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Further Results on Output Tracking Problem of Uncertain Nonlinear Systems with High-Order Nonlinearities

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Abstract

This paper considers the global practical tracking problem by state feedback for a class of high-order non-linear systems with more general uncertainties, to which the existing control methods are inapplicable. We successfully propose a new tracking control design scheme for the system studied by introducing sign function and necessarily modifying the method of adding a power integrator. It is shown that the designed controller guarantees that all states of the resulting closed-loop system are globally bounded and the tracking error remains prescribed arbitrarily small after a finite time.

***Keywords**: uncertain nonlinear systems, practical output tracking, sign function, state feedback*

1. Introduction

The output tracking problem is one of most important subjects in control theory and its applications, and it has been extensively studied for the last three decades. Its basic problem is to design a feedback control law making the controlled output track a given reference signal as much as possible. The output tracking in the usual case is in the sense of "*asymptotic*" where the tracking error converges to zero as time goes to the infinity, and this *asymptotic* tracking problem for time-invariant linear systems was completely solved about thirty years ago.

The corresponding problem for nonlinear systems was also carried out by a number of researchers at least over the past twenty years [1, 2], *etc.*

However, in the case of inherently nonlinear systems where the linearized systems may not be stabilizable and/or detectable, the tracking problem and even the stabilization problem become much more complicated and difficult to solve.

Thus, to overcome this difficulty, a new concept called practical output tracking for tracking problem has been introduced and various results in the framework of the new concept have been reported see [3-9], as well as the references therein.

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This paper deals with the practical output tracking problem with a state feedback for a class of high-order nonlinear systems having the following form:

$$\begin{aligned} \dot{z}_1 &= z_2^{p_1} + \phi_1(t, z, u) \\ \dot{z}_2 &= z_3^{p_2} + \phi_2(t, z, u) \\ &\vdots \\ \dot{z}_{n-1} &= z_n^{p_{n-1}} + \phi_{n-1}(t, z, u), \\ \dot{z}_n &= u + \phi_n(t, z, u), \\ y &= z_1, \end{aligned} \tag{1}$$

where $z = (z_1, \ldots, z_n)^{\mathrm{T}} \in R^n$ and $u \in R$ are the system state and the control input, respectively. For $i = 1, \ldots, n$, $\phi_i(t, z, u)$ are unknown continuous functions and $p_i \in R_{odd}^{\geq 1} := \{p/q \in R^+ : p$ and q are odd integers, $p \geq q\}$ $(i = 1, \ldots, n-1)$ are said to be the high orders of the system, with p_n obviously equal to one (which is not a limitation since we can easily set $v := u^{p_n}$ in the case of non-unity p_n).

We first introduce definition of the practical output tracking problem.

Consider the system (1) and assume that the reference signal $y_r(t)$ be a time-varying C^1-bounded on $[0, \infty)$. Then, the *global practical output tracking problem* by a *state controller* is formulated as follows: For any given real number $\varepsilon > 0$, design a continuous controller having the following structure

$$u = u(z, y_r(t)), \tag{2}$$

such that

(i) all the states of the closed-loop system (1) and (2) are well-defined on $[0, \infty)$ and globally bounded;

(ii) the global practical output tracking is achieved, that is, for every $z(0) \in R^n$ there is a finite time $T := T_{(\varepsilon, z(0))} > 0$, such that the output $y(t)$ of the closed loop system (1) with (2) satisfies

$$|y(t) - y_r(t)| = |z_1(t) - y_r(t)| < \varepsilon, \qquad \forall t \geq T > 0. \tag{3}$$

The problem of output tracking control of nonlinear systems is one of the most important and challenging problems in the field of nonlinear control and lots of efforts have been made during the last decades, see [1-11], as well as the references therein. With the help of the nonlinear output regulator theory [1], [2] and the method of adding a power integrator [12-14], series of research results have been obtained [3-5]. For details, in [3], practical output tracking via smooth state feedback for nonlinear systems was considered. Further, in [7-9], the practical output feedback tracking problem was also investigated for a class of nonlinear systems with higher-order growing unmeasurable states, extending the results on stabilization in [15-18].

In [7-9], the following condition on the uncertain term $\phi_i(\cdot)$ is assumed:

$$|\phi_i(t, z, u)| \leq C\left(|z_1|^{(r_i+\tau)/r_1} + \cdots + |z_i|^{(r_i+\tau)/r_i}\right) + C \tag{4}$$

where $C > 0$, $\tau > 0$ or $-2/p_1 p_2 \cdots p_{n-1}(2n+1) < \tau < 0$ are constants and r_i's are defined as $r_1 = 1,\ r_{i+1} p_i = r_i + \tau > 0, \quad i = 1, \ldots, n$. However, (4) needs the condition of $\tau = l/m$ with *l* being an even integer and *m* being an odd integer, which results in $(r_i + \tau)/r_j$ in (4) being always a ratio of odd integers. Naturally, an interesting problem may be proposed:

(a) *Is it possible to relax the assumption on* τ *in* (4)? (b) *Under the weaker assumption, can one design an output tracking controller*?

In this paper, by introducing the sign function approach, and overcoming several troublesome obstacles in the design and analysis procedure, we focus on solving the above problem under the assumption of the restriction on τ being relaxed to any real number.

2. Mathematical Preliminaries

At first, we give the following notations which will be used in this study.

Notations: R^+ denotes the set of all the nonnegative real numbers and R^n denotes the real n-dimensional space. For any vector $x=(x_1,\ldots,x_n)^T\in R^n$, denote

$$\bar{x}_i := (x_1,\ldots,x_i)^T \in R^i,\ \ i=1,\ldots,n,\ \ \|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}\,.$$

A sign function $\operatorname{sgn}(x)$ is defined as: $\operatorname{sgn}(x)=1$ if $x>0$, $\operatorname{sgn}(x)=0$ if $x=0$, and $\operatorname{sgn}(x)=-1$ if $x<0$.

In order to solve the global practical output tracking problem, we made the following assumption:

Assumption1 For $i=1,\ldots,n$, there are smooth functions $\psi_{ij}(z_1,\ldots,z_i),\ j=1,2$ and $\tau\in\left(-1\Big/\sum_{l=1}^{n} p_1\cdots p_{l-1},0\right)$ such that

$$\left|\phi_i(t,z,u)\right| \le \psi_{i1}(z_1,\ldots,z_i)\left(\left|z_1\right|^{(r_i+\tau)/r_1}+\cdots+\left|z_i\right|^{(r_i+\tau)/r_i}\right)+\psi_{i2}(z_1,\ldots,z_i) \quad (5)$$

where r_i's defined as

$$r_1=1,\quad r_{i+1}p_i=r_i+\tau>0,\quad i=1,\ldots,n. \quad (6)$$

Now, we introduce six technical lemmas which will play an important role and be frequently used in the later control design.

Lemma1[3]. For any real numbers $x\ge 0,\ y>0$ and $m\ge 1$, the following inequality holds:

$$x \le y + \left(x/m\right)^m \left(\left(m-1\right)/y\right)^{m-1}.$$

Lemma2[19]. For all $x,y\in R$ and a constant $p\ge 1$ the following inequalities holds:

(i) $\left|x+y\right|^p \le 2^{p-1}\left|x^p+y^p\right|$,

$$\left(|x|+|y|\right)^{1/p} \le |x|^{1/p}+|y|^{1/p} \le 2^{(p-1)/p}\left(|x|+|y|\right)^{1/p}$$

If $p\in R_{odd}^{\ge 1}$, then

(ii) $\left|x-y\right|^p \le 2^{p-1}\left|x^p-y^p\right|$,

$$\left|x^{1/p}-y^{1/p}\right| \le 2^{(p-1)/p}\left|x-y\right|^{1/p}.$$

Lemma3[19]. Let c, d be positive constants. Then, for any real-valued function $\gamma(x,y) > 0$, the following inequality holds:

$$|x|^c |y|^d \le \frac{c}{c+d}\gamma(x,y)|x|^{c+d} + \frac{d}{c+d}\gamma^{-c/d}(x,y)|y|^{c+d}.$$

Lemma4[20]. For $x, y \in R$ and $0 < p \le 1$ the following inequality holds

$$\left(|x|+|y|\right)^p \le |x|^p + |y|^p.$$

When $p = a/b \le 1,$ where $a > 0$ and $b > 0$ are *odd* integers

$$\left|x^p + y^p\right| \le 2^{1-p}\left|x+y\right|^p.$$

Lemma5[21]. If $p = a/b \in R_{\text{odd}}^{\ge 1}$ with $a \ge b \ge 1$ being some real numbers, then for any $x, y \in R$

$$\left|x^p - y^p\right| \le 2^{1-1/b}\left|\text{sgn}(x)|x|^a - \text{sgn}(y)|y|^a\right|^{1/b}$$

Lemma6[6]. If $f:\ [a,b] \to R\ (a \le b)$ is monotone continuous and satisfies $f(a) = 0$, then

$$\left|\int_a^b f(x)dx\right| \le |f(b)| \cdot |b-a|.$$

3. Construction of Tracking Control

In this section, we will present a recursive design approach to construct the tracking control for system (1). For simplicity, we denote $\text{sgn}(x)|x|^\alpha := [x]^\alpha$ for any $\alpha \in R^+$ and $x \in R$.

The following theorem is the main result of this paper.

Theorem 1. Let $y_r(t)$ be a reference signal whose derivative $\dot{y}_r(t)$ is also bounded. Then, under Assumption1, the global practical output tracking problem of the system (1) is solvable by a continuous state feedback controller of the form (2).

Proof: The inductive proof relies on the simultaneous construction of a C^1 Lyapunov function which is positive define and proper, as well as a homogeneous-like controller at each iteration.

Let $x_1 = z_1 - y_r$ and given $x_i = z_i,\ i = 2,\dots,n$. Then, we have

$$\begin{aligned}
\dot{x}_1 &= x_2^{p_1} + \phi_1(t, x_1 + y_r, x_2, \dots, x_n, u) - \dot{y}_r(t),\\
\dot{x}_2 &= x_3^{p_2} + \phi_2(t, x_1 + y_r, x_2, \dots, x_n, u),\\
&\ \vdots\\
\dot{x}_{n-1} &= x_n^{p_{n-1}} + \phi_{n-1}(t, x_1 + y_r, x_2, \dots, x_n, u),\\
\dot{x}_n &= u + \phi_n(t, x_1 + y_r, x_2, \dots, x_n, u),\\
y &= x_1 + y_r.
\end{aligned} \tag{7}$$

Initial Step. Let $\xi_1 = [x_1]$ and construct the Lyapunov function as

$$V_1(x_1) = W_1(\bar{x}_1) = \int_0^{x_1} s^{2-r_2 p_1} ds,$$

where

$$W_1\left(\bar{x}_1\right)=\int_{x_1^*}^{x_1}\left[\left[s\right]^{1/r_1}-\left[x_1^*\right]^{1/r_1}\right]^{2-r_2p_1}ds$$

and $x_1^*\equiv 0$ for convenience. Note that V_1 is C^1, positive definite and proper.
A direct calculation gives

$$\dot{V}_1(x_1)=x_1^{2-r_2p_1}\left[x_2^{p_1}+\phi_1(t,x_1+y_r,x_2,\ldots,x_n,u)-\dot{y}_r(t)\right]. \qquad (8)$$

Since $y_r(t)$ and $\dot{y}_r(t)$ are bounded and by Assumption1 and Lemmas1-6, it can be shown that there is a smooth functions $\tilde{\psi}_1(x_1)$ such that

$$\begin{aligned}\left|\phi_1(t,x+y_r,x_2\ldots,x_n,u)-\dot{y}_r\right| &\le \psi_{11}(x_1+y_r)\left|x_1+y_r\right|^{(r_1+\tau)/r_1}+\psi_{12}(x_1+y_r)+M\\ &\le \tilde{\psi}_{11}(x_1)\left|x_1\right|^{(r_1+\tau)/r_1}+\tilde{\psi}_{11}(x_1)M^{(r_1+\tau)/r_1}+\tilde{\psi}_{12}(x_1)+M\\ &\le \tilde{\psi}_1(x_1)\left|x_1\right|^{(r_1+\tau)/r_1}+\kappa_1(x_1)\end{aligned}$$

satisfying

$$\dot{V}_1(x_1)\le x_1^{2-r_2p_1}x_2^{*p_1}+x_1^{2-r_2p_1}\left(x_2^{p_1}-x_2^{*p_1}\right)+x_1^{2}\left(\tilde{\psi}_{11}(x_1)+\alpha_1(x_1)\right)+\delta,$$

where $\delta>0$ is any real constant,

$$\alpha_1(x_1)=\left(\frac{\left(2-r_2p_1\right)\kappa_1(x_1)}{2}\right)^{2/(2-r_2p_1)}\left(\frac{r_2p_1}{\left(2-r_2p_1\right)\delta}\right)^{r_2p_1/(2-r_2p_1)}$$

and

$$\kappa_1(x_1)\ge\tilde{\psi}_{11}(x_1)M^{(r_1+\tau)/r_1}+\tilde{\psi}_{12}(x_1)+M\ .$$

Define a smooth positive function $\tilde{\kappa}_1(x_1)$ such that $\tilde{\kappa}_1(x_1)\ge\alpha_1(x_1)+\kappa_1(x_1)$. Then, we have

$$\dot{V}_1(x_1)\le\left[\xi_1\right]^{2-r_2p_1}x_2^{*p_1}+\left[\xi_1\right]^{2-r_2p_1}\left(x_2^{p_1}-x_2^{*p_1}\right)+x_1^2\tilde{\kappa}_1(x_1)+\delta.$$

If we take the virtual controller x_2^* as

$$x_2^*=-\beta_1^{r_2}\left(x_1\right)x_1^{r_2}=-\beta_1^{r_2}\left(x_1\right)\left[\xi_1\right]^{r_2}, \qquad (9)$$

where $\beta_1\left(x_1\right)\ge\left(n+\tilde{\kappa}_1(x_1)\right)^{1/r_2p_1}$, then it follows that

$$\dot{V}_1(x_1)\le -nx_1^2+\left[\xi_1\right]^{2-r_2p_1}\left(x_2^{p_1}-x_2^{*p_1}\right)+\delta. \qquad (10)$$

Inductive Step. Suppose at the (k-1)-th step, there is a C^1, positive definite and proper Lyapunov function $V_{k-1}\left(x_1,\ldots,x_{k-1}\right)$, which is positive definite and proper, and a set of virtual controllers $x_1^*,\ldots,x_k^*$ defined by

$$\begin{aligned}
&x_1^*=0, && \xi_1=\left[x_1\right]^{1/r_1}-\left[x_1^*\right]^{1/r_1},\\
&x_2^*=-\beta_1^{r_2}\left(\bar{x}_1\right)\left[\xi_1\right]^{r_2}, && \xi_2=\left[x_2\right]^{1/r_2}-\left[x_2^*\right]^{1/r_2},\\
&\vdots && \\
&x_k^*=-\beta_{k-1}^{r_k}(\bar{x}_{k-1})\left[\xi_{k-1}\right]^{r_k}, && \xi_k=\left[x_k\right]^{1/r_k}-\left[x_k^*\right]^{1/r_k},
\end{aligned} \qquad (11)$$

with $\beta_i(\bar{x}_i), 1 \le i \le k-1$, being smooth positive functions, such that

$$\begin{aligned}\dot{V}_{k-1}(\bar{x}_{k-1}) \le &-(n-k+2)\left(\xi_1^2+\cdots+\xi_{k-1}^2\right)\\ &+\left[\xi_{k-1}\right]^{2-r_k p_{k-1}}\left(x_k^{p_{k-1}}-x_k^{*p_{k-1}}\right)+(k-1)\delta.\end{aligned} \tag{12}$$

We claim that (12) also holds at *Step k*. To prove this claim, we choose the following Lyapunov function

$$V_k(\bar{x}_k) = V_{k-1}(\bar{x}_{k-1}) + W_k(\bar{x}_k) \tag{13}$$

where

$$W_k(\bar{x}_k) = \int_{x_k^*}^{x_k}\left[\left[s\right]^{1/r_k} - \left[x_k^*\right]^{1/r_k}\right]^{2-r_{k+1}p_k} ds\,.$$

Noting that $2 - r_{k+1}p_k \ge 1$and using a similar method as in [21], $V_k(\cdot)$ can be shown to be C^1, proper and positive definite. Moreover, we can obtain

$$\frac{\partial W_k}{\partial x_i} = -\left(2-r_{k+1}p_k\right)\int_{x_k^*}^{x_k}\left[\left[s\right]^{1/r_k} - \left[x_k^*\right]^{1/r_k}\right]^{1-r_{k+1}p_k} ds\,\frac{\partial\left(\left[x_k^*\right]^{1/r_k}\right)}{\partial x_i}, \tag{14}$$

$$\frac{\partial W_k}{\partial x_k} = \left[\left[x\right]^{1/r_k} - \left[x_k^*\right]^{1/r_k}\right]^{2-r_{k+1}p_k} = \left[\xi_k\right]^{2-r_{k+1}p_k} \tag{15}$$

where $i = 1,\ldots,k-1$, and there is a known constant $L > 0$, such that

$$W_k \ge L\left(x_k - x_k^*\right)^{2-r_{k+1}p_k}. \tag{16}$$

Using (12)–(15), it follows that

$$\begin{aligned}\dot{V}_k(\bar{x}_k) \le &-(n-k+2)\sum_{i=1}^{k-1}\xi_i^2 + \left[\xi_{k-1}\right]^{2-r_k p_{k-1}}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right)\\ &+(k-1)\delta + \left[\xi_k\right]^{2-r_{k+1}p_k}\left(x_{k+1}^{*p_k} + \psi_k(\cdot)\right)\\ &+\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\left(x_{i+1}^{p_i} + \psi_i(\cdot)\right) + \left[\xi_k\right]^{2-r_{k+1}p_k}\left(x_{k+1}^{p_k} - x_{k+1}^{*p_k}\right)\end{aligned} \tag{17}$$

for a virtual controller $x_{k+1}^{*p_k}$ to be determined later. In order to proceed further, an appropriate bounding estimate should be given for the last three terms on the right hand side of inequality (17). This is accomplished in the following three facts whose technical proofs are given in the appendix.

Fact 1: There exists a positive constant a_k such that

$$\left[\xi_{k-1}\right]^{2-r_k p_{k-1}}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \le \frac{1}{3}\xi_{k-1}^2 + a_{k1}\xi_k^2\,.$$

Fact 2: There exists a nonnegative smooth function $b_k(\bar{x}_k)$ such that

$$\left[\xi_k\right]^{2-r_{k+1}p_k}\psi_k(t, x_1+y_r, x_2, \ldots, x_k, u) \le \frac{1}{3}\sum_{i=1}^{k-1}\xi_i^2 + b_k(\bar{x}_k)\xi_k^2 + \frac{1}{2}\delta.$$

Fact 3: There exists a nonnegative smooth functions $c_k(\bar{x}_k)$ such that

$$\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}\left(x_{i+1}^{p_i}+\psi_i(\cdot)\right)\le\frac{1}{3}\sum_{i=1}^{k-1}\xi_i^2+c_k(\overline{x}_k)\xi_k^2+\frac{1}{2}\delta .$$

Substituting the results of the previous into (17), we arrive at

$$\begin{aligned}\dot{V}_k\left(\overline{x}_k\right)\le -(n-k+1)\sum_{i=1}^{k-1}\xi_i^2+\left[\xi_k\right]^{2-r_{k+1}p_k}x_{k+1}^{*p_k}\\ +\tilde{\kappa}_k(\overline{x}_k)\xi_k^2+\left[\xi_k\right]^{2-r_{k+1}p_k}\left(x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right)+k\delta,\end{aligned}\tag{18}$$

where

$$\tilde{\kappa}_k(\overline{x}_k)=a_k+b_k(\overline{x}_k)+c_k(\overline{x}_k)$$

is a smooth positive function.

Now, it easy to see that the virtual controller

$$x_{k+1}^{*}\ =-\beta_k^{r_{k+1}}(\overline{x}_k)\left[\xi_k\right]^{r_{k+1}},\tag{19}$$

where $\beta_k(\overline{x}_k)\ge\left(\left(n-k+1\right)+\tilde{\kappa}_k(\overline{x}_k)\right)^{1/r_{k+1}p_k}$ is a smooth function, renders

$$\dot{V}_k\left(x_1,\ldots,x_k\right)\le -(n-k+1)\sum_{i=1}^{k}\xi_i^2+\left[\xi_k\right]^{2-r_{k+1}p_k}\left(x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right)+k\delta .$$

This completes the inductive step.

Using the inductive argument above, we can conclude that at the n-th step, there exists a continuous state feedback controller of the form

$$u=x_{n+1}^{*}=-\beta_n^{r_{n+1}}(\overline{x}_n)\left[\xi_n\right]^{r_{n+1}}\tag{20}$$

with the C^1, proper and positive definite Lyapunov function $V_n\left(x_1,x_2\ldots,x_n\right)$ constructed via the inductive procedure, we arrive at

$$\dot{V}_n\left(x_1,\ldots,x_n\right)\le -\sum_{i=1}^{n}\xi_i^2+n\delta.\tag{21}$$

Noting that $\tau\in\left(-1\big/\sum_{l=1}^{n}p_1\cdots p_{l-1},0\right)$ and $r_{k+1}p_k=r_k+\tau$, we have $0<r_{k+1}p_k<1$. Moreover, recall that $V(x_1,\ldots,x_n)=\sum_{k=1}^{n}W_k(x_1,\ldots,x_k)$, where W_k's are defined in (13). Then, it follows from Lemma5, we have

$$W_k(\overline{x}_k)\le\left|x_k-x_k^{*}\right|\left|\xi_k\right|^{2-r_{k+1}p_k}\le 2^{1-r_k}\left|\xi_k\right|^{2-\tau}\le 2\left(\left|\xi_k\right|^2\right)^{\sigma}.\tag{22}$$

So we have the following estimate:

$$V_n(\overline{x}_n)=\sum_{k=1}^{n}W_k(\overline{x}_k)\le 2\sum_{k=1}^{n}\left|\xi_k\right|^{\sigma}.\tag{23}$$

Let $\sigma=(2-\tau)/2$. By $\tau\in\left(-1\big/\sum_{l=1}^{n}p_1\cdots p_{l-1},0\right)$, $1/\sigma\in\left(0,1\right)$. With (21) and (23) in mind, by Lemma4, it is not difficult to obtain that

$$\dot{V}_n\left(\overline{x}_n\right)\le -\left(V_n\left(\overline{x}_n\right)/2\right)^{1/\sigma}+n\delta\tag{24}$$

It will show that the state $x(t)$ of closed-loop system (7) is well-defined on $\left[0,+\infty\right)$ and globally bounded. First, introduce the following set

$$S := \left\{ x(t) \in R^n \,\middle|\, V_n(\bar{x}_n) \geq 2\left(2n\delta\right)^{\sigma} \right\}, \tag{25}$$

and let $x(t)$ be the trajectory of (7) with an initial state $x(0)$. If $x(t) \in S$, then it follows from (25) that

$$\dot{V}_n\left(\bar{x}_n\right) \leq -\left(V_n\left(\bar{x}_n\right)/2\right)^{1/\sigma} + n\delta \leq -n\delta < 0. \tag{26}$$

This implies that, as long as $x(t) \in S$, $V_n(x(t))$ is strictly decreasing with time t, and hence $x(t)$ must enter the complement set $R^n - S$ in a finite time $T \geq 0$ and stay there forever. Thus, the solution $x(t)$ of the system (7) is well-defined and globally bounded on $\left[0,+\infty\right)$. Next, it will be shown that

$$\left|y(t) - y_r(t)\right| = \left|z_1(t) - y_r(t)\right| < \varepsilon, \qquad \forall t \geq T > 0. \tag{27}$$

This is also easily shown from (15), (24) and by tuning the parameter δ:

$$\left|y(t) - y_r(t)\right| = \left|x_1(t)\right| \leq V_n(x(t)) \leq 2\left(2n\delta\right)^{\sigma} < \varepsilon.$$

Therefore, for any $\varepsilon > 0$, there is globally practical output-tracking such that (27) holds. This completes the proof of Theorem 1.

4. Conclusions

In this paper, the global practical tracking problem has been studied for a class of high-order non-linear systems with more general uncertainties and presented a continuous state feedback output tracking controller for a class of high-order nonlinear systems under weaker condition. The controller guarantees that the states of the closed-loop system are globally bounded, while the tracking error can be bounded by any given positive number after a finite time. It should be noted that the proposed controller can only work well when the whole state vector is measurable. Therefore, a natural and more interesting problem is how to design output feedback output tracking controller for the systems studied in the paper if only partial state vector being measurable, which is now under our further investigation.

Appendix

Proof of Fact 1: Noting that $\tau \in \left(-1\Big/\sum_{l=1}^{n} p_1 \cdots p_{l-1}, 0\right)$ and $r_k p_{k-1} = r_{k-1} + \tau$, we have $0 < r_k p_{k-1} < 1$. Using (11), it follows from Lemma5 that

$$\begin{aligned} \left| x_k^{p_{k-1}} - x_k^{*p_{k-1}} \right| &= \left| \left(x_k^{1/r_k}\right)^{r_k p_{k-1}} - \left(x_k^{*1/r_k}\right)^{r_k p_{k-1}} \right| \\ &\leq 2^{1-r_k p_{k-1}} \left| \left[x_k\right]^{1/r_k} - \left[x_k^*\right]^{1/r_k} \right|^{r_k p_{k-1}} \\ &= 2^{1-r_k p_{k-1}} \left|\xi\right|^{r_k p_{k-1}}. \end{aligned} \tag{A1}$$

By (A.1) and Lemma3, it can be obtained that

$$\begin{aligned}\left[\xi_{k-1}\right]^{2-r_k p_{k-1}}\left(x_k^{p_{k-1}}-x_k^{*p_{k-1}}\right) &\le 2^{1-r_k p_{k-1}}\left|\xi_{k-1}\right|^{2-r_k p_{k-1}}\left|\xi_k\right|^{r_k p_{k-1}} \\ &\le \frac{1}{3}\xi_{k-1}^2+a_k\xi_k^2\end{aligned} \tag{A2}$$

where $a_k > 0$ is a constant.

Proof of Fact 2 : According to (11), Assumption 1, and Lemma2, it follows that

$$\begin{aligned}&\left|\phi_k(x_1+y_r,x_2,\ldots,x_k)\right| \\ &\le \psi_k(x_1+y_r,x_2\ldots,x_k)\left(\left|x_1+y_r\right|^{(r_1+\tau)/r_1}+\left|x_2\right|^{(r_2+\tau)/r_2}+\cdots+\left|x_k\right|^{(r_k+\tau)/r_k}\right) \\ &\le \tilde{\psi}_k(x_1,\ldots,x_k)\sum_{j=1}^{k}\left(\left|\xi_j\right|+\beta_{j-1}\left|\xi_{j-1}\right|\right)^{r_{k+1}p_k}+\tilde{\psi}_k(x_1,\ldots,x_k)M^{r_{k+1}p_k} \\ &\le 2^{1-r_{k+1}p_k}\tilde{\psi}_k(x_1,\ldots,x_k)\sum_{j=1}^{k}\left(\left|\xi_j\right|^{r_{k+1}p_k}+\beta_{j-1}^{r_{k+1}p_k}\left|\xi_{j-1}\right|^{r_{k+1}p_k}\right) \\ &\qquad +\tilde{\psi}_k(x_1,\ldots,x_k)M^{r_{k+1}p_k} \\ &\le \bar{\psi}_k(x_1,\ldots,x_k)\sum_{j=1}^{k}\left|\xi_j\right|^{r_{k+1}p_k}+\bar{\psi}_k(x_1,\ldots,x_k)M^{r_{k+1}p_k}\end{aligned} \tag{A3}$$

where $\beta_0=0,\ \xi_0=0$ and $\bar{\psi}_k(\cdot)=2^{1-r_{k+1}p_k}\sum_{j=1}^{k}\left(1+\beta_{j-1}^{r_{k+1}p_k}\right)\tilde{\psi}_k(\cdot)\ge 0$ is a smooth function.

Using (A.3) and Lemmas3 and 5, we obtain,

$$\begin{aligned}&\left[\xi_k\right]^{2-r_{k+1}p_k}\phi_k(x_1+y_r,x_2,\ldots,x_k) \\ &\le \bar{\psi}_k(x_1,\ldots,x_k)\sum_{j=1}^{k}\left|\xi_k\right|^{2-r_{k+1}p_k}\left|\xi_j\right|^{r_{k+1}p_k} \\ &\qquad +\bar{\psi}_k(x_1,\ldots,x_k)\left|\xi_k\right|^{2-r_{k+1}p_k}M^{r_{k+1}p_k} \\ &\le \frac{1}{3}\sum_{j=1}^{k-1}\xi_j^2+b_k(x_1,\ldots,x_k)\xi_k^2+\frac{1}{2}\delta,\end{aligned} \tag{A4}$$

where $b_k(x_1,\ldots,x_k)>0$ is a smooth function.

Proof of Fact 3 : Note that

$$\left[x_{j+1}^{*}\right]^{1/r_{j+1}}=-\beta_j\xi_j=-\sum_{j=1}^{k-1}B_j\left[x_j\right]^{1/r_j}, \tag{A5}$$

where $B_j=\beta_{k-1}\cdots\beta_j,\ \ j=1,\ldots,k-1.$

Using (A.5), after simple calculations, it is not hard to obtain that for $j=1,\ldots,k-1$,

$$\frac{\partial\left[x_k^{*}\right]^{1/r_k}}{\partial x_j}=-\sum_{j=1}^{k-1}\frac{\partial B_l}{\partial x_j}\left[x_j\right]^{1/r_j}-\frac{1}{r_j}B_j\left|x_j\right|^{1/r_j-1}. \tag{A6}$$

By (14), (15), (A.3), (A.6), and Lemmas2 and 5, we get

$$\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}\dot{x}_j=\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}\left(x_{j+1}^{p_j}+\psi_j(\cdot)\right)$$

$$=-\left(2-r_{k+1}p_k\right)\int_{x_k^*}^{x_k}\left([s]^{1/r_k}-\left[x_k^*\right]^{1/r_k}\right)^{2-r_{k+1}p_k}ds$$

$$\cdot\sum_{j=1}^{k-1}\frac{\partial\left[x_k^{*1/r_k}\right]}{\partial x_j}\left(x_{j+1}^{p_j}+\psi_j(\cdot)\right)$$

$$\le 2^{2-r_k}\left|\xi_k\right|^{1-\tau}\sum_{j=1}^{k-1}\left|\frac{\partial\left[\alpha_{k-1}^{1/r_k}\right]}{\partial x_j}\right|\left(\left|x_{j+1}\right|^{p_j}+\left|\psi_j(\cdot)\right|\right)$$

$$\le\sum_{j=1}^{k-1}\bar{B}_j\left|\xi_k\right|^{1-\tau}\left|\xi_j+\beta_{j-1}\xi_{j-1}\right|^{1-r_j}$$

$$\cdot\left(\left|\xi_{jL1}+\beta_j\xi_j\right|^{r_{j+1}p_j}+\bar{\psi}_j(\cdot)\sum_{l=1}^{j}\left|\xi_j\right|^{r_{j+1}p_j}+\bar{\psi}_j(\cdot)M\right)$$

$$\le 2^{2-r_k}\left|\xi_k\right|^{1-\tau}\sum_{j=1}^{k-1}\left|\sum_{l=1}^{k-1}\frac{\partial B_l}{\partial x_j}\left|x_j\right|+\frac{1}{r_j}B_j\left|x_j\right|^{1/r_j-1}\right|\left(\left|x_{j+1}\right|^{p_j}+\left|\psi_j(\cdot)\right|\right)$$

$$\le\sum_{j=1}^{k-1}\bar{B}_j\left|\xi_k\right|^{1-\tau}\left|x_j\right|^{1/r_j-1}\left(\left|x_{j+1}\right|^{p_j}+\left|\psi_j(\cdot)\right|\right),$$

where $\bar{B}_j\ge 0$ is a smooth function.

Noting that $r_{j+1}p_j=r_j+\tau$, by using Lemma3, we have

$$\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}\dot{x}_j=\sum_{j=1}^{k-1}\frac{\partial W_k}{\partial x_j}\left(x_{j+1}^{p_j}+\psi_j(\cdot)\right)$$

$$\le\frac{1}{3}\sum_{j=1}^{k-1}\xi_j^2+c_k(\cdot)\xi_j^2+\frac{1}{2}\delta,$$

where $c_k(\cdot)$ is a smooth function.

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