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Synchronization of a Novel Class of Fractional-Order Uncertain Chaotic Systems via Adaptive Sliding Mode Controller

Aminreza Riahi[1], Naeimadeen Noghredani[2] ,Mahnoosh Shajiee[3] and Naser Pariz[4]

[1,3,4] Department of Electrical Engineering, Faculty of Engineering, Ferdowsi University of Mashhad (FUM) campus, Azadi Sq., P.O. Box: 9177948974, Mashhad, Khorasan Razavi, Iran
[2]Department of Electrical Engineering, Ferdows Branch, Islamic Azad University, Ferdows, Iran
[1]ar.riahi@stu.um.ac.ir, [2]naeim.noghredani@ferdowsiau.ac.ir ,[3]ma.shajiee@stu.um.ac.ir,[4]n-pariz@um.ac.ir

Abstract

In this work an adaptive sliding mode controller in the presence of uncertainty, as well as the external disturbance is considered. A concise introduction and investigation of the dynamic behavior of a novel class of chaotic systems with fractional order derivatives for synchronization is presented. It is supposed that the high bounds of uncertainty and external disturbance are unknown. The proposed controller is designed based on error dynamics and acceptable adaptive laws. The sliding mode dynamic stability and the condition to start sliding are proved by Lyapunov stability theory. With this new proposed approach, Chen and Lorenz system with fractional order derivatives are synchronized. Finally, simulation results with MATLAB software showed that the designed comparative sliding mode controller was able to synchronize chaotic systems with fractional order derivatives in the presence of the mentioned adverse factors. The main characteristic of the proposed method compared to other methods is providing acceptable adaptive laws for satisfactory functioning against uncertainty and external disturbance and eliminate the chattering phenomenon for synchronization of non-identical chaotic systems with fractional order derivatives.

Keywords*: synchronization, fractional-order chaotic systems, adaptive sliding mode control, uncertainty, external disturbance*

1. Introduction

Chaos phenomenon is one of the growing areas of study and research. Among its high usage and widely popular contexts are laser fields [1-3], medical fields [4-6], earthquake [7], chemical reactors [8], mathematics [9-11], economic systems [12-14], and so on.

Today, the control of chaos phenomenon can be regarded as an interesting subject, attracting wide attention from the researchers.

Fractional calculus can be called either a new or an old science. Until recently, most of the researchers have not been aware of the existence of such a branch of science; however, nowadays, due to the interest of the scientists and mathematicians in the science, the related research speed has increased dramatically. Among the areas that are of interest to researchers in fractional order science are mechanics, electricity, mathematics, biology and so on [15-

18]. Controlling chaotic systems with fractional order derivatives is also one of the most popular areas of fractional order science.

Examples include investigations into the chaotic behavior of fractional-order horizontal platform systems [19] and many published articles on fractional-order chaotic systems [20-23]. Among the hottest topics that have been investigated and explored widely in the field of nonlinear science are synchronization and control of chaotic systems. For example, a sliding mode controller is designed in [24] for the synchronization of chaotic systems in the presence of uncertainty and external disturbance. Article [25] defines an adaptive hybrid complex projective synchronization method to synchronize two chaotic complex systems. Finite-time hybrid projective synchronization for the unified chaotic system has been provided in [26]. The finite-time master–slave synchronization and parameter identification problem for uncertain Lurie systems based on the finite-time stability theory and the adaptive control method have been investigated in [27]. Faieghi and Delavari studied chaotic synchronization of Genesio-Tesi system utilizing two strategies; active control and sliding mode [28].

This paper is organized as follows: First, a new class of chaotic systems with fractional order derivatives is introduced then, an adaptive sliding mode controller is designed to synchronize fractional-order chaotic systems in master-slave structure. In addition, in the following parts, adaptive sliding mode controller asymptotic stability is investigated in the presence of uncertainty and external disturbances. The simulation results finally proved the effectiveness of the proposed controller against the mentioned adverse factors.

2. Preliminaries

Derivative operator - integrator is characterized by ${}_aD_t^q$, a combination of differential-integral operator used in the calculations. The operator is a symbol to represent the fractional integral and fractional derivative expressed in a phrase, which is defined as follows:

$$
{}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q} & q > 0 \\ 1 & q = 0 \\ \int_a^t (dt)^{-q} & q < 0 \end{cases} \tag{1}
$$

where q is the fractional order. There are various definitions for fractional derivative and integral. The most common definitions are Grunwald–Letnikov definition, Riemann–Liouville definition and Caputo definition. In the rest of this paper, Riemann-Liouville (RL) definition of derivative is used. RL derivative in the order of q is explained below ([30]):

$$
{}_0D_t^q f(t) = D^q f(t) = \frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{q+m-1}} d\tau \tag{2}
$$

where m is the first integer which is not less than q, i.e. $m-1 \le q < m$ and $\Gamma(.)$ is the well-known Euler's gamma function

$$
\Gamma(P) = \int_0^\infty t^{p-1} e^{-t} dt \quad ; \quad \Gamma(P+1) = P\Gamma(P) \tag{3}
$$

Lemma 2.1 (Barbalat's lemma [35]). If $\eta : R \to R$ is a uniformly continuous function for $t \geq 0$ and if the limit of the integral $\int_0^t \eta(\omega) d\omega$ exists and is finite, then $\lim_{t \to \infty} \eta(t) = 0.$

Lemma 2.2. The following equality is valid for every positive scalar α and given scalar β.

$$\beta \tanh(\alpha\beta) = \left|\beta \tanh(\alpha\beta)\right| = \left|\beta\right|\left|\tanh(\alpha\beta)\right| \geq 0. \tag{4}$$

Proof. From the definition of $\tanh(\alpha) = \dfrac{e^{\alpha} - e^{-\alpha}}{e^{\alpha} + e^{-\alpha}}$ then $\beta \tanh(\alpha\beta)$ is as follows:

$$\beta \tanh(\alpha\beta) = \frac{\beta(e^{\alpha\beta} - e^{-\alpha\beta})}{e^{\alpha\beta} + e^{-\alpha\beta}} = \frac{\beta(e^{2\alpha\beta} - 1)}{e^{2\alpha\beta} + 1} \tag{5}$$

And given that $\begin{cases} e^{2\alpha\beta} - 1 \leq 0, & if \quad \beta \leq 0 \\ e^{2\alpha\beta} - 1 > 0, & if \quad \beta > 0 \end{cases}$ so it can be obtained $\beta(e^{2\alpha\beta} - 1) \geq 0$. So the following inequality is true

$$\beta \tanh(\alpha\beta) = \frac{\beta(e^{2\alpha\beta} - 1)}{e^{2\alpha\beta} + 1} \geq 0. \tag{6}$$

Because of the fact that if $ab \geq 0$ for every scalars a and b, $ab = \left|ab\right| = \left|a\right|\left|b\right| \geq 0$ holds, the following inequality can be obtained;

$$\beta \tanh(\alpha\beta) = \left|\beta \tanh(\alpha\beta)\right| = \left|\beta\right|\left|\tanh(\alpha\beta)\right| \geq 0. \tag{7}$$

3. System Description

A class of three-dimensional fractional-order chaotic systems is given by [30-33]:

$$\begin{cases} \dfrac{d^{q_1} x}{dt^{q_1}} = y.f(x, y, z) + z.\xi(x, y, z) - \alpha x, \\ \dfrac{d^{q_2} y}{dt^{q_2}} = g(x, y, z) - \beta y, \\ \dfrac{d^{q_3} z}{dt^{q_3}} = y.h(x, y, z) - x.\xi(x, y, z) - \gamma z, \end{cases} \tag{8}$$

where $q_i \quad (i = 1, 2, 3)$ are fractional orders satisfying $0 < q_i < 1; \quad x, y$ and z are state variables. Each of the four functions $f(.), g(.), h(.)$ and $\xi(.)$ is considered as continuation nonlinear vector functions belonging to $R^3 \to R$ space, and α, β, γ are known constants, for any negative or positive values.

Remark 3.1. If $q_1 = q_2 = q_3 = q$, fractional-order system (8), is called a commensurate fractional-order system. Otherwise, it is called incommensurate fractional-order system.

Remark 3.2. Note that many fractional-order chaotic systems belong to the class characterized by (8). Examples include the fractional-order financial system and the unified chaotic system of fractional-order version (including the fractional-order Chen system, fractional-order Lu's system). Table 1 shows that these fractional-order chaotic models can be described by the proposed system (8).

In this paper, The master system in the presence of uncertainty $\Delta g(x_1, x_2, x_3)$ and external disturbances $d_1(t)$ is considered as follows:

$$
\begin{aligned}
\frac{d^{q_1} x_1}{dt^{q_1}} &= x_2 f(x_1, x_2, x_3) + x_3 \xi(x_1, x_2, x_3) - \alpha x_1 \\
\frac{d^{q_2} x_2}{dt^{q_2}} &= g(x_1, x_2, x_3) - \beta x_2 + \Delta g(x_1, x_2, x_3) + d_1(t) \\
\frac{d^{q_3} x_3}{dt^{q_3}} &= x_2 h(x_1, x_2, x_3) - x_1 \xi(x_1, x_2, x_3) - \gamma x_3
\end{aligned}
$$
(9)

Where x_1, x_2, x_3 are state variables. Also, adding a control input $u(t)$, uncertainty $\Delta g(y_1, y_2, y_3)$ and external disturbances $d_2(t)$ to the second state equation of system (8), the slave system would be as follows:

$$
\begin{aligned}
\frac{d^{q_1} y_1}{dt^{q_1}} &= y_2 f(y_1, y_2, y_3) + y_3 \xi(y_1, y_2, y_3) - \alpha y_1 \\
\frac{d^{q_2} y_2}{dt^{q_2}} &= g(y_1, y_2, y_3) - \beta y_2 + \Delta g(y_1, y_2, y_3) + d_2(t) - u(t) \\
\frac{d^{q_3} y_3}{dt^{q_3}} &= y_2 h(y_1, y_2, y_3) - y_1 \xi(y_1, y_2, y_3) - \gamma y_3
\end{aligned}
$$
(10)

where $y_1, \mathrm{y}_2, \mathrm{y}_3$ are state variables.

Remark 3.3. Throughout this paper, it is assumed that $f(.), g(.), h(.)$ and $\xi(.)$ are required to ensure that the fractional-order system (10) with control input $u(t)$ has a unique solution in the time interval $[T, +\infty],\ T > 0$ for any given initial conditions.

Assumption 3.1. Uncertainties of $\Delta g(x_1, x_2, x_3)$ and $\Delta g(y_1, y_2, y_3)$ together with external disturbances $d_1(t)$ and $d_2(t)$ were presumed to be bounded. Then, there exist $\hat{\vartheta}_1, \hat{\rho}_1, \hat{\vartheta}_2$ and $\hat{\rho}_2$ positive constants are as following:

$$
|\Delta g(x_1, x_2, x_3)| < \hat{\vartheta}_1 \quad , \quad |d_1(t)| < \hat{\rho}_1 \quad , \quad |\Delta g(y_1, y_2, y_3)| < \hat{\vartheta}_2 \quad , \quad |d_2(t)| < \hat{\rho}_2
$$
$$
\forall x_1, x_2, x_3, y_1, y_2, y_3 \in R, \ \forall t \in [0, \infty)
$$
(11)

Assumption 3.2. The constants $\hat{\vartheta}_1, \hat{\rho}_1, \hat{\vartheta}_2$ and $\hat{\rho}_2$ are unknown.

4. Synchronization via Adaptive Sliding Mode Controller

Control the state of sliding mode contains three steps:

Reaching to the surface level (the required time to surface collide), the sliding level (the required time to sliding on a stable surface) and the stable state level (the origin).

In this paper, an adaptive sliding mode controller was designed based on error dynamics that guarantees the synchronization of two chaotic systems in the master-slave structure.

4.1 Design of switching surface

The most important issue of the sliding mode method is how to define the switching surface,

Table 1. List of Published Fractional-order Chaotic Systems, which can be Described by the Proposed General Model

Name	Model	$f(x,y,z)$	$g(x,y,z)$	$h(x,y,z)$	$\xi(x,y,z)$
Chen's system	$\begin{cases} D^{q_1}x = a(y-x) \\ D^{q_2}y = dx - xz + cy \\ D^{q_3}z = xy - bz \end{cases}$	a	$dx - xz$	x	0
Lorenz model	$\begin{cases} D^{q_1}x = a(y-x) \\ D^{q_2}y = x(b-z) - y \\ D^{q_3}z = xy - cz \end{cases}$	a	$x(b-z)$	x	0
Financial system	$\begin{cases} D^{q_1}x = z + (y-a)x \\ D^{q_2}y = 1 - by - x^2 \\ D^{q_3}z = -x - cz \end{cases}$	x	$1-x^2$	0	1
Lu's model	$\begin{cases} D^{q_1}x = a(y-x) \\ D^{q_2}y = -xz + cy \\ D^{q_3}z = xy - bz \end{cases}$	a	$-xz$	x	0
Liu system	$\begin{cases} D^{q_1}x = -ax - ey^2 \\ D^{q_2}y = -kxz + by \\ D^{q_3}z = mxy - cz \end{cases}$	$-ey$	$-kxz$	mx	0

that represents the desired system dynamics To propose an adaptive sliding mode control scheme, the sliding surface is defined as;

$$s(t) = D^{q_2-1}e_2 - \psi(t) \tag{12}$$

Where $D^{q_2-1}e_2$ and $\psi(t)$ functions are described;

$$D^{q_2}e_2 = D^{q_2}y_2 - D^{q_2}x_2 \tag{13}$$

$$\begin{aligned}\dot{\psi}(t) = &[\tilde{c}_1 x_1 f(x_1, x_2, x_3) + \tilde{c}_2 x_3 h(x_1, x_2, x_3) + kx_2] \\ &-[\tilde{c}_1 y_1 f(y_1, y_2, y_3) + \tilde{c}_2 y_3 h(y_1, y_2, y_3) + ky_2]\end{aligned} \quad (14)$$

in which k is assumed to be an arbitrary positive constant, $\tilde{c}_1$ and $\tilde{c}_2$ can be chosen to make the system converge to the sliding surface faster.

Remark 4.1. Adding kx_2 and ky_2 in (14), we have one degree of freedom choosing any arbitrary positive k as one of our controller gain. Comparing using only β in [32], having k instead of β, we can ensure that the dynamic of the system on the sliding surface will be stabilized quickly.

Two conditions of $s(t) = 0$ and $\dot{s}(t) = 0$ are met if the system situation variables be provided on the sliding surface, leading to establish the relation (15).

$$\dot{s}(t) = D^{q_2} e_2 - \dot{\psi}(t) = 0 \quad (15)$$

So, the following equation can be derived

$$\begin{aligned}D^{q_2} y_2 - D^{q_2} x_2 = \dot{\psi}(t) = &[\tilde{c}_1 x_1 f(x_1, x_2, x_3) + \tilde{c}_2 x_3 h(x_1, x_2, x_3) + kx_2] \\ &-[\tilde{c}_1 y_1 f(y_1, y_2, y_3) + \tilde{c}_2 y_3 h(y_1, y_2, y_3) + ky_2]\end{aligned}$$

(16)

According to the master system (10) and slave system (9) equation and sliding surface derivative (9), the control law equation is as below.

$$\begin{aligned}u_{eq}(t) = &\, g(y_1, y_2, y_3) - g(x_1, x_2, x_3) + \beta(x_2 - y_2) + \Delta g(y_1, y_2, y_3) - \Delta g(x_1, x_2, x_3) \\ &+ d_2(t) - d_1(t) + \tilde{c}_1 y_1 f(y_1, y_2, y_3) + \tilde{c}_2 y_3 h(y_1, y_2, y_3) + ky_2 \\ &- \tilde{c}_1 x_1 f(x_1, x_2, x_3) - \tilde{c}_2 x_3 h(x_1, x_2, x_3) - kx_2\end{aligned}$$

(17)

The next step is to develop a switching control law to satisfy the sliding condition. The discontinuous reaching law is chosen as follows:

$$u_r(t) = \eta sign(s) \quad (18)$$

Where $sign(s) = \begin{cases} 1 & s > 0 \\ 0 & s = 0 \\ -1 & s < 0 \end{cases}$ and η is the switching gain achieved by the following adaptive law;

$$\dot{\eta} = l\,|s| \quad ; \quad \eta(0) = \eta_0 \quad (19)$$

where l is a positive constant and η_0 is the initial value of the update vector parameter η. Using equation (17) and (18), we design the following controller;

$$\begin{aligned}u(\mathrm{t}) = &\, u_{eq}(t) + u_r(t) \\ = &\, g(y_1, y_2, y_3) - g(x_1, x_2, x_3) + \beta(x_2 - y_2) + \Delta g(y_1, y_2, y_3) - \Delta g(x_1, x_2, x_3) \\ &+ d_2(t) - d_1(t) + \tilde{c}_1 y_1 f(y_1, y_2, y_3) + \tilde{c}_2 y_3 h(y_1, y_2, y_3) + ky_2 \\ &- \tilde{c}_1 x_1 f(x_1, x_2, x_3) - \tilde{c}_2 x_3 h(x_1, x_2, x_3) - kx_2 + \eta sign(s)\end{aligned}$$

(20)

Although the uncertainty of $\Delta g(x_1,x_2,x_3)$ and $\Delta g(y_1,y_2,y_3)$, together with the external turbulences $d_1(t)$ and $d_2(t)$ are practically unknown functions, they must be known to apply the control law (20). To overcome this problem, the control law (20) is assumed to be rewritten with the help of assumption 3.2 as follows:

$$u(t)=g(y_1,y_2,y_3)-g(x_1,x_2,x_3)+\beta(x_2-y_2)+\tilde{c}_1 y_1 f(y_1,y_2,y_3)$$
$$+\tilde{c}_2 y_3 h(y_1,y_2,y_3)+ky_2-\tilde{c}_1 x_1 f(x_1,x_2,x_3)-\tilde{c}_2 x_3 h(x_1,x_2,x_3)-kx_2$$
$$+(\vartheta_2+\rho_2)sign(s)-(\vartheta_1+\rho_1)sign(s)+\eta sign(s)$$

(21)

Where $\vartheta_1,\vartheta_2,\rho_1,\rho_2$ are estimations for $\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\rho}_1,\hat{\rho}_2$ respectively. To tackle $\Delta g(x_1,x_2,x_3)$, $\Delta g(y_1,y_2,y_3)$, $d_1(t)$ and $d_2(t)$ appropriate adaptive laws can be proposed as;

$$\dot{\vartheta}_1=\lambda_1|s| \quad ; \quad \vartheta_1(0)=\vartheta_{1^0}, \qquad \dot{\vartheta}_2=\lambda_3|s| \quad ; \quad \vartheta_2(0)=\vartheta_{2^0}$$
$$\dot{\rho}_1=\lambda_2|s| \quad ; \quad \rho_1(0)=\rho_{1^0} \quad , \quad \dot{\rho}_2=\lambda_4|s| \quad ; \quad \rho_2(0)=\rho_{2^0}$$

(22)

Where λ_1, λ_2, λ_3 and λ_4 are positive constants and ϑ_{1^0}, , ρ_{1^0} , ϑ_{2^0} and ρ_{2^0} are the initial values of the update parameters $\vartheta_1,\rho_1,\vartheta_2,\rho_2$.

Remark 4.2. In this study, the system has been disrupted by the negative factors, uncertainty and external disturbances, which are assumed to be bounded. The purpose behind the various conducted research was the application of known bounds which were believed to have a given constant [32,34]. However, it is not always possible to assume that the uncertainty and external disturbance are bounded with known constants. In this case, the bounds of the uncertainty and external disturbance are estimated by the adaptive laws (22). This is the solution to this problem that bounds are unknown.

Theorem 4.1. The proposed sliding surface is asymptotically stable by applying the controller (21) and adaptive rules (19), (22), and in the presence of uncertainty and external disturbance if $\hat{\vartheta}_1>0$, $\hat{\rho}_1>0$, $\vartheta_1<0$, $\rho_1<0$, $\hat{\eta}>0$ be chosen.

Proof. By adopting Lyapunov function $V=s^2+\frac{1}{\lambda_1}(\vartheta_1-\hat{\vartheta}_1)^2+\frac{1}{\lambda_2}(\rho_1-\hat{\rho}_1)^2+\frac{1}{\lambda_3}(\vartheta_2-\hat{\vartheta}_2)^2+\frac{1}{\lambda_4}(\rho_2-\hat{\rho}_2)^2+\frac{1}{l}(\eta-\hat{\eta})^2$, it is guaranteed that the state jumps at the switching instants.

$$\dot{V}=2\left\{s\dot{s}+\frac{1}{\lambda_1}(\vartheta_1-\hat{\vartheta}_1)\dot{\vartheta}_1+\frac{1}{\lambda_2}(\rho_1-\hat{\rho}_1)\dot{\rho}_1+\frac{1}{\lambda_3}(\vartheta_2-\hat{\vartheta}_2)\dot{\vartheta}_2+\frac{1}{\lambda_4}(\rho_2-\hat{\rho}_2)\dot{\rho}_2+\frac{1}{l}(\eta-\hat{\eta})\dot{\eta}\right\}$$

(23)

With the comparative rule application of (19) and (20), and the controller (21) in equation (23) we have;

$$\dot{V} = 2\{s[g(y_1,y_2,y_3)-\beta y_2+\Delta g(y_1,y_2,y_3)+d_2(t)-g(y_1,y_2,y_3)+g(x_1,x_2,x_3)-\beta(x_2-y_2)$$
$$-\tilde{c}_1 y_1 f(y_1,y_2,y_3)-\tilde{c}_2 y_3 h(y_1,y_2,y_3)-ky_2+\tilde{c}_1 x_1 f(x_1,x_2,x_3)+\tilde{c}_2 x_3 h(x_1,x_2,x_3)+kx_2$$
$$-(\vartheta_2+\rho_2)sign(s)+(\vartheta_1+\rho_1)sign(s)-\eta sign(s)-g(x_1,x_2,x_3)+\beta x_2-\Delta g(x_1,x_2,x_3)-d_1(t)$$
$$-\tilde{c}_1 x_1 f(x_1,x_2,x_3)-\tilde{c}_2 x_3 h(x_1,x_2,x_3)-kx_2+\tilde{c}_1 y_1 f(y_1,y_2,y_3)+\tilde{c}_2 y_3 h(y_1,y_2,y_3)+ky_2]$$
$$+(\vartheta_1-\hat{\vartheta}_1)|s|+(\rho_1-\hat{\rho}_1)|s|+(\vartheta_2-\hat{\vartheta}_2)|s|+(\rho_2-\hat{\rho}_2)|s|+(\eta-\hat{\eta})|s|\}$$
$$=2\{s[(\Delta g(y_1,y_2,y_3)+d_2(t))-(\Delta g(x_1,x_2,x_3)+d_1(t)]-(\vartheta_2+\rho_2)|s|+(\vartheta_1+\rho_1)|s|-\eta|s|$$
$$+(\vartheta_1-\hat{\vartheta}_1)|s|+(\rho_1-\hat{\rho}_1)|s|+(\vartheta_2-\hat{\vartheta}_2)|s|+(\rho_2-\hat{\rho}_2)|s|+(\eta-\hat{\eta})|s|\}$$
(24)

It is obvious that

$$\dot{V} \le 2\{|s|[(|\Delta g(y_1,y_2,y_3)|+|d_2(t)|)-(|\Delta g(x_1,x_2,x_3)|+|d_1(t)|)]-(\vartheta_2+\rho_2)|s|$$
$$+(\vartheta_1+\rho_1)|s|-\eta|s|+(\vartheta_1-\hat{\vartheta}_1)|s|+(\rho_1-\hat{\rho}_1)|s|+(\vartheta_2-\hat{\vartheta}_2)|s|+(\rho_2-\hat{\rho}_2)|s|+(\eta-\hat{\eta})|s|\}$$
(25)

By Assumptions 3.1 and 3.2, one can derive:

$$\dot{V} \le 2\{(\hat{\vartheta}_2+\hat{\rho}_2)|s|-(\hat{\vartheta}_1+\hat{\rho}_1)|s|-(\vartheta_2+\rho_2)|s|+(\vartheta_1+\rho_1)|s|-\eta|s|$$
$$+(\vartheta_1-\hat{\vartheta}_1)|s|+(\rho_1-\hat{\rho}_1)|s|+(\vartheta_2-\hat{\vartheta}_2)|s|+(\rho_2-\hat{\rho}_2)|s|+(\eta-\hat{\eta})|s|\}$$
$$=2\left\{-2\hat{\vartheta}_1|s|-2\hat{\rho}_1|s|+2\vartheta_1|s|+2\rho_1|s|-\hat{\eta}|s|\right\}<0$$
(26)

As it was observed, to obtain $\dot{V}<0$, choose the gains $\hat{\vartheta}_1>0$, $\hat{\rho}_1>0$, $\vartheta_1<0$, $\rho_1<0$, $\hat{\eta}>0$.

Remark 4.3. Since the designed controller here is an adaptive sliding mode, $\tanh(\tau s)$ function is applied rather than $sign(s)$ function to avoid the adverse chattering phenomenon where τ is a positive scalar. So the adaptive sliding mode controller law can be proposed as:

$$u(t)=g(y_1,y_2,y_3)-g(x_1,x_2,x_3)+\beta(x_2-y_2)+\tilde{c}_1 y_1 f(y_1,y_2,y_3)+$$
$$\tilde{c}_2 y_3 h(y_1,y_2,y_3)+ky_2-\tilde{c}_1 x_1 f(x_1,x_2,x_3)-\tilde{c}_2 x_3 h(x_1,x_2,x_3)-kx_2$$
$$+(\vartheta_2+\rho_2)sign(s)-(\vartheta_1+\rho_1)sign(s)+\mu\tanh(\tau s)$$
(27)

With

$$\dot{\mu}=l_1|s\tanh(\tau s)| \quad ; \qquad \mu(0)=\mu_0$$
(28)

where l_1 is a positive constant and μ_0 is the initial value of the update vector parameter μ.

Theorem 4.2. The proposed sliding surface is asymptotically stable with the application of controller (27) and adaptive laws (22), (28), and in the presence of uncertainty and external disturbance if $\psi>0$, $\hat{\rho}_1>0$, $\vartheta_1<0$, $\rho_1<0$ is chosen.

Proof. Let's consider the following Lyapunov candidate:

$$V = s^2 + \frac{1}{\lambda_1}(\vartheta_1 - \hat{\vartheta}_1)^2 + \frac{1}{\lambda_2}(\rho_1 - \hat{\rho}_1)^2 + \frac{1}{\lambda_3}(\vartheta_2 - \hat{\vartheta}_2)^2 + \frac{1}{\lambda_4}(\rho_2 - \hat{\rho}_2)^2 + \frac{1}{l_1}(\mu - \psi)^2$$

(29)

Like the proof of Theorem 4.1, we can derive;

$$\dot{V} \le 2\{(\hat{\vartheta}_2 + \hat{\rho}_2)|s| - (\hat{\vartheta}_1 + \hat{\rho}_1)|s| - (\vartheta_2 + \rho_2)|s| + (\vartheta_1 + \rho_1)|s| - \mu s \tanh(\tau s)$$
$$+(\vartheta_1 - \hat{\vartheta}_1)|s| + (\rho_1 - \hat{\rho}_1)|s| + (\vartheta_2 - \hat{\vartheta}_2)|s| + (\rho_2 - \hat{\rho}_2)|s| + (\mu - \psi)|s \tanh(\tau s)|\}$$

(30)

Equation (30) is rewritten using Lemma 2.2 as follows;

$$\dot{V} \le 2\{-2\hat{\rho}_1|s| + 2\vartheta_1|s| + 2\rho_1|s| - \mu|s \tanh(\tau s)| + (\mu - \psi)|s \tanh(\tau s)|\}$$
$$= 2\{-2\hat{\rho}_1|s| + 2\vartheta_1|s| + 2\rho_1|s| - \psi|s||\tanh(\tau s)|\} < 0$$

(31)

It means that the designed sliding surface is asymptotically stable with an appropriate selection of adaptive gains. Thus, the proof is achieved completely.

5. Simulation Results

In this section, fractional-order Chen systems and fractional-order Lorenz systems were examined using two proposed examples of the adaptive sliding mode controller performance. The output results support theoretical ones.

Example1.

Fractional-order chen chaotic system is a subset of chaotic systems mentioned in Table 1. The related state equation is as follows:

$$D^{q_1}x = a(y - x)$$
$$D^{q_2}y = dx - xz + cy$$
$$D^{q_3}z = xy - bz \qquad (32)$$

Where $(a, b, c, d) = (35, 3, 28, -7)$.

Chaotic behavior has been shown without uncertainty, external disturbance and input in Fig.1. Systems of Master (9) in the presence of uncertainty $\Delta g(x_1, x_2, x_3) = 0.5\cos(3\pi x_2)$ and external disturbance $d_1(t) = 0.7\sin 3\mathrm{t}$, and systems of slave (10) in the presence of uncertainty $\Delta g(y_1, y_2, \mathrm{y}_3) = 0.6\sin(\pi y_2)$ and external disturbance $d_2(t) = 0.8\cos 2\mathrm{t}$ and controller (27) were considered. To synchronize two fractional order, chen systems in the master-slave structure for master system with the initial conditions of $\left[{}_0D_t^{-0.1}x_1(0),\ {}_0D_t^{-0.1}x_2(0),\ {}_0D_t^{-0.05}x_3(0)\right]^T = [-9, -5, 14]^T$,and the slave system with initial conditions of $\left[{}_0D_t^{-0.1}y_1(0),\ {}_0D_t^{-0.1}y_2(0),\ {}_0D_t^{-0.05}y_3(0)\right]^T = [1, 7, -8]^T$, and fractional-order $q = [0.9, 0.9, 0.95]$ for both systems were proposed. As it was noted above in Remark 3.2, systems of (9) and (10) are called incommensurate fractional-order systems. There exist appropriate positive constants $\hat{\vartheta}_1, \hat{\vartheta}_2, \hat{\rho}_1, \hat{\rho}_2$ such that;

$$|\Delta g(x_1,x_2,x_3)| < \hat{\vartheta}_1 \quad , \quad |d_1(t)| < \hat{\rho}_1 \quad , \quad |\Delta g(y_1,y_2,y_3)| < \hat{\vartheta}_2 \quad , \quad |d_2(t)| < \hat{\rho}_2 \tag{33}$$

Based on the sliding surface (12) and the proposed controller law (27) for synchronization, the following functions are achieved:

$$s(t) = D^{q_2-1}e_2 - \psi(t) \tag{34}$$

$$\dot{\psi}(t) = [\tilde{c}_1 ax_1 + \tilde{c}_2 x_1x_3 + kx_2] - [\tilde{c}_1 ay_1 + \tilde{c}_2 y_1y_3 + ky_2] \tag{35}$$

$$\begin{aligned} u(t) &= dy_1 - y_1y_3 + cy_2 - dx_1 + x_1x_3 - cx_2 + \beta(x_2 - y_2) + \tilde{c}_1 ay_1 + \tilde{c}_2 y_1y_3 + ky_2 \\ &- \tilde{c}_1 ax_1 - \tilde{c}_2 x_1x_3 - kx_2 + (\vartheta_2 + \rho_2)sign(s) - (\vartheta_1 + \rho_1)sign(s) + \mu \tanh(0.1s) \end{aligned} \tag{36}$$

So, adaptive laws as $\dot{\vartheta}_1 = 0.1|s|$, $\dot{\rho}_1 = 0.4|s|$, $\dot{\vartheta}_2 = 0.3|s|$, $\dot{\rho}_2 = 0.7|s|$, $\dot{\mu} = 0.6|s \tanh(0.1s)|$ and $\tilde{c}_1 = \tilde{c}_2 = 1$ are determined. Vectors $\vartheta_1, \vartheta_2, \rho_1, \rho_2$ are also updated to the initial condition $[2.1, 2.1, 2.1, 2.1]$. Simulation results for $k = 3.976$ are shown in Figs 2-5. Fig.2 shows the convergence of the errors to zero. Fig. 3 shows synchronization of system variable via effective adaptive sliding mode controller $x_1, x_2, x_3, y_1, y_2, y_3$. The time responses of the update vector parameters $\vartheta_1, \vartheta_2, \rho_1, \rho_2, \mu$ are shown in Fig.4. The adaptive parameter values are clearly seen to be bounded. The time histories of the applied control input (27) are plotted in Fig.5.

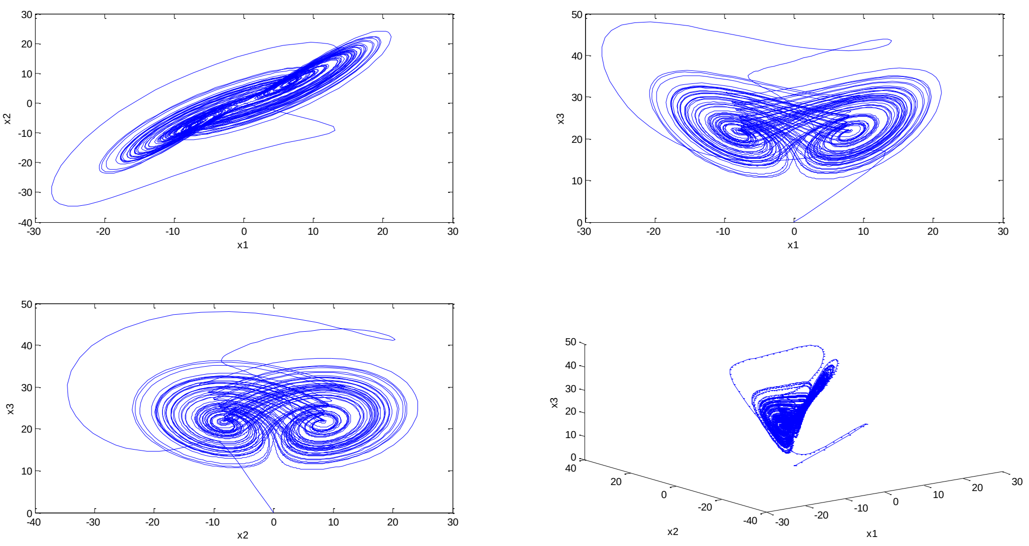

Figure 1. Phase Diagram of Chen System with Fractional Order *q*= [0.9, 0.9, 0.95]

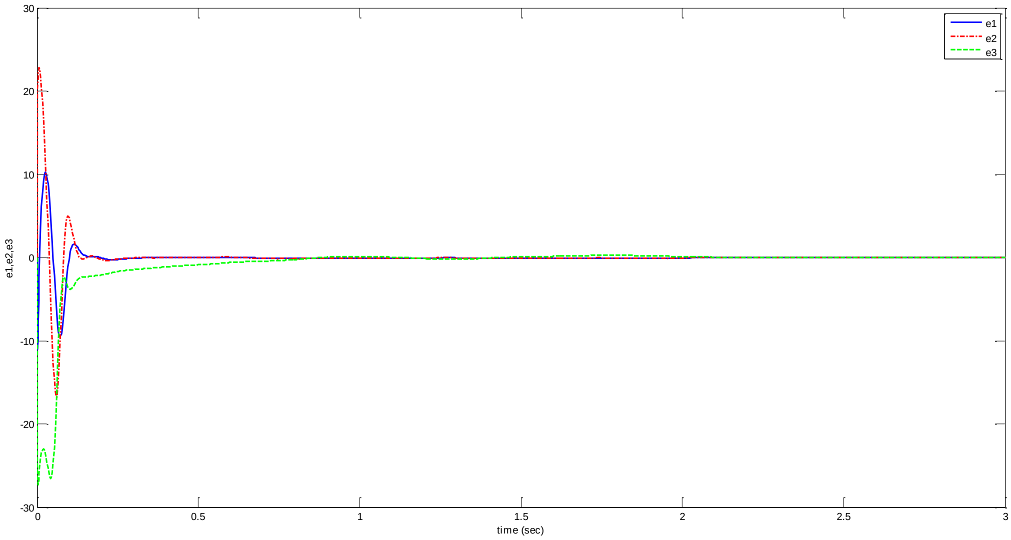

Figure 2. The State Trajectories of Error

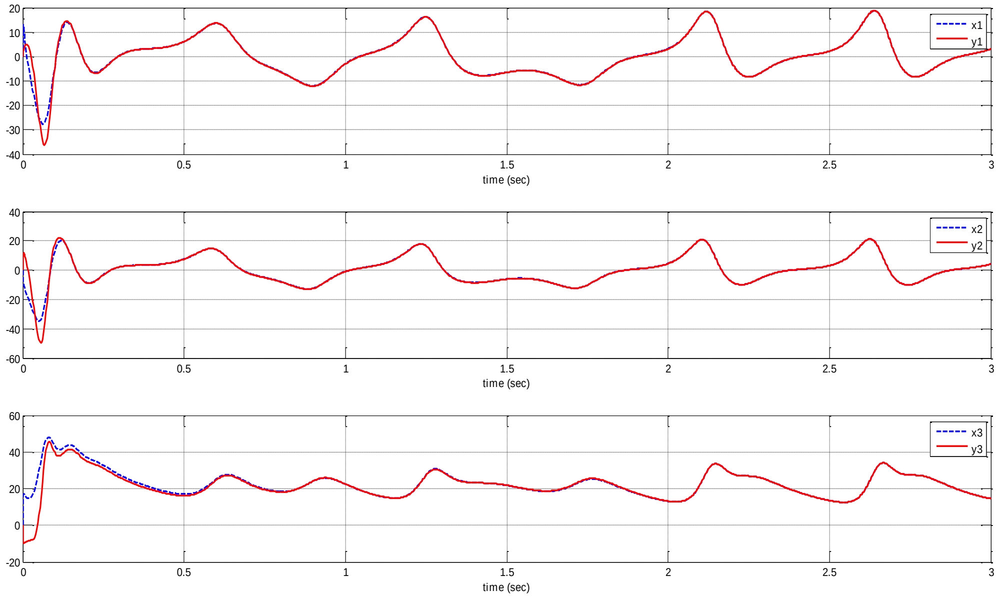

Figure 3. Synchronization of the State Variables

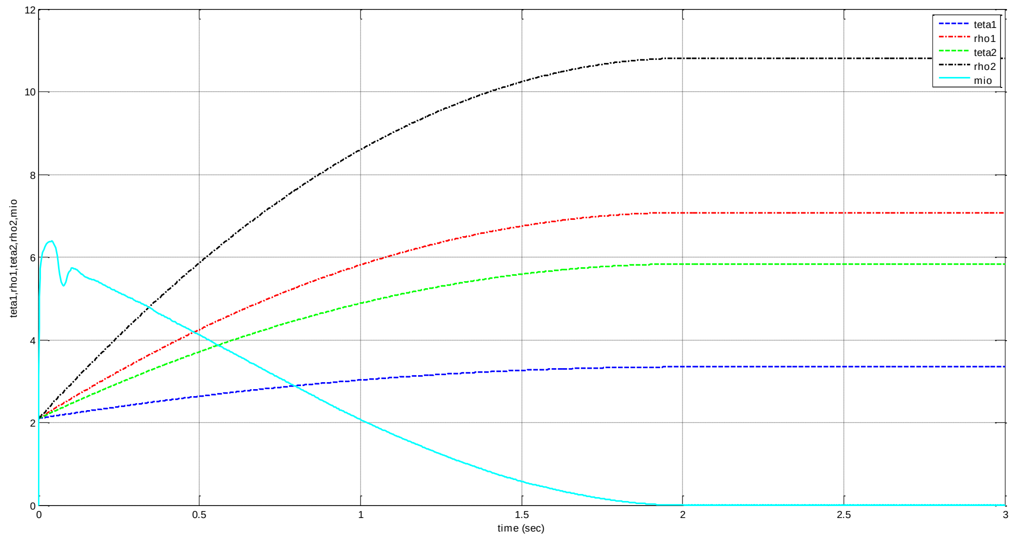

Figure 4. The Time Responses of the Update Vector Parameters $\vartheta_1, \vartheta_2, \rho_1, \rho_2, \mu$

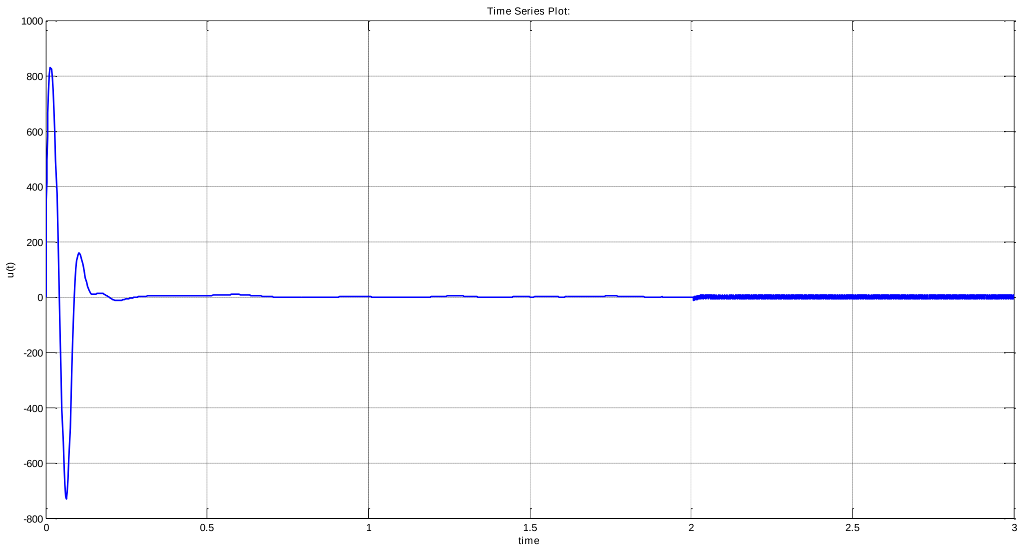

Figure 5. Control Function in Synchronization Procedure

Example 2.

The proposed controller effectiveness in the presence of external disturbance and uncertainty for fractional-order Lorenz chaotic system taken from table 1 are examined in this example. Fractional-order Lorenz chaotic system equations are as follows:

$$\begin{aligned} D^{q_1}x &= a(y-x) \\ D^{q_2}y &= x(b-z)-y \\ D^{q_3}z &= xy-cz \end{aligned} \tag{37}$$

Where $(a,b,c)=(10,28,\frac{8}{3})$.

Chaotic behavior has been shown without uncertainty, external disturbance and input in Fig.6.

Systems of Master (9) in the presence of uncertainty $\Delta g(x_1,x_2,x_3)=0.7\sin(2\pi x_1)$ and external disturbance $d_1(t)=0.1\cos 3\text{t}$, and systems of slave (10) in the presence of uncertainty $\Delta g(y_1,y_2,\text{y}_3)=0.4\cos(\pi y_3)$ and external disturbance $d_2(t)=0.9\sin 4\text{t}$ and controller (27) are considered. To synchronize, two fractional order Lorenz systems in the master-slave structure for master system with the initial conditions of $\left[{}_0D_t^{-0.003}x_1(0),\ {}_0D_t^{-0.002}x_2(0),\ {}_0D_t^{-0.001}x_3(0)\right]^T=[-8,9,-9]^T$,and the slave system with initial conditions of $\left[{}_0D_t^{-0.003}y_1(0),\ {}_0D_t^{-0.002}y_2(0),\ {}_0D_t^{-0.001}y_3(0)\right]^T=[7,-15,14]^T$, and fractional-order $q=[0.997,0.998,0.999]$ for both systems are presented. As it was noted above in Remark 3.2, systems of (9) and (10) are called incommensurate fractional-order systems. There exist appropriate positive constants $\hat{\vartheta}_1,\hat{\vartheta}_2,\hat{\rho}_1,\hat{\rho}_2$ such that;

$$\left|\Delta g(x_1,x_2,x_3)\right|<\hat{\vartheta}_1\quad,\quad\left|d_1(t)\right|<\hat{\rho}_1\quad,\quad\left|\Delta g(y_1,y_2,y_3)\right|<\hat{\vartheta}_2\quad,\quad\left|d_2(t)\right|<\hat{\rho}_2 \tag{33}$$

Based on the sliding surface (12) and the proposed controller law (27) for synchronization, the following functions are obtained:

$$s(t)=D^{q_2-1}e_2-\psi(t) \tag{38}$$

$$\dot{\psi}(t)=[\tilde{c}_1 ax_1+\tilde{c}_2 x_1x_3+kx_2]-[\tilde{c}_1 ay_1+\tilde{c}_2 y_1y_3+ky_2] \tag{39}$$

$$u(t)=by_1-y_1y_3-bx_1+x_1x_3+\beta(x_2-y_2)+\tilde{c}_1 ay_1+\tilde{c}_2 y_1y_3+ky_2$$
$$-\tilde{c}_1 ax_1-\tilde{c}_2 x_1x_3-kx_2+(\vartheta_2+\rho_2)sign(s)-(\vartheta_1+\rho_1)sign(s)+\mu\tanh(0.1\text{s}) \tag{40}$$

So, adaptive laws as $\dot{\vartheta}_1=0.1|s|$, $\dot{\rho}_1=0.4|s|$, $\dot{\vartheta}_2=0.3|s|$, $\dot{\rho}_2=0.7|s|$, $\dot{\mu}=0.6|s\tanh(0.1\text{s})|$ and $\tilde{c}_1=\tilde{c}_2=1$ were determined. Vectors $\vartheta_1,\vartheta_2,\rho_1,\rho_2$ were also updated to the initial condition $[2.1,2.1,2.1,2.1]$. Simulation results for $k=1.143$ were shown in Figs 7-10. Fig.7 shows the convergence of the errors to zero. In Fig. 8 the effectiveness of the adaptive sliding mode controller to synchronize of system variables x_1,x_2,x_3,y_1,y_2,y_3 is verified. The time responses of the update vector parameters $\vartheta_1,\vartheta_2,\rho_1,\rho_2,\mu$ are shown in Fig.9. The adaptive parameter values are clearly seen to be bounded. The time histories of the applied control input (27) are plotted in Fig.10.

For performance validation of proposed operation scenario using Matlab/Simulink. According to the above examples, the proper functioning of the designed controller to uncertainty and external disturbance is obvious.

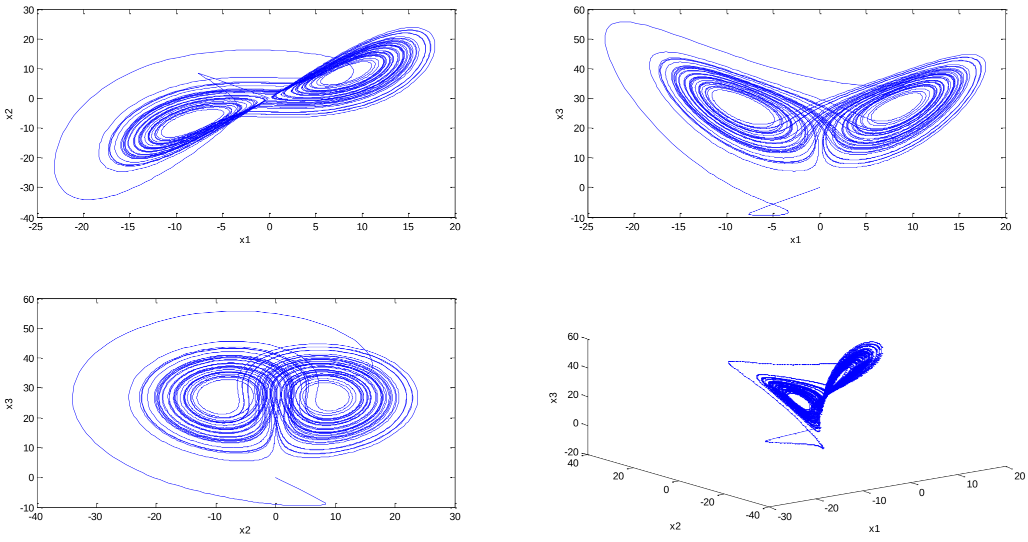

Figure 6. Phase Diagram of Lorenz System with Fractional Order q= [0.997, 0.998, 0.999]

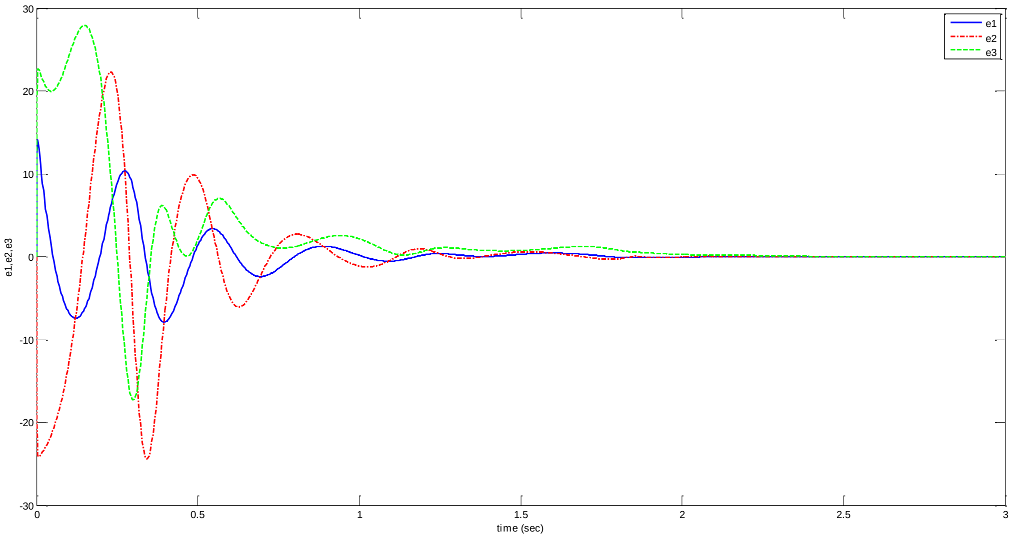

Figure 7. The State Trajectories of Error System

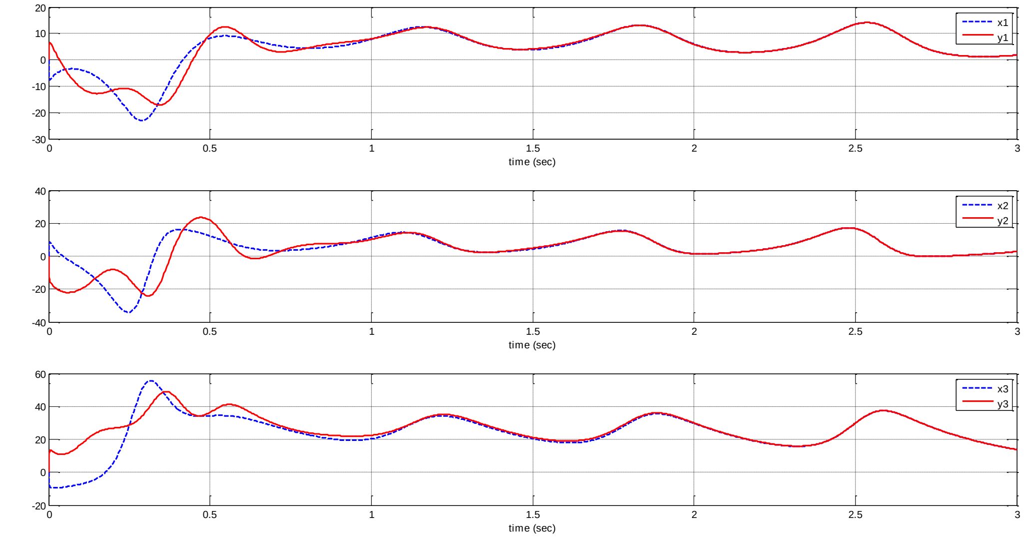

Figure 8. Result of Synchronizatin of the State Variables

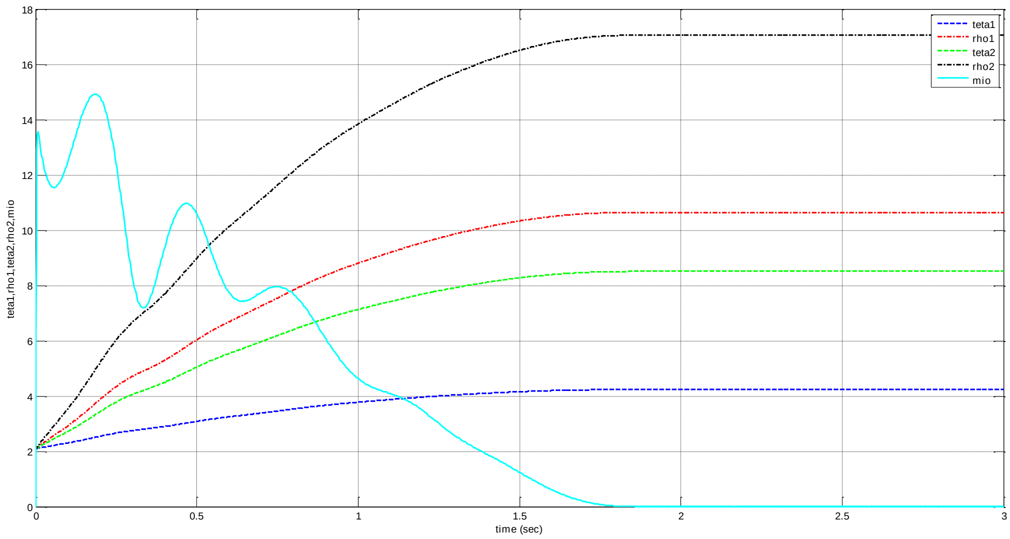

Figure 9. The Time Responses of the Update Vector Parameters

$\vartheta_1, \vartheta_2, \rho_1, \rho_2, \mu$

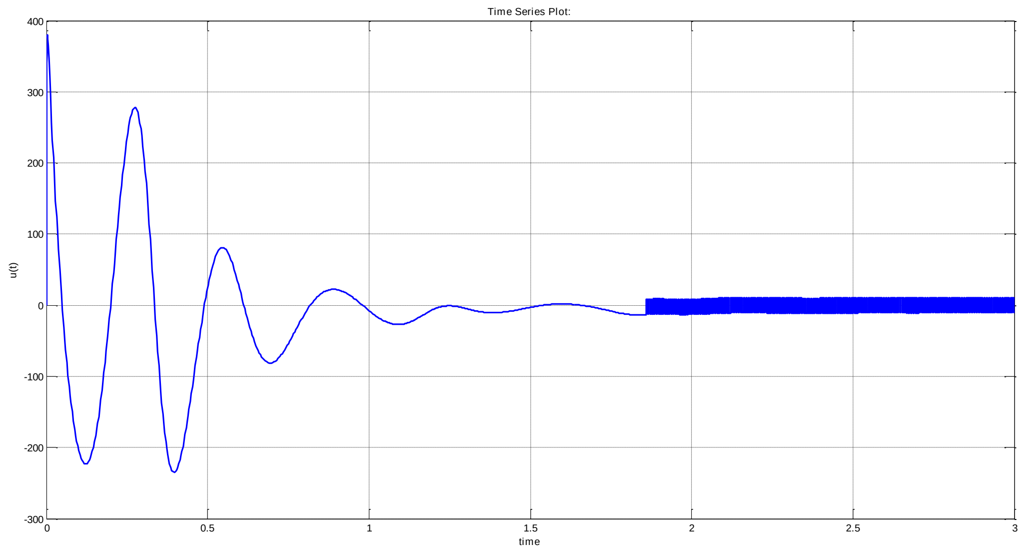

Figure 10. Control Function in Synchronization Procedure

6. Conclusion

In this paper, an innovative adaptive sliding mode synchronizer controller is proposed for fractional-order chaotic system in master-slave structure in the presence of uncertainty and external disturbance. Sliding surface and adaptive laws are applied to design this controller. Moreover, a novel class of fractional-order systems has been introduced. Proper adaptive laws are designed here to counter against uncertainty and external disturbance, helping the system to do synchronization more appropriately.

It is also assumed that the bounds of uncertainties and external disturbance are unknown.

The optimal performance of the controller in mitigating the chattering phenomenon is quite evident with proving the proposed adaptive sliding mode controller stability.

References

[1] N Li, W. Pan, S. Xiang, L. Yan, B. Luo and X, Zou, "Influence of statistical distribution properties on ultrafastrandom-number generation using chaotic semiconductor lasers", Optik, vol. 125, (2014), pp. 3555-3558.

[2] W. Gao, "Study on statistical properties of chaotic laser light", Phys. Lett. A., **(2012)**, pp. 331: 292-97.

[3] S. Banerjee, S. Mukhopadhyay and A.L. Rondoni, "Multi-image encryption based on synchronization of chaotic lasers and iris authentication", Opt Laser Eng., vol. 50, **(2012)**, 950-57.

[4] T, Lee, "Verifier-based three-party authentication schemesusing extended chaotic maps for data exchange intelecare medicine information systems", computer methods and programs in biomedicine, vol. 117, **(2014)**, pp. 646-472.

[5] A, Provata, P. Katsaloulis and D.A. Verganelakis, "Dynamics of chaotic maps for modelling themultifractal spectrum of human brain", Diffusion Tensor Images, Chaos, Solitons & Fractals, vol. 45, **(2012)**, pp. 174-180.

[6] Y. Sarbaz, F. Towhidkhah, A. Jafari and S. Gharibzadeh, "Do the chaotic features of gait change in Parkinson's disease? Journal of Theoretical Biology, vol. 307, **(2012)**, pp. 160-167.

[7] A.A. Liolios and A.K. Boglou, "Chaotic behaviour in the non-linear optimal control of unilaterally contacting building systems during earthquakes", Chaos, Solitons and Fractals, vol. 17, **(2003)**, pp. 493-498.

[8] M. Pe´rez-Polo and M. Pe´rez-Molina, "Saddle-focus bifurcation and chaotic behavior of a continuous stirred tank reactor using PI control", Chemical Engineering Science, vol. 74, **(2012)**, pp. 79-92.

[9] S.M. Hosseinalipour, A. Tohidi, M. Shokrpour and N.M. Nouri, "Introduction of a chaotic dough mixer, part A: mathematical modeling and numerical simulation", Journal of Mechanical Science and Technology, vol. 27, no. 5, **(2013)**, pp. 1329-1339.
[10] J. Kupka, "Some chaotic and mixing properties of fuzzified dynamical Systems", Information Sciences, vol. 279, **(2014)**, pp. 642-653.
[11] O.E. Turgut, M.S. Turgut and M.T. Coban, "Chaotic quantum behaved particle swarm optimization algorithm for solving nonlinear system of equations", Computers and Mathematics with Applications, vol. 68, **(2014)**, pp. 508-530.
[12] M. Akhmet, Z. Akhmetova and M.O. Fen, "Chaos in economic models with exogenous shocks", Journal of Economic Behavior & Organization, vol. 106, **(2014)**, pp. 95-108.
[13] I. Pan, A. Korre, S. Das and S. Durucan, "Chaos suppression in a fractional order financial system using intelligent regrouping PSO based fractional fuzzy control policy in the presence offractional Gaussian noise", Nonlinear Dyn., vol. 70, **(2012)**, pp. 2445-2461.
[14] I. Pan and S. Das, "Multi-objective active control policy design for commensurate and incommensurate fractional order chaotic financial systems, Applied Mathematical Modelling", **(2014)**
[15] R. Toledo-Hernandez, V. Rico-Ramirez, G.A. Iglesias-Silva and M.U. Diwekar, "A fractional calculus approach to the dynamic optimization of biological reactive systems", Part I: Fractional models for biological reactions, Chemical Engineering Science, vol. 117, **(2014)**, pp. 217-228.
[16] J.R. Wang and X. Li, "Periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses", J Applied Math Computer, **(2013)**.
[17] A. Duarte-Mermoud, N. Aguila-Camacho, A.J. Gallegos and R. Castro-Linares, "Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems", Common Nonlinear Sci. Number Simulation, vol. 22, **(2015)**, pp. 650-659.
[18] F. Cort´es and M.J. Elejabarrieta, "Finite element formulations for transient dynamic analysis in structural systems with viscoelastic treatments containing fractional derivative models", International Journal for Numerical Methods in Engineering, vol. 69, **(2007)**, pp. 2173-2195.
[19] M.P. Aghababa, "Chaotic behavior in fractional-order horizontal platform systems and its suppression using a fractional finite-time control strategy", Journal of Mechanical Science and Technology, vol. 28, no. 5, **(2014)**, pp. 1875-1880.
[20] M. Srivastava, S.K. Agrawal, K. Vishal and S. Das, "Chaos control of fractional order Rabinovich–Fabrikant system and synchronization between chaotic and chaos controlled fractional order Rabinovich–Fabrikant system", Applied Mathematical Modelling, vol. 38, (2014), pp. 61–3372.
[21] M.R. Faieghi, H. Delavari and D. Baleanu, "Control of an uncertain fractional-order Liu system via fuzzy fractional-order sliding mode control", Journal of Vibration and Control, vol. 18, no. 9, **(2011)**, pp. 1366-1374.
[22] Y. Xu, H. Wang, D. Liu and H. Huang, "Sliding mode control of a class of fractional chaotic systems in the presence of parameter perturbations", Journal of Vibration and Control, **(2013)**, pp. 1-14.
[23] B. Wang, J. Xue, D. Chen, "Takagi-Sugeno fuzzy control for a wide class of fractional-order chaotic systems with uncertain parameters via linear matrix inequality", Journal of Vibration and Control, **(2014)**, pp. 1-14.
[24] M.P. Aghababa and H. Feizi, "Design of a sliding mode controller for synchronizing chaotic systems with parameter and model uncertainties and external disturbances", Transactions of the Institute of Measurement and Control, vol. 34, no. 8, **(2012)**, pp. 990-997.
[25] Q. Wei, X. Wang and X. Hu, "Adaptive hybrid complex projective synchronization of chaotic complex system", Transactions of the Institute of Measurement and Control, vol. 36, no. 8, **(2014)**, pp. 1093-1097.
[26] Q. Wei, X. Wang and X. Hu, "Finite-time hybrid projective synchronization of unified chaotic system", Transactions of the Institute of Measurement and Control, **(2014)**, pp. 1-5.
[27] T. Wang, S. Zhao, W. Zhou and W. Yu, "Finite-time master slave synchronization and parameter identification for uncertain Lurie systems", ISA Transactions, vol. 53, **(2014)**, pp. 1184-1190.
[28] M.R. Faieghi and H. Delavari, "Chaos in fractional-order Genesio–Tesi system and its Synchronization", Common Nonlinear Sci. Number Simulation, vol. 17, **(2012)**, pp. 731-741.
[29] C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue and V. Feliu, "Fractional-order systems and controls", Springer, **(2010)**.
[30] M.R. Faieghi, H. Delavari and D. Baleanu, "A note on stability of sliding mode dynamics in suppression of fractional-order chaotic systems", Computers and Mathematics with Applications, vol. 66, **(2013)**, pp. 832-837.
[31] M. Roopaei, B.R. Sahraei and T.C. Lin, "Adaptive sliding mode control in a novel class of chaotic systems", Common Nonlinear Sci. Number Simulation, vol. 15, **(2010)**, pp. 4158-4170.
[32] C. Yin, S.M. Zhong and W.F. Chen, "Design of sliding mode controller for a class of fractional-order chaotic systems", Common Nonlinear Sci. Number Simulation, vol. 17, **(2012)**, pp. 356-366.

[33] P.M. Aghababa, Comments on "Control of a class of fractional-order chaotic systems via sliding mode", Nonlinear Dyn., vol. 67, **(2012)**, pp. 903-908.
[34] Z. Wang, X. Huang and H. Shen, "Control of an uncertain fractional order economic system via adaptive sliding mode", Neurocomputing, vol. 83, **(2012)**, pp. 83–88.
[35] M.R. Dastranj, A. Riahi and M. Moghaddas, "Synchronization of Chaotic Fractional-Order Lu-Lu Systems with Active Sliding Mode Control", International Journal of Control and Automation, vol. 7, no. 9, **(2014)**, pp. 131-138.
[36] H.K. Khalil, "Nonlinear System", third ed., Prentice Hall, New Jersey, **(2002)**.

Authors

Aminreza Riahi, He received his B.S. and M.S. Degree in Control & Automation Engineering from Islamic Azad University,Gonabad Branch, respectively. He is cuurently a Ph.D. Student at Ferdowsi University of Mashhad and a member of Young Researchers and Elite Club IAU in Iran. His research fields include Fractional-order systems, control engineering and soft computing, especially on complex nonlinear systems.

Naeimadeen Noghredani, He received his B.S. degree in Electronic Engineering 2011 at Islamic Azad University, Birjand and M.S. degree in Control and Automation Engineering in 2014 from Islamic Azad University,Gonabad Branch.He is currently a member of Young Researchers and Elite Club IAU in Iran. His research interests include complex systems, control systems and chaotic fractional-order derivative systems.

Mahnoosh Shajiee, She received her B.S. degree in Electronic Engineering from the Sadjad Industrial University,Mashhad, and the M.S. in Control Engineering from Ferdowsi University Of Mashhad. She has one year's experience of working as a research and teacher assistant at Ottawa University,Canada, She is currently a Ph.D. candidate at Ferdowsi University of Mashhad. Her research interests focus on Medical Engineering, Control of Wind Turbine and RF Circuit Analysis.

Naser Pariz, He received his B.S. degree in 1987 in Electrical Engineering and M.S. and Ph.D. Degrees in 1991 and 2001 in Electrical Engineering specializing in Control from Ferdowsi University of Mashhad, Mashhad, Iran. His Doctoral dissertation concerned on nonlinear analysis of power systems.Currently, he is Full Professor at Department of Electrical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran. His main research interests include Nonlinear Systems analysis, Switching Systems and Power System analysis.

