 2^{\mu} k_{1\min}$, $\tilde{k}_2 = 2^{\mu} k_{2\min}$ and $\tilde{k} = \min\{\tilde{k}_1, \tilde{k}_2\}$ 。

Therefore, according to Lemma 1, for a given initial state $x(0) = x_0$, x_1 and z will converge to 0 in finite time T, which is the settling time. According to the definitions of $\varphi(x_1)$ and z, when $x_1 = 0$, $z = 0$ and $x_2 = 0$, the closed-loop system (18) is finite-time stable.

Remark 1. The gains K_1 and K_2 are obtained to guarantee the control performances by the trail and error method.

Remark 2. It is obvious that the derivative of $Sig(x_1)^{\alpha}$ is infinite when x_{1i}=0 and $\dot{x}_{1i} \neq 0$ in $\dot{\varphi}_1(x_1)$. To avoid this singularity problem, a threshold should be set to judge the singularity. Thus, the following definition of $\dot{\varphi}(x_1)$ is modified:

$$\dot{\varphi}_i(x_{1i}) = \begin{cases} -k_{1i}\alpha\left|x_{1i}\right|^{\alpha-1}\dot{x}_{1i}, & if\ \left|x_{1i}\right| \ge \lambda\ and\ \dot{x}_{1i} \neq 0 \\ -k_{1i}\alpha\left|\Delta_i\right|^{\alpha-1}\dot{x}_{1i}, & if\ \left|x_{1i}\right| < \lambda\ and\ \dot{x}_{1i} \neq 0 \\ 0 & , if\ \dot{x}_{1i} = 0 \end{cases} \tag{21}$$

where λ and Δ_i are both small positive constants, x_{1i} is the ith element of vector x_1, and $\dot{\varphi}_i(x_{1i})$ is the ith element of vector $\dot{\varphi}(x_1)$.

3.2. Observer Design

Defining that $p_1 = q, p_2 = \dot{q}$, and p_1 is measurable while p_2 is unmeasurable, the observer for p_2 is designed in the following, with $\hat{p}_1, \hat{p}_2$ being the state estimations.

The control law (17) is redefined as

$$\tau = M(q)\left[\ddot{q}_d - x_1 - K_2 Sig(\hat{z})^{\alpha} + \dot{\varphi}(x_1) - K_3\,\mathrm{sgn}(\hat{z})\right] + C(q,\dot{\hat{q}})\dot{\hat{q}} + G(q) \tag{22}$$

where K_3 = diag(k_{31}, k_{32}, …,k_{3n}) is a positive definite diagonal matrix, and the error variable (15) can be expressed as

$$\hat{z} = \hat{x}_2 - \varphi(x_1) = \hat{p}_2 - \dot{q}_d - \varphi(x_1) \tag{23}$$

Thus, we can obtain

$$\begin{cases} \dot{p}_1 = p_2 \\ \dot{p}_2 = F\left(p_1, p_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) \end{cases} \tag{24}$$

where

$$F\left(p_1, p_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) = U(p_1, \hat{p}_2, \ddot{q}_d) + M^{-1}(q)\left(C(p_1, \hat{p}_2)\hat{p}_2 - C(p_1, p_2)p_2\right) \tag{25}$$

$$U(p_1, \hat{p}_2, \ddot{q}_d) = \ddot{q}_d - x_1 - K_2 Sig(\hat{z})^{\alpha} + \dot{\varphi}(x_1) - K_3\,\mathrm{sgn}(\hat{z}) \tag{26}$$

The nonlinear observer is designed as

$$\begin{cases} \dot{\hat{p}}_1 = \hat{p}_2 + L_1 Sig(\tilde{p}_1)^{\beta_1} \\ \dot{\hat{p}}_2 = \hat{F}\left(p_1, \hat{p}_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) + L_2 Sig(\tilde{p}_1)^{\beta_2} \end{cases} \tag{27}$$

where $\hat{F}\left(p_1, \hat{p}_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) = U(p_1, \hat{p}_2, \ddot{q}_d)$, $\tilde{p}_1 = p_1 - \hat{p}_1$, $\tilde{p}_2 = p_2 - \hat{p}_2$, $0 < \beta_2 < 1$, β_1 $=(1+\beta_2)/2$, L_1 and L_2 are both positive definite and diagonal matrixes. By defining L_1=diag(l_{11}, l_{12}, …, l_{1n}), L_2=diag(l_{21}, l_{22}, …, l_{2n}), and setting the initial $\hat{p}_1 = p_1$ and $\hat{p}_2 = 0$, we obtain the error equation

$$\begin{cases} \dot{\tilde{p}}_1 = \tilde{p}_2 - L_1 Sig(\tilde{p}_1)^{\beta_1} \\ \dot{\tilde{p}}_2 = -L_2 Sig(\tilde{p}_1)^{\beta_2} + \tilde{F}(p_1, p_2, \hat{p}_2) \end{cases} \tag{28}$$

where

$$\begin{aligned} \tilde{F}(p_1, p_2, \hat{p}_2) &= F\left(p_1, p_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) - \hat{F}\left(p_1, \hat{p}_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) \\ &= M^{-1}(q)\left(C(p_1, \hat{p}_2)\hat{p}_2 - C(p_1, p_2)p_2\right) \end{aligned} \tag{29}$$

Thus, the closed-loop system (18) is rewritten as

$$\begin{cases} \dot{x}_1 = -K_1 Sig(x_1)^{\alpha} + \hat{z} \\ \dot{\hat{z}} = -K_2 Sig(\hat{z})^{\alpha} - x_1 + \tilde{F}(p_1, p_2, \hat{p}_2) - K_3\,\mathrm{sgn}(\hat{z}) \end{cases} \tag{30}$$

In fact, $\tilde{F}(p_1, p_2, \hat{p}_2)$ is bounded, here, we suppose that $\left\|\tilde{F}(p_1, p_2, \hat{p}_2)\right\| \le f$, where f is a positive constant.

Theorem 4.1. *Considering the robotic system* (10), *and designing the control law and the observer as* (22) *and* (27) *respectively, the equilibriums at the origin of the closed-loop systems* (30) *and* (28) *are globally finite-time stable.*

Proof:

Step 1: The controlled system is finite-time stable.

Selecting the Lyapunov function

$$V = \frac{1}{2}x_1^{\mathrm{T}}x_1 + \frac{1}{2}\hat{z}^{\mathrm{T}}\hat{z} \tag{31}$$

and differentiating V with respect to time, by combining (12), we can obtain

$$\begin{aligned} \dot{V} &= x_1^{\mathrm{T}}\dot{x}_1 + \hat{z}^{\mathrm{T}}\dot{\hat{z}} \\ &= x_1^{\mathrm{T}}\left(-K_1 Sig(x_1)^{\alpha} + \hat{z}\right) + \hat{z}^{\mathrm{T}}[-K_2 Sig(\hat{z})^{\alpha} - x_1 + \tilde{F}(p_1, p_2, \hat{p}_2) - K_3\,\mathrm{sgn}(\hat{z})] \\ &= -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} + x_1^{\mathrm{T}}\hat{z} + \hat{z}^{\mathrm{T}}[-K_2 Sig(\hat{z})^{\alpha} - x_1 + \tilde{F}(p_1, p_2, \hat{p}_2) - K_3\,\mathrm{sgn}(\hat{z})] \\ &= -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} - \hat{z}^{\mathrm{T}}K_1 Sig(\hat{z})^{\alpha} - \hat{z}^{\mathrm{T}}\left(K_3\,\mathrm{sgn}(\hat{z}) - \tilde{F}(p_1, p_2, \hat{p}_2)\right) \end{aligned} \tag{32}$$

If we select $k_{3i} \ge f$, there comes

$$\begin{aligned} \dot{V} &= -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} - z^{\mathrm{T}}K_1 Sig(\hat{z})^{\alpha} - \hat{z}^{\mathrm{T}}\left(K_3\,\mathrm{sgn}(\hat{z}) - \tilde{F}(t, p_1, p_2, \hat{p}_2)\right) \\ &\le -x_1^{\mathrm{T}}K_1 Sig(x_1)^{\alpha} - z^{\mathrm{T}}K_1 Sig(\hat{z})^{\alpha} \\ &\le -\tilde{k}_1\left(\frac{1}{2}\sum_{i=1}^{n} x_{1i}^2\right)^{\mu} - \tilde{k}_2\left(\frac{1}{2}\sum_{i=1}^{n} \hat{z}_i^2\right)^{\mu} \\ &= -\tilde{k}V^{\mu} \end{aligned} \tag{33}$$

Obviously, according to Lemma 1, for a given initial state $x(0) = x_0$, x_1 and z will converge to 0 in finite time T, T is the settling time. According to the definitions of $\varphi(x_1)$ and z, when $x_1 = 0$ and $z = 0$, $x_2 = 0$, the closed-loop system (30) is finite-time stable.

Step 2: The observer is also finite-time convergent.

Noting that the estimation error system (28) is not homogeneous, we consider the following system

$$\begin{cases} \dot{\tilde{p}}_1 = \tilde{p}_2 - L_1 Sig(\tilde{p}_1)^{\beta_1} \\ \dot{\tilde{p}}_2 = -L_2 Sig(\tilde{p}_1)^{\beta_2} \end{cases} \tag{34}$$

It is obvious that the error system (34) is a homogeneous system of degree $k = (\beta_2 / \beta_1) - 1 < 0$ with a dilation of $(1/\beta_1, 1)$. According to Lemma 2, there are two important conditions under which the zero solution of this system is finite-time stable. Therefore, we first prove that the zero solution of the homogeneous system (34) is asymptotically stable.

To this end, we select the Lyapunov function

$$V = \sum_{i=1}^{n} l_{2i} \left|\tilde{p}_{1i}\right|^{\beta_2+1} + \frac{1+\beta_2}{2} \tilde{p}_2^{\mathrm{T}} \tilde{p}_2 \tag{35}$$

The derivative of V along the trajectories of (34) is

$$\begin{aligned} \dot{V} &= \left(1+\beta_2\right) \dot{\tilde{p}}_1^{\mathrm{T}} L_2 Sig\left(\tilde{p}_1\right)^{\beta_2} + \left(1+\beta_2\right) \tilde{p}_2^{\mathrm{T}} \dot{\tilde{p}}_2 \\ &= \left(1+\beta_2\right)\left(\tilde{p}_2 - L_1 Sig(\tilde{p}_1)^{\beta_1}\right)^{\mathrm{T}} L_2 Sig\left(\tilde{p}_1\right)^{\beta_2} + \left(1+\beta_2\right) \tilde{p}_2^{\mathrm{T}} \left(-L_2 Sig(\tilde{p}_1)^{\beta_2}\right) \\ &= -\left(1+\beta_2\right)\left(Sig(\tilde{p}_1)^{\beta_1}\right)^{\mathrm{T}} L_1^T L_2 Sig\left(\tilde{p}_1\right)^{\beta_2} \\ &= -\left(1+\beta_2\right) \sum_{i=1}^{n} l_{1i} l_{2i} \left|\tilde{p}_{1i}\right|^{\beta_1+\beta_2} \\ &\le 0 \end{aligned} \tag{36}$$

Such that the $\dot{V}$ in Eq. (36) is negative semi-definite. Now noting that $\dot{V} \equiv 0$ implies $\tilde{p}_1 \equiv 0$, which implies $\tilde{p}_2 \equiv 0$ using (35), according to LaSalle's invariant set theorem, the zero solution of the homogeneous system (34) is asymptotically stable.

Then, we use Lemma 4 to reveal the local finite-time stability of the closed-loop system (28). Since $\tilde{F}(p_1, p_2, \hat{p}_2)$ is smooth and $k < 0$, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \frac{\tilde{F}(p_1, p_2, p_2 - \varepsilon^{k+r_2} \tilde{p}_2)}{\varepsilon^{k+r_2}} = \lim_{\varepsilon \to 0} \frac{M^{-1}(q)\left(C(p_1, p_2 - \varepsilon^{r_2} \tilde{p}_2)\left(p_2 - \varepsilon^{r_2} \tilde{p}_2\right) - C(p_1, p_2) p_2\right)}{\varepsilon^{k+r_2}} \\ &= \lim_{\varepsilon \to 0} \frac{M^{-1}(q)\left(-C(p_1, p_2)\varepsilon^{r_2} \tilde{p}_2\right)}{\varepsilon^{k+r_2}} \\ &= -M^{-1}(q)\left(C(p_1, p_2)\tilde{p}_2\right) \lim_{\varepsilon \to 0} \varepsilon^{-k} \\ &= 0 \end{aligned} \tag{37}$$

Therefore, according to Lemma 2, we can obviously obtain the local finite-time stability of the closed-loop system(28).

Finally, by invoking Lemma 3, we obtain the global finite-time stability of (28). Thus, the proof is completed.

Remark 3. The control law and the observer can be separately designed, *i.e.* the separation principle is automatically satisfied.

4. Observer-Based Finite-Time Control Considering Uncertainties

In fact, the precise values of robot dynamic parameters are difficult or even impossible to acquire due to the existance of measuring error, payload variation and external disturbance. Here, we assume that the actual parameters can be written as

$$\begin{aligned} &M(q) = M_0(q) + \Delta M(q), \\ &C(q,\dot{q}) = C_0(q,\dot{q}) + \Delta C(q,\dot{q}), \\ &G(q) = G_0(q) + \Delta G(q) \end{aligned} \tag{38}$$

where $M_0(q), C_0(q,\dot{q})$ and $G_0(q)$ are all estimated terms; $\Delta M(q), \Delta C(q,\dot{q})$ and $\Delta G(q)$ are all uncertain terms. Then, the dynamic model of robotic manipulator can be written as

$$M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + G_0(q) = \tau + \rho(q,\dot{q},\ddot{q}) \tag{39}$$

where $\rho(q,\dot{q},\ddot{q}) = -\Delta M(q)\ddot{q} - \Delta C(q,\dot{q})\dot{q} - \Delta G(q) \in \mathbf{R}^n$ is the lumped system uncertainty, which is assumed to be bounded by the following function:

$$\rho(q,\dot{q},\ddot{q}) \le b_0 + b_1\|q\| + b_2\|\dot{q}\|^2 \tag{40}$$

where $b_0 > 0$, $b_1 > 0$ and $b_2 > 0$.

Equation(10) can be expressed as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = M^{-1}(q)\left[\tau - C(q,\dot{q})(\dot{q}) - G(q)\right] + w - \ddot{q}_d \end{cases} \tag{41}$$

where $w = M_0^{-1}(q)\rho$. It is obvious that $M_0(q)$ is positive and bounded, so that w is also bounded. Therefore, supposing that $\|w\| \le d$ in this paper, where d is a positive constant. Theorem 5.1 holds.

Theorem 5.1. *Considering the robotic system* (10)*, the control law is redesigned as*

$$\tau = M_0(q)\left[\ddot{q}_d - x_1 - K_2 Sig(\hat{z})^{\alpha} + \dot{\varphi}(x_1) - K_3\,\text{sgn}(\hat{z})\right] + C_0(q,\dot{\hat{q}})\dot{\hat{q}} + G_0(q) \tag{42}$$

and the sliding mode observer is designed as

$$\begin{cases} \dot{\hat{p}}_1 = \hat{p}_2 + L_1 Sig(\tilde{p}_1)^{\beta_1} \\ \dot{\hat{p}}_2 = \hat{F}\left(p_1, \hat{p}_2, U(p_1, \hat{p}_2, \ddot{q}_d)\right) + L_3\,\text{sgn}(\tilde{p}_1) \end{cases} \tag{43}$$

where $L_3=diag(l_{31}, l_{32}, \ldots, l_{3n})$ is a positive definite diagonal matrix. Thus, the equilibriums at the origin of the closed-loop system composed of (41), (42) *and* (24), (43) *are globally finite-time stable*.

Remark 4. In the controller (42), we select $k_{3i} > d$ to guarantee the robustness of the closed-loop system. The proof is similar to that of Theorem 4.1.

Remark 5. In the presence of uncertainties, to guarantee the finite-time stability and robustness of the observer, the observer (43) [17] is developed, in which the $L_2 Sig(\tilde{p}_1)^{\beta_2}$ in (27) is substituted with $L_3\,\text{sgn}(\tilde{p}_1)$.

Remark 6. The parameters of the observer should be selected as:

$$\begin{cases} L_2 > f^+ \\ L_1 > \left(\dfrac{2}{L_2 - f^+}\right)^{1-\beta_1} \dfrac{\left(L_2 + f^+\right)\left(1+\lambda\right)}{1-\lambda} \end{cases} \tag{44}$$

where λ is a selected constant, $0 < \lambda < 1$, and f^+ is the supremum of $\tilde{F}(p_1, p_2, \hat{p}_2)$.

By combining (24), we can obtain the estimation error system

$$\begin{cases} \dot{\tilde{p}}_1 = \tilde{p}_2 - L_1 Sig(\tilde{p}_1)^{\beta_1} \\ \dot{\tilde{p}}_2 = -L_3 \operatorname{sgn}(\tilde{p}_1) + \tilde{F}(p_1, p_2, \hat{p}_2) \end{cases} \tag{45}$$

where

$$\tilde{F}(p_1, p_2, \hat{p}_2) = M_0^{-1}(q)\left(\rho + C_0(p_1, \hat{p}_2)\hat{p}_2 - C_0(p_1, p_2)p_2\right) \tag{46}$$

which is different from Eq. (29). However, we can also conclude that $\tilde{F}(p_1, p_2, \hat{p}_2)$ is bounded. For the industrial robot system, $q, \dot{q}, \ddot{q} \in \mathbf{R}^n$ are all bounded. According to properties (11) and (12), we have

$$\begin{aligned} \left\|\tilde{F}(t, p_1, p_2, \hat{p}_2)\right\| &= \left\|M_0^{-1}(q)\left(\rho + C_0(p_1, \hat{p}_2)\hat{p}_2 - C_0(p_1, p_2)p_2\right)\right\| \\ &\le \left\|M_0^{-1}(q)\right\|\left\|\left(\rho + C_0(p_1, \hat{p}_2)\hat{p}_2 - C_0(p_1, p_2)p_2\right)\right\| \\ &\le \left\|M_0^{-1}(q)\right\|\left(\|\rho\| + \|C_0(p_1, \hat{p}_2)\hat{p}_2 - C_0(p_1, p_2)p_2\|\right) \\ &\le \left\|M_0^{-1}(q)\right\|\left(\|\rho\| + C_M\|\hat{p}_2 - p_2\|\right) \\ &\le \left\|M_0^{-1}(q)\right\|\left(d + C_M\|\tilde{p}_2\|\right) \\ &< f^+ \end{aligned} \tag{47}$$

where f^+ is an existing constant, such that the above-mentioned inequality holds.

Remark 7. Since the proof of the finite-time convergence of the observer (43) is the same as the literature [17], so it is omitted in this paper.

Remark 8. In order to eliminate "chattering" phenomenon, the sign function is replaced by the following function in practice, which but not affect the finite-time stability of the closed-loop system:

$$sat(z) = \begin{cases} \operatorname{sgn}(\frac{z}{\delta}), \ if \ \left|\frac{z}{\delta}\right| \ge 1 \\ \frac{z}{\delta}, \quad others \end{cases} \tag{48}$$

where δ is a smaller positive constant called the boundary layer thickness.

5. Simulations

In this section, a two-link planar robot shown in Figure 1 is used to test the effectiveness of the proposed observer-based finite-time control methods. The two-link planar robotic manipulator is described as

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -2b\dot{q}_2, & -b\dot{q}_2 \\ b\dot{q}_1 \ , & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \tag{49}$$

where

$$\begin{aligned} &m_{11} = p_1 + p_2 + 2p_3\cos q_2 - 2p_4\sin q_2, \\ &m_{12} = p_2 + p_3\cos q_2 - p_4\sin q_2, \\ &m_{22} = p_2, \\ &b = p_3\sin q_2 + p_4\cos q_2, \\ &f_1 = f_{v1}\dot{q}_1 + f_{c1}sgn(\dot{q}_1), \\ &f_2 = f_{v2}\dot{q}_2 + f_{c2}sgn(\dot{q}_2), \end{aligned}$$

p_1, p_2, p_3 and p_4 are the minimum inertia parameters of robot, f_{v1}, f_{c1} are respectively the viscous, Coulomb friction coefficients of Joint 1, while f_{v2}, f_{c2} are respectively those of Joint 2.

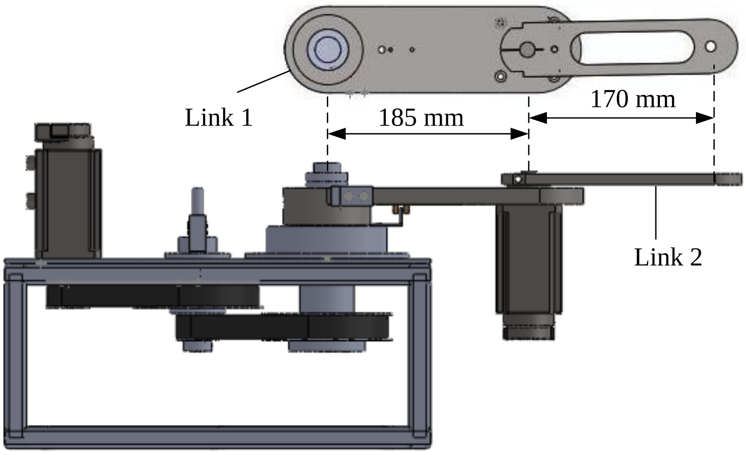


Figure 1. Two-Link Planar Robot

The actual values of these above-mentioned parameters are p_1=0.0398 kgm^2, p_2=0.0026 kgm^2, p_3=–0.0015 kgm^2, p_4=0.0081 kgm^2, f_{v1}=0.534684, f_{c1}=0.81958, f_{v2}=0.001 and f_{c2}=0.002. The estimated values are p_1=0.0286 kgm^2, p_2=0.01624 kgm^2, p_3=–0.00045 kgm^2, p_4=0.004923 kgm^2, f_{v1}=0.634384, f_{c1}=0.91748, f_{v2}=0.01196 and f_{c2}=0.044647. The reference signals are given by q_{d1}=sin($2\pi t$) and q_{d2}=sin($2\pi t$). The initial values of the system are selected as $q_1(0)$=0.2 rad and $q_2(0)$=0.2 rad. The external disturbance τ_d = [0.2sin(10t), 0.1sin(10t)]^T.

In order to demonstrate the advantages of this proposed controllers with observer, we compare the controllers with the finite-time inverse tracking controller proposed by Su[14]. The finite-time inverse tracking control (FIDC) law is given by

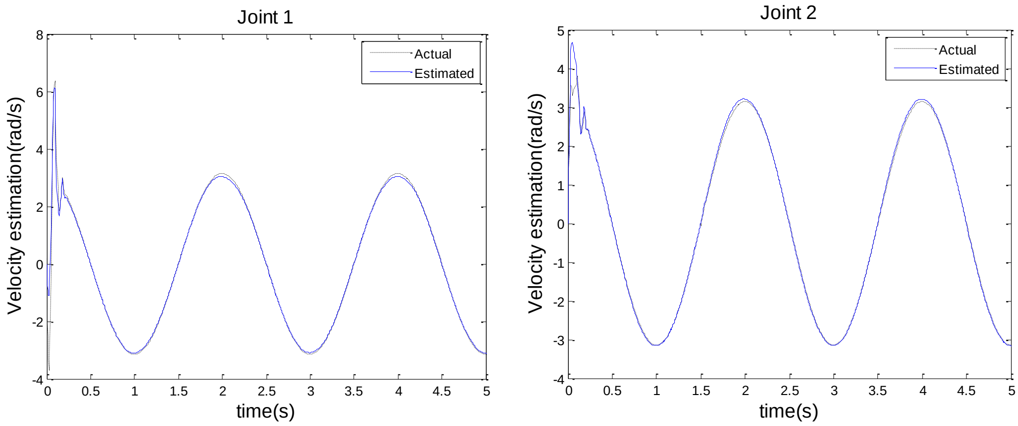
$$\tau = M_0(q)\left[\ddot{q}_d - K_p Sig(e)^{\alpha_1} - K_d Sig(\dot{e})^{\alpha_2}\right] + C_0(q,\dot{q})\dot{q} + F_0(\dot{q}) \qquad (50)$$

where K_p=diag(1000,1000), K_d=diag(500,500), α_1=0.7, $\alpha_2=2\alpha_1/(\alpha_1+1)$ = 0.8235 and $F_0=[f_1, f_2]^T$. All states in this controller of [14] are assumed to be measurable.

The two proposed observers (Theorems 4.1 and 5.1) are both considered in this paper to illustrate the robustness of the proposed methods, although the uncertainties are not considered in Theorem 4.1.

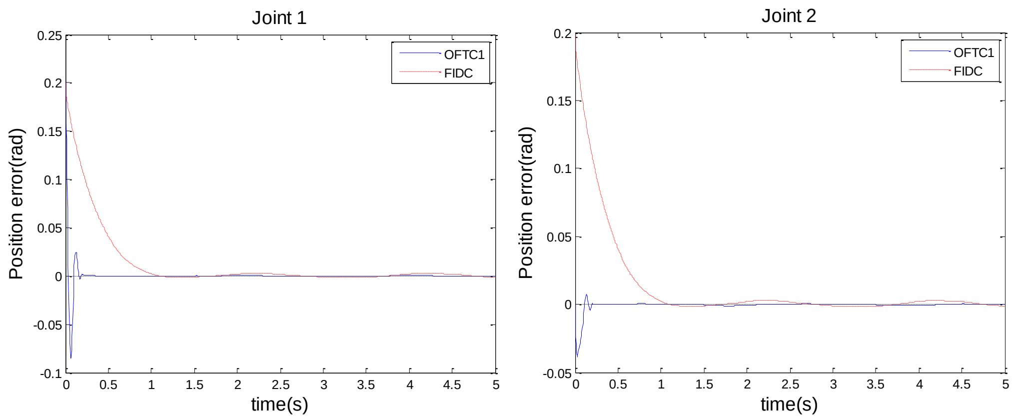
Simulation 1 (Theorem 4.1, OFTC1)

The control parameters are selected as K_1=diag(50, 50), K_2=diag(50, 50), K_3=diag(20, 20), α = 0.8 and Δ = 0.01, and the observer parameters are selected as L_1=diag(30, 50), L_2=diag(500, 800), β_1 = 0.7 and $\beta_2 = 2\beta_2 - 1$=0.4.



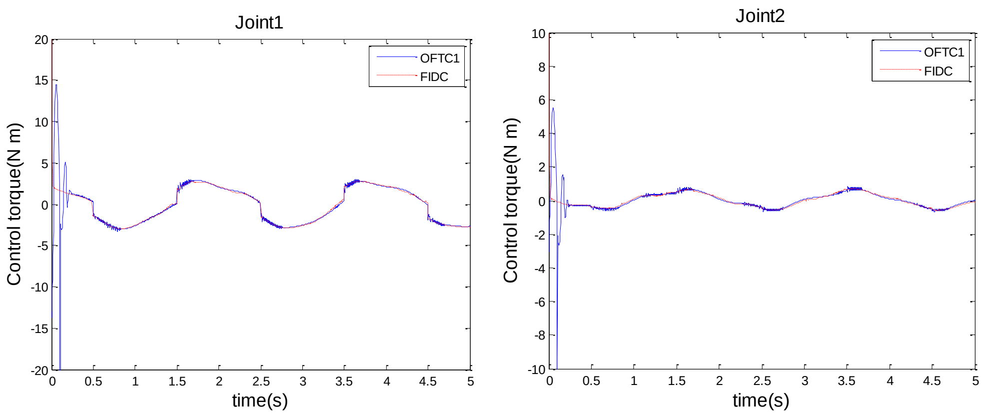
(a) (b)

Figure 2. Velocity Estimation (OFTC1)



(a) (b)

Figure 3. Trajectory Tracking Error (OFTC1)



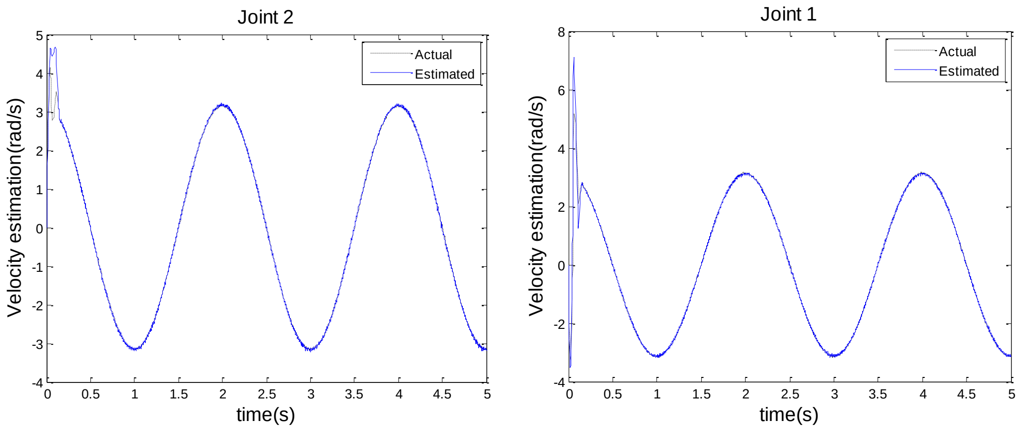
(a) (b)

Figure 4. Control Input Torque (OFTC1)

Figure 2 shows the actual and the estimated joint velocities. Figure 3 shows the trajectory tracking error (OFTC1). As shown in Figure 3, joint positions exhibit better transient performance, and smaller error is remarked in a steady-state regime. It is obviously that the observer helps to estimate joint velocities fast and accurately . Figure 4 shows the control input torques of each joint.

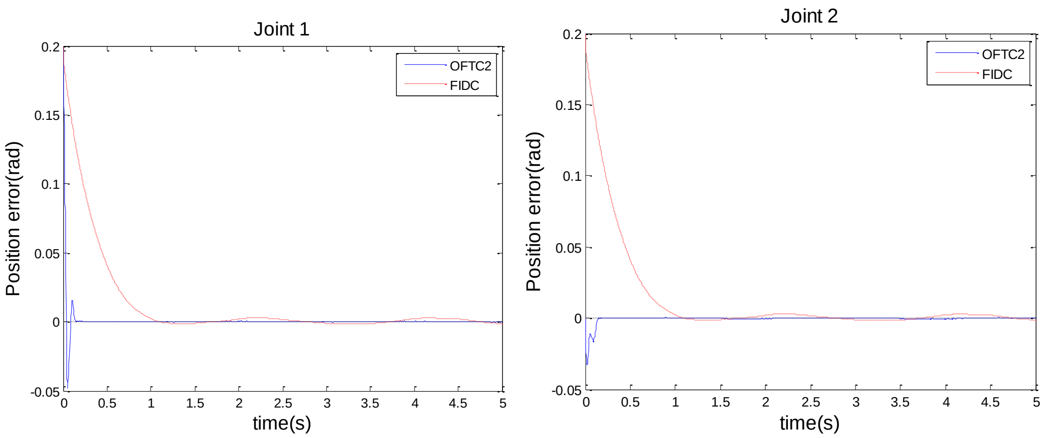
Simulation 2 (Theorem 5.1, OFTC2)

The control parameters are selected as K_1=diag(50, 50), K_2=diag(50, 50), K_3=diag(20, 20), α = 0.8 and Δ = 0.01, and the observer parameters are selected as L_1=diag(30, 50), L_3=diag(30, 20) and β_1 = 0.7.



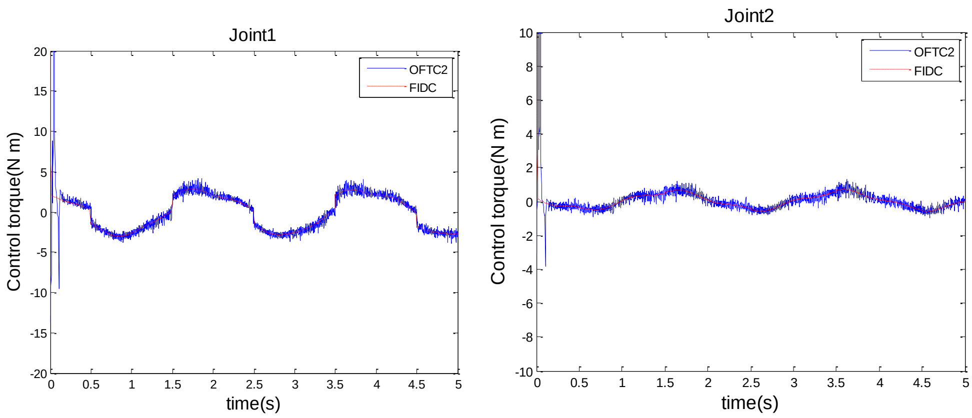
(a) (b)

Figure 5. Velocity Estimation (OFTC2)



(a) (b)

Figure 6. Trajectory Tracking Error (OFTC2)



(a) (b)

Figure 7. Control Input Torque (OFTC2)

Figure 5 shows the actual and the estimated joint velocities. It is clearly that the observer helps to estimate joint velocities fast and accurately. Figure 6 shows the position tracking errors and Figure 7 shows the control input torques.

From these figures, we can see that the proposed controller OFTC is more robust to dynamic uncertainties and external disturbances than FIDC. Moreover, it can be easily found that the tracking performance of FIDC greatly degrades when there is no exact and available dynamic knowledge of robotic manipulators, even though the velocities in FIDC are assumed to be measurable. However, there are a few weak “chattering” phenomena in the OFTC method. Therefore, in order to eliminate the chattering, it is practically important to select an appropriate boundary layer δ in (48). In summary, the OFTC methods are suitable for the robotic manipulators with uncertainties, and are of high performances such as fast response, high tracking precision and strong robustness.

6. Conclusions

This paper develops two observer-based finite-time control methods for robotic manipulators without or with uncertainties. The introduction of the nonlinear observer guarantees the finite-time convergence of the unmeasurable velocities. Meanwhile, the closed-loop system is also finite-time stable so that high-performance control is achieved. A simulation on a two-link robotic manipulator verifies the effectiveness of the proposed methods.

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