

# Improved Stability Criteria for Neural Networks with Additive Time-varying Delay Components

Jianmin Jiao

*College of Mathematics and Information Science, Baoji University of Arts and Sciences, Baoji, 721013, China*  
*[jmjiao@126.com](mailto:jmjiao@126.com)*

## Abstract

*This paper investigated the problem of improved delay-dependent stability criteria for neural networks with two additive time-varying delay components. By constructing a more general type of Lyapunov functional and using effective mathematical technique to estimate the upper bound on the time derivative of the Lyapunov functional, some new delay-dependent stability criteria are derived in terms of linear matrix inequalities. The resulting stability criteria are of fewer matrix variables and less conservative than some existing ones. Numerical examples are given to illustrate the effectiveness and less conservativeness of the proposed method.*

**Keywords:** *Neural networks, stability, additive time-varying delays, linear matrix inequality (LMI)*

## 1. Introduction

In the past few decades, neural networks have been investigated extensively because of their successful applications in various scientific areas, such as pattern recognition, signal processing, image processing, associative memories, and so on [1,2]. It is well known that time delay is unavoidably existent in various real-world systems, and it is often a major cause of oscillation and instability [3-5]. Thus, the stability analysis for time delay neural networks has received considerable attention in recent years. The stability criteria for time delay neural networks can be classified into two types: delay-independent stability criteria [6-8] and delay-dependent stability criteria [9-25]. Generally speaking, the delay-dependent stability criteria are less conservative than delay-independent ones, especially when the size of time-delays is small. Many efforts have been made to the research of delay-dependent stability problems for neural networks, and various methods are utilized for getting less conservative results, such as free weighting matrix method [9-13], augmented Lyapunov functional method [14-16], discretized Lyapunov functional method [17], delay-partitioning method [18-22], delay-slope-dependent method [23], and reciprocally convex method [24,25]. Note that the delay-dependent stability results in [6-25] can only provide stability conditions for neural networks with a single delay in the state. Recently, a new model for linear systems with two additive time-varying delays was proposed in [26]. The new model has a strong application background in networked control systems [27,28]. Some delay-dependent stability criteria for linear systems with two additive time-varying delay components are derived in [26-30]. In [31], Zhao et al. have initially investigated the globally asymptotic stability analysis of neural networks with two additive time-varying delay components by using the free-weighting matrix method. In [32,33], by constructing modified Lyapunov functionals, some improved delay-dependent stability criteria were obtained by using the convex polyhedron method. [34] and [35] have also considered the stability problem for neural networks with two additive time-varying delay components by constructing augmented Lyapunov functionals which contain triple integral terms, and some delay-dependent stability criteria

were obtained by employing the reciprocally convex method and the delay-partitioning method, respectively. In [36], by defining a more general Lyapunov functional, a simplified delay-dependent stability criterion is established. Yet it is worth pointing out that, though the works above were elegant, there still exist some points waiting for the improvement. Firstly, the information of neuron activation function is not adequately considered, which may lead to much conservatism. Secondly, many matrix variables are introduced in the obtained results, which brings a large computational burden.

Motivated by the preceding discussions, by introducing a more general type of Lyapunov functional in this paper, some improved delay-dependent stability criteria for neural networks with two additive time-varying delay components are derived. Since more information of the time-varying delays are used, our method can get less conservative results compared to some existing literature. Furthermore, since reciprocally convex method is employed, the computational complexity is also reduced. The achieved stability criteria are expressed by a set of LMIs, which can be easily checked by Matlab LMI toolbox. Two numerical examples are provided to show the effectiveness and less conservativeness of the proposed method.

**Notations:** Throughout this paper,  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space, while  $\mathbb{R}^{n \times m}$  refers to the set of all real matrices with  $m$  rows and  $n$  columns.  $A^T$  represents the transpose of the matrix  $A$ . For real symmetric matrices  $X$  and  $Y$ , the notation  $X \geq Y$  (respectively,  $X > Y$ ) means matrix  $X - Y$  is positive-semidefinite (respectively, positive-definite).  $I_n$  is the  $n$ -dimensional identity matrix.

## 2. Problem Description and Preliminaries

Consider the following delayed neural network with two additive time-varying delay components

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - d_1(t) - d_2(t))) + J, \quad (1)$$

where  $x(\cdot) = [x_1(\cdot), x_2(\cdot), \dots, x_n(\cdot)]^T \in \mathbb{R}^n$  is the state vector associated with the  $n$  neurons,  $g(x(\cdot)) = [g_1(x_1(\cdot)), g_2(x_2(\cdot)), \dots, g_n(x_n(\cdot))]^T \in \mathbb{R}^n$  denotes the neuron activation function with  $g(0) = 0$ ;  $C = \text{diag}\{c_1, c_2, \dots, c_n\}$  is a diagonal matrix representing self-feedback term,  $A, B \in \mathbb{R}^{n \times n}$  are the connection weight matrix and the delayed connection weight matrix, respectively,  $J = [J_1, J_2, \dots, J_n]^T \in \mathbb{R}^n$  is a constant input vector.  $d_1(t)$  and  $d_2(t)$  are continuous time-varying functions that represent the two delay components in the state which satisfy

$$0 \leq d_i(t) \leq d_i, \quad 0 \leq \dot{d}_i(t) \leq \mu_i (i = 1, 2), \quad (2)$$

where  $d_i, \mu_i (i = 1, 2)$  are known constants. Naturally, we denote

$$d(t) = d_1(t) + d_2(t), \quad (3)$$

$$\dot{d} = \dot{d}_1 + \dot{d}_2, \quad (4)$$

$$\mu = \mu_1 + \mu_2. \quad (5)$$

In addition, it is assumed that each neuron activation function  $g_i(\cdot) (i = 1, 2, \dots, n)$  is bounded and satisfies the following condition

$$k_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq k_i^+, \quad \forall s_1, s_2 \in \mathbb{R}, s_1 \neq s_2, i = 1, 2, \dots, n, \quad (6)$$

where  $k_i^-, k_i^+ (i = 1, 2, \dots, n)$  are known real constants.

It is easy to prove the existence and uniqueness of the equilibrium of system (1) by employing the well-known Brouwer's fixed point theorem [1]. Assume that

$x^* = [x_1^*, x_2^*, \dots, x_n^*]^T$  is an equilibrium, by choosing the coordinate transformation  $z(\cdot) = x(\cdot) - x^*$ , system (1) can be converted to

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t - d_1(t) - d_2(t))), \quad (7)$$

where  $z(\cdot) = [z_1(\cdot), z_2(\cdot), \dots, z_n(\cdot)]^T$  is the state vector of the transformed system, and the transformed neuron activation function is  $f(z(\cdot)) = [f_1(z_1(\cdot)), f_2(z_2(\cdot)), \dots, f_n(z_n(\cdot))]^T$ , where  $f_i(z_i(\cdot)) = g_i(z_i(\cdot) + x_i^*) - g_i(x_i^*) (i = 1, 2, \dots, n)$ . According to (6), one can obtain that

$$k_i^- \leq \frac{f_i(z_i(t))}{z_i(t)} \leq k_i^+, f_i(0) = 0, \forall z_i(t) \neq 0, \quad (8)$$

which implies that

$$[f_i(z_i(t)) - k_i^- z_i(t)][f_i(z_i(t)) - k_i^+ z_i(t)] \leq 0, \quad (9)$$

$$[f_i(z_i(t - d(t))) - k_i^- z_i(t - d(t))][f_i(z_i(t - d(t))) - k_i^+ z_i(t - d(t))] \leq 0, \quad (10)$$

where  $t > 0$  and  $i = 1, 2, \dots, n$ .

**Remark 1** It is easy to see that the assumed conditions (6) and (8) of neuron activation functions for systems (1) and (7) include those used conditions in [31-36] as their special cases when  $k_i^- = 0$  and  $k_i^+ > 0$ .

To the end this section, we introduce the following lemmas, which will play an important role in the proof of the main results.

**Lemma 1** (Jensen inequality [3]) For any matrix  $R > 0$ , scalars  $\gamma_1$  and  $\gamma_2$  satisfying  $\gamma_2 > \gamma_1$ , a vector function  $\omega : [\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n$  such that the integrations concerned are well defined, then the following inequality holds

$$-(\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} \omega^T(s) R \omega(s) ds \leq - \int_{\gamma_1}^{\gamma_2} \omega^T(s) ds R \int_{\gamma_1}^{\gamma_2} \omega(s) ds.$$

**Lemma 2** (Reciprocally convex lemma [4]) For any real numbers  $\alpha_1 > 0, \alpha_2 > 0$  with  $\alpha_1 + \alpha_2 = 1$ , vectors  $\zeta_1, \zeta_2$ , and matrices  $R > 0, M$  satisfying

$$\begin{bmatrix} R & M \\ M^T & R \end{bmatrix} \geq 0,$$

then the following inequality holds

$$-\frac{1}{\alpha_1} \zeta_1^T R \zeta_1 - \frac{1}{\alpha_2} \zeta_2^T R \zeta_2 \leq - \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^T \begin{bmatrix} R & M \\ M^T & R \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}.$$

### 3. Main Results

In this sequel, a more general type of Lyapunov functional is proposed to derive new delay delay-dependent stability criteria for neural network (7) with additive time-varying delay components satisfying (2). Before introducing the main results, following notations are defined for simplicity.

$$\begin{aligned}
 K_1 &= \text{diag} \{k_1^-, k_2^-, \dots, k_n^-\}, \quad K_2 = \text{diag} \{k_1^+, k_2^+, \dots, k_n^+\}, \\
 \Sigma_1 &= \text{diag} \{k_1^- k_1^+, k_2^- k_2^+, \dots, k_n^- k_n^+\}, \quad \Sigma_2 = \text{diag} \{k_1^- + k_1^+, k_2^- + k_2^+, \dots, k_n^- + k_n^+\}, \\
 \xi(t) &= [z^T(t) \quad f^T(z(t)) \quad f^T(z(t-d(t))) \quad z^T(t-d_1) \quad z^T(t-d_2) \quad z^T(t-d_1(t)) \quad z^T(t-d_2(t)) \\
 &\quad z^T(t-d(t)) \quad z^T(t-d_1(t)-d_2) \quad z^T(t-d_1-d_2(t)) \quad z^T(t-d)]^T, \\
 e_0 &= -Ce_1 + Ae_2 + Be_3, \quad e_i = [0_{n \times (i-1)n} \quad I_n \quad 0_{n \times (11-i)n}] (i = 1, 2, \dots, 11).
 \end{aligned}$$

In the following, we present a new stability criterion.

**Theorem 1** For given scalars  $d_i > 0, \mu_i > 0 (i = 1, 2)$ , the system (7) with two additive time-varying delay components satisfying (2) is globally asymptotically stable if there exist matrices  $P > 0, Q_i > 0 (i = 1, 2, \dots, 9), R_1 > 0, R_2 > 0$ , positive diagonal matrices  $\Lambda_j = \text{diag} \{\lambda_{j1}, \lambda_{j2}, \dots, \lambda_{jn}\} \geq 0, \Delta_j = \text{diag} \{\delta_{j1}, \delta_{j2}, \dots, \delta_{jn}\} \geq 0 (j = 1, 2)$ , and matrices  $M_k (k = 1, 2, 3, 4)$  with appropriate dimensions such that the following LMIs hold

$$\Xi = \Xi_1 + \Xi_2 + \Xi_3 + \Xi_4 < 0, \tag{11}$$

$$\begin{bmatrix} R_1 & M_i \\ M_i^T & R_1 \end{bmatrix} \geq 0 (i = 1, 2), \tag{12}$$

$$\begin{bmatrix} R_2 & M_j \\ M_j^T & R_2 \end{bmatrix} \geq 0 (j = 1, 2), \tag{13}$$

where

$$\begin{aligned}
 \Xi_1 &= e_1^T (P - K_1 \Lambda_1 + K_2 \Lambda_2) e_0 + e_0^T (P - \Lambda_1 K_1 + \Lambda_2 K_2) e_1 + e_2^T (\Lambda_1 - \Lambda_2) e_0 + e_0^T (\Lambda_1 - \Lambda_2) e_2, \\
 \Xi_2 &= e_1^T (Q_1 + Q_2 + Q_3 + Q_4 + Q_5) e_1 + e_2^T Q_6 e_2 - (1 - \mu) e_3^T Q_6 e_3 + e_4^T (-Q_1 + Q_8) e_4 + e_5^T (-Q_2 + Q_7) e_5 \\
 &\quad - (1 - \mu_1) e_6^T Q_3 e_6 - (1 - \mu_2) e_7^T Q_4 e_7 - (1 - \mu) e_8^T Q_5 e_8 - (1 - \mu_1) e_9^T Q_7 e_9 - (1 - \mu_2) e_{10}^T Q_8 e_{10} - e_{11}^T Q_9 e_{11},
 \end{aligned}$$

$$\begin{aligned}
 \Xi_3 &= de_0^T (R_1 + R_2) e_0 \\
 &\quad - \frac{1}{d_1} \begin{bmatrix} e_1 - e_6 \\ e_9 - e_{11} \end{bmatrix}^T \begin{bmatrix} R_1 & M_1 \\ M_1^T & R_1 \end{bmatrix} \begin{bmatrix} e_1 - e_6 \\ e_9 - e_{11} \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} e_6 - e_8 \\ e_8 - e_9 \end{bmatrix}^T \begin{bmatrix} R_1 & M_2 \\ M_2^T & R_1 \end{bmatrix} \begin{bmatrix} e_6 - e_8 \\ e_8 - e_9 \end{bmatrix} \\
 &\quad - \frac{1}{d_1} \begin{bmatrix} e_7 - e_8 \\ e_8 - e_{10} \end{bmatrix}^T \begin{bmatrix} R_2 & M_3 \\ M_3^T & R_2 \end{bmatrix} \begin{bmatrix} e_7 - e_8 \\ e_8 - e_{10} \end{bmatrix} - \frac{1}{d_2} \begin{bmatrix} e_1 - e_7 \\ e_{10} - e_{11} \end{bmatrix}^T \begin{bmatrix} R_2 & M_4 \\ M_4^T & R_2 \end{bmatrix} \begin{bmatrix} e_1 - e_7 \\ e_{10} - e_{11} \end{bmatrix}, \\
 \Xi_4 &= - \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}^T \begin{bmatrix} 2\Delta_1 \Sigma_1 & -\Delta_1 \Sigma_2 \\ -\Sigma_2 \Delta_1 & 2\Delta_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} e_8 \\ e_3 \end{bmatrix}^T \begin{bmatrix} 2\Delta_2 \Sigma_1 & -\Delta_2 \Sigma_2 \\ -\Sigma_2 \Delta_2 & 2\Delta_2 \end{bmatrix} \begin{bmatrix} e_8 \\ e_3 \end{bmatrix}.
 \end{aligned}$$

**Proof** Construct a new class of Lyapunov functional as follow

$$V(z(t)) = \sum_{i=1}^3 V_i(z(t)), \tag{14}$$

where

$$V_1(z(t)) = z^T(t) P z(t) + 2 \sum_{i=1}^n \int_0^{z_i(t)} [\lambda_{1i} (f_i(s) - k_i^- s) + \lambda_{2i} (k_i^+ s - f_i(s))] ds,$$

$$\begin{aligned}
 V_2(z(t)) &= \int_{t-d_1}^t z^T(s)Q_1z(s)ds + \int_{t-d_2}^t z^T(s)Q_2z(s)ds + \int_{t-d_1(t)}^t z^T(s)Q_3z(s)ds + \int_{t-d_2(t)}^t z^T(s)Q_4z(s)ds \\
 &+ \int_{t-d(t)}^t (z^T(s)Q_5z(s) + f^T(z(s))Q_6f(z(s)))ds + \int_{t-d(t)-d_2}^{t-d_2} z^T(s)Q_7z(s)ds \\
 &+ \int_{t-d_1-d_2(t)}^{t-d_1} z^T(s)Q_8z(s)ds + \int_{t-d}^t z^T(s)Q_9z(s)ds,
 \end{aligned}$$

$$V_3(z(t)) = \int_{-d}^0 \int_{t+\theta}^t \dot{z}^T(s)(R_1 + R_2)\dot{z}(s)dsd\theta.$$

Taking the time-derivatives of functionals  $V_i(z(t))(i = 1, 2, 3)$  along the trajectories of system (7), we have

$$\begin{aligned}
 \dot{V}_1(z(t)) &= 2z^T(t)P\dot{z}(t) + 2\sum_{i=1}^n [\lambda_{1i}(f_i(z_i(t)) - k_i^-z_i(t)) + \lambda_{2i}(k_i^+z_i(t) - f_i(z_i(t)))]\dot{z}(t) \\
 &= 2z^T(t)(P - K_1A_1 + K_2A_2)\dot{z}(t) + 2f^T(z(t))(A_1 - A_2)\dot{z}(t) \\
 &= \xi^T(t)\Xi_1\xi(t).
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 \dot{V}_2(z(t)) &= z^T(t)(Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_9)z(t) + f^T(z(t))Q_6f(z(t)) \\
 &- (1 - \dot{d}(t))f^T(z(t-d(t)))Q_6f(z(t-d(t))) + z^T(t-d_1)(-Q_1 + Q_8)z(t-d_1) \\
 &+ z^T(t-d_2)(-Q_2 + Q_7)z(t-d_2) - (1 - \dot{d}_1(t))z^T(t-d_1(t))Q_3z(t-d_1(t)) \\
 &- (1 - \dot{d}_2(t))z^T(t-d_2(t))Q_4z(t-d_2(t)) - (1 - \dot{d}(t))z^T(t-d(t))Q_5z(t-d(t)) \\
 &- (1 - \dot{d}_1(t))z^T(t-d_1(t)-d_2)Q_7z(t-d_1(t)-d_2) \\
 &- (1 - \dot{d}_2(t))z^T(t-d_1-d_2(t))Q_8z(t-d_1-d_2(t)) - z^T(t-d)Q_9z(t-d) \\
 &\leq z^T(t)(Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_9)z(t) + f^T(z(t))Q_6f(z(t)) \\
 &- (1 - \mu)f^T(z(t-d(t)))Q_6f(z(t-d(t))) + z^T(t-d_1)(-Q_1 + Q_8)z(t-d_1) \\
 &+ z^T(t-d_2)(-Q_2 + Q_7)z(t-d_2) - (1 - \mu_1)z^T(t-d_1(t))Q_3z(t-d_1(t)) \\
 &- (1 - \mu_2)z^T(t-d_2(t))Q_4z(t-d_2(t)) - (1 - \mu)z^T(t-d(t))Q_5z(t-d(t)) \\
 &- (1 - \mu_1)z^T(t-d_1(t)-d_2)Q_7z(t-d_1(t)-d_2) \\
 &- (1 - \mu_2)z^T(t-d_1-d_2(t))Q_8z(t-d_1-d_2(t)) - z^T(t-d)Q_9z(t-d) \\
 &= \xi^T(t)\Xi_2\xi(t).
 \end{aligned} \tag{16}$$

$$\begin{aligned}
 \dot{V}_3(z(t)) &= d\dot{z}^T(t)(R_1 + R_2)\dot{z}(t) - \int_{t-d}^t \dot{z}^T(s)R_1\dot{z}(s)ds - \int_{t-d}^t \dot{z}^T(s)R_2\dot{z}(s)ds \\
 &= d\dot{z}^T(t)(R_1 + R_2)\dot{z}(t) - \int_{t-d_1(t)}^t \dot{z}^T(s)R_1\dot{z}(s)ds - \int_{t-d(t)}^{t-d_1(t)} \dot{z}^T(s)R_1\dot{z}(s)ds \\
 &- \int_{t-d_1(t)-d_2}^{t-d(t)} \dot{z}^T(s)R_1\dot{z}(s)ds - \int_{t-d}^{t-d_1(t)-d_2} \dot{z}^T(s)R_1\dot{z}(s)ds - \int_{t-d_2(t)}^t \dot{z}^T(s)R_2\dot{z}(s)ds \\
 &- \int_{t-d(t)}^{t-d_2(t)} \dot{z}^T(s)R_2\dot{z}(s)ds - \int_{t-d_1-d_2(t)}^{t-d(t)} \dot{z}^T(s)R_2\dot{z}(s)ds - \int_{t-d}^{t-d_1-d_2(t)} \dot{z}^T(s)R_2\dot{z}(s)ds.
 \end{aligned}$$

By Lemma 1, one can obtain

$$\begin{aligned}
 \dot{V}_3(z(t)) &\leq d\dot{z}^T(t)(R_1 + R_2)\dot{z}(t) \\
 &- \frac{1}{d_1(t)} \left[ \int_{t-d_1(t)}^t \dot{z}(s)ds \right]^T R_1 \left[ \int_{t-d_1(t)}^t \dot{z}(s)ds \right] - \frac{1}{d_1 - d_1(t)} \left[ \int_{t-d}^{t-d_1(t)-d_2} \dot{z}(s)ds \right]^T R_1 \left[ \int_{t-d}^{t-d_1(t)-d_2} \dot{z}(s)ds \right] \\
 &- \frac{1}{d_2(t)} \left[ \int_{t-d(t)}^{t-d_2(t)} \dot{z}(s)ds \right]^T R_2 \left[ \int_{t-d(t)}^{t-d_2(t)} \dot{z}(s)ds \right] - \frac{1}{d_2 - d_2(t)} \left[ \int_{t-d_1(t)-d_2}^{t-d(t)} \dot{z}(s)ds \right]^T R_2 \left[ \int_{t-d_1(t)-d_2}^{t-d(t)} \dot{z}(s)ds \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{d_1(t)} \left[ \int_{t-d(t)}^{t-d_2(t)} \dot{z}(s) ds \right]^T R_2 \left[ \int_{t-d(t)}^{t-d_2(t)} \dot{z}(s) ds \right] - \frac{1}{d_1 - d_1(t)} \left[ \int_{t-d_1-d_2(t)}^{t-d(t)} \dot{z}(s) ds \right]^T R_2 \left[ \int_{t-d_1-d_2(t)}^{t-d(t)} \dot{z}(s) ds \right] \\
 & -\frac{1}{d_2(t)} \left[ \int_{t-d_2(t)}^t \dot{z}(s) ds \right]^T R_2 \left[ \int_{t-d_2(t)}^t \dot{z}(s) ds \right] - \frac{1}{d_2 - d_2(t)} \left[ \int_{t-d}^{t-d_1-d_2(t)} \dot{z}(s) ds \right]^T R_2 \left[ \int_{t-d}^{t-d_1-d_2(t)} \dot{z}(s) ds \right] \\
 & = d \xi^T(t) e_0^T (R_1 + R_2) e_0 \xi(t) \\
 & - \frac{1}{d_1} \frac{1}{\alpha_1} \xi^T(t) (e_1 - e_6)^T R_1 (e_1 - e_6) \xi(t) - \frac{1}{d_1} \frac{1}{\alpha_2} \xi^T(t) (e_9 - e_{11})^T R_1 (e_9 - e_{11}) \xi(t) \\
 & - \frac{1}{d_2} \frac{1}{\alpha_3} \xi^T(t) (e_6 - e_8)^T R_1 (e_6 - e_8) \xi(t) - \frac{1}{d_2} \frac{1}{\alpha_4} \xi^T(t) (e_8 - e_9)^T R_1 (e_8 - e_9) \xi(t) \\
 & - \frac{1}{d_1} \frac{1}{\alpha_1} \xi^T(t) (e_7 - e_8)^T R_2 (e_7 - e_8) \xi(t) - \frac{1}{d_1} \frac{1}{\alpha_2} \xi^T(t) (e_8 - e_{10})^T R_2 (e_8 - e_{10}) \xi(t) \\
 & - \frac{1}{d_2} \frac{1}{\alpha_3} \xi^T(t) (e_1 - e_7)^T R_2 (e_1 - e_7) \xi(t) - \frac{1}{d_2} \frac{1}{\alpha_4} \xi^T(t) (e_{10} - e_{11})^T R_2 (e_{10} - e_{11}) \xi(t),
 \end{aligned}$$

where

$$\alpha_1 = \frac{d_1(t)}{d_1}, \alpha_2 = \frac{d_1 - d_1(t)}{d_1}, \alpha_3 = \frac{d_2(t)}{d_2}, \alpha_4 = \frac{d_2 - d_2(t)}{d_2}.$$

From Lemma 2 and LMIs (12), (13), we have

$$\dot{V}_3(z(t)) \leq \xi^T(t) \Xi \xi(t). \tag{17}$$

On the other hand, from (9) and (10), it yields that

$$\begin{aligned}
 0 & \leq -2 \sum_{i=1}^n \delta_{1i} [f_i(z_i(t)) - k_i^- z_i(t)] [f_i(z_i(t)) - k_i^+ z_i(t)] \\
 & = - \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix}^T \begin{bmatrix} 2A_1 \Sigma_1 & -A_1 \Sigma_2 \\ -\Sigma_2 A_1 & 2A_1 \end{bmatrix} \begin{bmatrix} z(t) \\ f(z(t)) \end{bmatrix} \\
 & = -\xi^T(t) \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \begin{bmatrix} 2A_1 \Sigma_1 & -A_1 \Sigma_2 \\ -\Sigma_2 A_1 & 2A_1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \xi(t),
 \end{aligned} \tag{18}$$

and

$$\begin{aligned}
 0 & \leq -2 \sum_{i=1}^n \delta_{2i} [f_i(z_i(t-d(t))) - k_i^- z_i(t-d(t))] [f_i(z_i(t-d(t))) - k_i^+ z_i(t-d(t))] \\
 & = - \begin{bmatrix} z(t-d(t)) \\ f(z(t-d(t))) \end{bmatrix}^T \begin{bmatrix} 2A_2 \Sigma_1 & -A_2 \Sigma_2 \\ -\Sigma_2 A_2 & 2A_2 \end{bmatrix} \begin{bmatrix} z(t-d(t)) \\ f(z(t-d(t))) \end{bmatrix} \\
 & = -\xi^T(t) \begin{bmatrix} e_8 \\ e_3 \end{bmatrix} \begin{bmatrix} 2A_2 \Sigma_1 & -A_2 \Sigma_2 \\ -\Sigma_2 A_2 & 2A_2 \end{bmatrix} \begin{bmatrix} e_8 \\ e_3 \end{bmatrix} \xi(t).
 \end{aligned} \tag{19}$$

Hence, according to (15)-(19), we can obtain

$$\dot{V}(z(t)) \leq \xi^T(t) \Xi \xi(t).$$

Thus, system (7) is globally asymptotically stable. This completes the proof.

**Remark 2** Theorem 1 provides a novel delay-dependent stability criteria for neural networks with two additive time-varying delay components. It is noted that unlike the Lyapunov functionals chosen in [31-36], the constructed Lyapunov functional (14) takes the states  $z(t-d_1(t)-d_2)$  and  $z(t-d_1-d_2(t))$  into account, which make full use of the information of the neuron activation functions and the involved time-varying delay. So,

the Lyapunov functional in our paper is more general than that in [31-36], and the stability criterion in our paper may be less conservative.

**Remark 3** It is worth mentioning that in [31-36], the free-weighting matrices technique is utilized to discuss the stability problem for neural networks with two additive time-varying delay components. However, some matrices do not reduce the conservatism of the resulting criterion; on the contrary, they make the stability analysis more complicated and lead to a significant increase in the computational demand. While in our proof, we enlarge the cross terms appeared in the derivative of Lyapunov functional by using reciprocally convex method [5]. Thus, the criterion derived in Theorem 1 has less decision variables than the ones in [31-36].

When the informations of the derivative of time-varying delays are unknown, a rate-independent corollary for the delays  $d_1(t)$  and  $d_2(t)$  satisfying  $0 \leq d_i(t) \leq d_i (i = 1, 2)$  is derived by setting  $Q_j = 0 (j = 3, 4, \dots, 8)$  in the proof of Theorem 1.

**Corollary 1** For given scalars  $d_i > 0 (i = 1, 2)$ , the system (7) with two additive time-varying delay components satisfying  $0 \leq d_i(t) \leq d_i (i = 1, 2)$  is globally asymptotically stable if there exist matrices  $P > 0$ ,  $Q_1 > 0, Q_2 > 0, Q_9 > 0$ ,  $R_1 > 0, R_2 > 0$ , positive diagonal matrices  $A_j = \text{diag} \{ \lambda_{j1}, \lambda_{j2}, \dots, \lambda_{jn} \} \geq 0$ ,  $A_j = \text{diag} \{ \delta_{j1}, \delta_{j2}, \dots, \delta_{jn} \} \geq 0 (j = 1, 2)$ , and matrices  $M_k (k = 1, 2, 3, 4)$  with appropriate dimensions such that the LMIs (12),(13) and following LMI hold

$$\Xi_1 + \tilde{\Xi}_2 + \Xi_3 + \Xi_4 < 0, \quad (20)$$

where

$$\tilde{\Xi}_2 = e_1^T (Q_1 + Q_2 + Q_9) e_1 - e_4^T Q_1 e_4 - e_5^T Q_2 e_5 - e_{11}^T Q_9 e_{11},$$

and

$\Xi_1, \Xi_3, \Xi_4$  are defined as in Theorem 1.

**Remark 4** It is worth pointing out that the obtained results in this paper are all formulated in terms of LMIs, they can be easily solved using any LMI toolbox like one of Matlab.

## 4. Numerical Examples

In this section, two numerical examples are given to illustrate the effectiveness and the less conservatism of obtained results in this paper.

**Example 1** Consider the neural network (7) with following parameters

$$C = \begin{bmatrix} 2.0 & 0 \\ 0 & 1.8 \end{bmatrix}, A = \begin{bmatrix} 1.2 & 1.0 \\ -1.0 & -1.2 \end{bmatrix}, B = \begin{bmatrix} 1.0 & 1.0 \\ -1.0 & 0.8 \end{bmatrix},$$

$$K_1 = \text{diag} \{ -0.2, -0.4 \}, K_2 = \text{diag} \{ 0.4, 0.8 \}.$$

**Case I :** For given  $d_1 = 1.0, \mu_1 = 0.3, \mu_2 = 0.6$ , by resorting to the Matlab LMI toolbox to solve the LMIs in Theorem 1, it is concluded that system (7) is globally asymptotically stable while  $d_2$  is up to 1.6620, and the corresponding feasible solutions of LMIs (11)-(13) are given as follows

$$P = \begin{bmatrix} 340.3995 & 29.5773 \\ 29.5773 & 437.4800 \end{bmatrix}, Q_1 = \begin{bmatrix} 290.8926 & 92.4557 \\ 92.4557 & 30.0660 \end{bmatrix}, Q_2 = \begin{bmatrix} 7.6643 & -29.1050 \\ -29.1050 & 126.2785 \end{bmatrix},$$

$$\begin{aligned}
 Q_3 &= \begin{bmatrix} 6.9822 & -27.6835 \\ -27.6835 & 127.94448 \end{bmatrix}, Q_4 = \begin{bmatrix} 290.2311 & 90.5342 \\ 90.5342 & 29.0593 \end{bmatrix}, Q_5 = \begin{bmatrix} 0.0233 & 0.0128 \\ 0.0128 & 0.0110 \end{bmatrix}, \\
 Q_6 &= 10^3 \times \begin{bmatrix} 0.5363 & -1.3635 \\ -1.3635 & 3.4672 \end{bmatrix}, Q_7 = \begin{bmatrix} 7.6619 & -29.1063 \\ -29.1063 & 126.2754 \end{bmatrix}, Q_8 = \begin{bmatrix} 290.8902 & 92.4543 \\ 92.4543 & 30.0629 \end{bmatrix}, \\
 Q_9 &= \begin{bmatrix} 74.0150 & 6.8621 \\ 6.8621 & 94.9614 \end{bmatrix}, R_1 = \begin{bmatrix} 9.4331 & -38.8144 \\ -38.8144 & 173.5194 \end{bmatrix}, R_2 = \begin{bmatrix} 92.1490 & 28.8032 \\ 28.8032 & 11.1457 \end{bmatrix}, \\
 A_1 &= \begin{bmatrix} 445.1171 & 0 \\ 0 & 917.0683 \end{bmatrix}, A_2 = \begin{bmatrix} 260.9814 & 0 \\ 0 & 242.2099 \end{bmatrix}, A_3 = 10^3 \times \begin{bmatrix} 2.3534 & 0 \\ 0 & 1.9197 \end{bmatrix}, \\
 A_4 &= \begin{bmatrix} 739.4622 & 0 \\ 0 & 212.7990 \end{bmatrix}, M_1 = \begin{bmatrix} -8.8791 & 39.7223 \\ 32.9453 & -156.8078 \end{bmatrix}, M_2 = \begin{bmatrix} -9.4320 & 38.9257 \\ 38.6997 & -173.5083 \end{bmatrix}, \\
 M_3 &= \begin{bmatrix} -92.1470 & -28.7258 \\ -28.8797 & -11.1387 \end{bmatrix}, M_4 = \begin{bmatrix} -91.4163 & -26.8136 \\ -30.3343 & -11.0515 \end{bmatrix}.
 \end{aligned}$$

**Case II:** For given  $d_1 = 1.0$  and unknown  $\mu_1, \mu_2$ , by resorting to the Matlab LMI toolbox to solve the LMIs in Corollary 1, it is concluded that system (7) is globally asymptotically stable while  $d_2$  is up to 0.9669, and the corresponding feasible solutions of LMIs (12),(13),(20) are given as follows

$$\begin{aligned}
 P &= \begin{bmatrix} 869.0976 & -29.8226 \\ -29.8226 & 999.4159 \end{bmatrix}, Q_1 = Q_2 = \begin{bmatrix} 0.0655 & 0.0120 \\ 0.0120 & 0.0484 \end{bmatrix}, Q_9 = 10^3 \times \begin{bmatrix} 0.9417 & 0.1586 \\ 0.1586 & 1.4051 \end{bmatrix}, \\
 R_1 &= \begin{bmatrix} 206.3890 & -43.2173 \\ -43.2173 & 400.7494 \end{bmatrix}, R_2 = \begin{bmatrix} 207.5793 & -43.4705 \\ -43.4705 & 403.0807 \end{bmatrix}, A_1 = 10^3 \times \begin{bmatrix} 1.2371 & 0 \\ 0 & 2.9284 \end{bmatrix}, \\
 A_2 &= \begin{bmatrix} 870.7120 & -43.2173 \\ -43.2173 & 648.6941 \end{bmatrix}, A_3 = 10^3 \times \begin{bmatrix} 8.0968 & 0 \\ 0 & 1.6775 \end{bmatrix}, A_4 = 10^3 \times \begin{bmatrix} 2.3738 & 0 \\ 0 & 1.1019 \end{bmatrix}, \\
 M_1 &= \begin{bmatrix} -206.1429 & 56.4754 \\ 29.8478 & -400.2845 \end{bmatrix}, M_2 = \begin{bmatrix} -206.1415 & 56.4748 \\ 29.8474 & -400.2829 \end{bmatrix}, \\
 M_3 &= \begin{bmatrix} -207.3318 & 56.8057 \\ 30.0234 & -402.6131 \end{bmatrix}, M_4 = \begin{bmatrix} -207.3305 & 56.8052 \\ 30.0229 & -402.6116 \end{bmatrix}.
 \end{aligned}$$

**Example 2** Consider the neural network (7) with following parameters

$$\begin{aligned}
 C &= \begin{bmatrix} 2.0 & 0 \\ 0 & 2.0 \end{bmatrix}, A = \begin{bmatrix} 1.0 & 1.0 \\ -1.0 & -1.0 \end{bmatrix}, B = \begin{bmatrix} 0.88 & 1.0 \\ 1.0 & 1.0 \end{bmatrix}, \\
 K_1 &= \text{diag} \{0, 0\}, K_2 = \text{diag} \{0.4, 0.8\}.
 \end{aligned}$$

Assuming  $\mu_1 = 0.7, \mu_2 = 0.1$ . Our purpose is to calculate the upper delay bound  $d_2$  of delay  $d_2(t)$ , or  $d_1$  of delay  $d_1(t)$ , when the other is known, below which the system (7) is globally asymptotically stable. The maximum value of upper delay bound  $d_2$  obtained by [32,34,35] and Theorem 1 in this paper for different  $d_1$  are listed in Table I. The maximum value of upper delay bound  $d_1$  obtained by [31,33,36] and Theorem 1 in this paper for different  $d_2$  are listed in Table II. From Table I and Table II we can see that by using Theorem 1, our obtained results are less conservative than those of [31-36].

Moreover, Theorem 1 in this paper contain fewer decision variables than those conditions in [31-36], which can be seen from Table III. Hence, it is mathematically less complex and computationally more efficient.



**Table 1. Allowable Upper Bound  $d_2$  for Various  $d_1$**

Methods	$d_1 = 0.8$	$d_1 = 1.0$	$d_1 = 1.2$
[32]	1.5666	1.3668	1.1664
[34]	2.0164	1.8203	1.6197
[35]	2.0961	1.8961	1.6961
Theorem 1	5.0811	3.7229	2.7561

**Table 2. Allowable Upper Bound  $d_1$  for Various  $d_2$**

Methods	$d_2 = 0.1$	$d_2 = 0.2$	$d_2 = 0.3$
[31]	2.1564	1.6464	1.4365
[36]	2.2571	2.1719	2.1074
[33]	2.7248	2.3325	2.2187
Theorem 1	5.3783	3.2321	2.5117

**Table 3. Number of Decision Variables**

Methods	Number of decision variables
[31]	$32.5n^2 + 8.5n$
[32]	$36.5n^2 + 8.5n$
[33]	$15.5n^2 + 11.5n$
[34]	$80.5n^2 + 8.5n$
[35]	$21.5n^2 + 8.5n$
[36]	$11n^2 + 10n$
Theorem 1	$10n^2 + 10n$

## 5. Conclusion

This paper deals with the problem of stability for neural networks with two additive time-varying delay components. By constructing a more general type of Lyapunov functional, and using reciprocally convex method, delay-dependent stability criteria are proposed in terms of LMIs for considered systems. Stability criteria are obtained by using more information of the time-varying delays and considering the relationship between the time-varying delays and their upper delay bounds, these results have been shown to be less conservative than some recently report results. Moreover, the proposed method decreases the computational complexity comparable to some existing methods. Two numerical examples are given to illustrate the applicability of the results. The method used in this paper can be further extended to cope with the robust stability problem for such systems and stability problem for delayed neural networks with interval time-varying delay.

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### Author



**Jianmin Jiao**, he received the M.S. in College of Mathematics and Information Science from Shaanxi Normal University, Xi'an, China, in 2009. He is currently an associate professor of College of Mathematics and Information Science, Baoji University of Arts and Sciences, China. His research interests include time-delay systems, neural networks, singular systems and robust control.

