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G^1 Continuity Method based on C-B Spline Curves and C-Bézier

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Abstract

C-B spline can not represent semi-circle and semi-elliptical arcs precisely. This paper will discuss the nature of C-B spline's and C-Bézier curve's endpoints. Then, on the basis of the analysis of their characteristics and by means of adding control points to make C-B spline dominate the first and last vertices of the polygon and get tangency with the first and end sides. Thus it has given out the G^1 splicing method for C-B spline and C-Bézier curve which can represent the semi-circle arcs and semi-elliptical arcs of C-B spline, hence enhancing the controlling and presenting capacity of C-B spline.

Keywords: *C-B spline, Bézier curve, G^1 splicing, Surface modeling*

1. Introduction

Nowadays more and more mechanical parts are designed in the form of surface modeling and processed on CNC machine. Those surfaces can be divided into several types according to its formations, such as translation surfaces, ruled surface, rotating surface which is the one of the most common surfaces and swept surface. However, Bézier and B-spline can not represent circle precisely so they are not suitable for describing rotating surfaces. And for other surfaces, the sectional curve or boundary line often contains arc, elliptic arc and parabolic arc that are designed precisely by drawings and that require high precision in manufacturing. Bézier and B-spline can not represent those surfaces precisely because it is unable to represent quadric curve except for parabola. However, adopting approximate methods can not only bring new troubles in processing and make simple problem complicated, but also produce design errors that have not existed before. If using another mathematical description separately for those quadratic curves, such as implicit equation, it will bring problems produced by implicit equation and lead to two different mathematics concurrent in geometric models and make the system become huger.

It is obvious that NURBS can represent quadric curve, so it can represent engineering surface precisely. The purpose of NURBS is that describing free-form curve and surface and quadric curve and surface uniformly. But NURBS is not just a simple expansion for Bézier and B-spline. The new problems that calculation becomes complicated, especially weight factor, parameterization and surface continuity which are still not solved, will emerge in the process of modeling. The existence of weight factor provides flexibility for representing surface, but puts forward higher requirements for designer and user and increases memory space. The advantages of NURBS that represent free-form curve and surface and quadric curve and surface uniformly are not exploited to the full, especially for those engineering surfaces that have simple formulation but include quadric curves. Therefore it is imperative to develop a new model to solve those problems. C-B splines are widely used in the geometry modeling because of its characteristics that the in-plant controlling capacity and so on. However, the C-B spline can not precisely describe the semi-circle and semi-elliptical arcs, which is a common problem in engineering. To solve the problem, this paper will focus on the G1 splicing method for C-B spline and C-Bézier

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curve. C-Bézier curves can represent semi-circle and semi-elliptical arcs precisely, thus by utilizing it and the G1 splicing method in the C-B spline modeling; we can solve the problem effectively.

2. Research Actualities

In view of the limits of NURBS model and to keep its geometric property and overcome the disadvantages, some new curve and surface models come into being and the mixed curve and surface model based on polynomial and non-polynomial mixed space is worthy of being mentioned. Those models have aroused the researcher's great interest and attention because they inherit the advantages of polynomial splines and also avoid the disadvantages of NURBS. At present, the study of mixed model mainly focuses on the following aspects:

(1). Helical Spline

As a symmetrical Tchebycheffian B-spline, helical spline, proposed in the form of Bézier, is linear combination of 1, t, sint, cost:

$s(t) = c1 + c2 + c3\sin t + c4\cos t \quad t \in I,$ whereby parameter intervals $I \leqq 2\pi$

Provided that control polygon of helical spline curve is b0b1b2b3, the form of helical spline is determined by the length of I, and when I's absolute value approaches zero, the curve is approximate to b0b1b2b3 while when I's absolute value approaches 2π, the curve is approximate to b0b3. Helical spline can represent straight line, circle and helical line precisely and its parameter is arc-length parameter. And the tensor product surface can represent helicoids, rotating surface, etc. However, helical spline can not represent higher ordered polynomial spline so that weakens the application in CAGD. For this connection, Zhang Jiwen in Zhejiang University puts forward C-B spline curves with uniform knot.

(2). C-curve

The fundamental principle of C-curve surface is that using base {1, u, sin u, cosu} to replace power-base {1, u, u2, u3} of cubic polynomial curvilinear equation. The theory of C-curve surface contains C- Fugerson[1], C- Bézier and C-B spline relative to traditional modeling method including Fugerson, Bézier, and B-spline. C-B spline theory, on one hand, has many advantages when describes free-form curve and surface of B-spline, on the other hand, can describe quadric curve and surface precisely and is suitable for constructing engineering surface.

Through theoretical analysis and engineering application of C-B spline, it can obtain that C-B spline has following characteristics in engineering curve and surface modeling:

First, it can represent free-form curve and surface and quadric curve and surface in a unified form; second, the introduction of controls parameter α, which is easy to select, has increased the ability to represent curve and surface. Adjustment of α has a regulating effect on the form of curve and surface; third, the required methods is less than NURBS when defines memory space for curve and surface; forth, C-B spline theory includes B-spline, so makes it easier to convert B-spline into C-B spline; fifth, the algorithm of curve and surface is much easier and the computation speed is faster than NURBS [2].

For those advantages, C-B spline has become a significant tool for geometric modeling in CAD/CAM system. However, C-B spline can only represent primary polynomial curve which restricts the application in CAD/CMD. So Chen Qinyu puts forward to κ-th order C-Bézier curve of Ω which is equal to span{1, t, t2,…,tk-2, sin t, cost}[3], and Lv Yonggang has made up general segmented κ-th order trigonometric polynomial B-spline base of Ω which is equal to span{ sin t, cost, tk-3, tk-4,…t, 1} and defined trigonometric polynomial B-spline curve[4]. And Wang Guozhao has given κ-th order (which κ is no less than 3) non-uniform trigonometric polynomial B-spline curve of Ω which is equal to span {1, t, t2… tk-3, cost, sin t} [5].

3. The Definition and Properties of C-B Spline

The C-B spline is formed by substituting basis $\begin{bmatrix}\sin t & \cos t & t & 1\end{bmatrix}$ for the basis of triple uniform B-spline curve equations $\begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}$. Its definition can be represented by a matrix form as: [6]

$$P_i(t)=B_0(t)b_i+B_1(t)b_{i+1}+B_2(t)b_{i+2}+B_3(t)b_{i+3}=$$

$$\frac{1}{2\alpha(1-C)}\begin{pmatrix}\sin t & \cos t & t & 1\end{pmatrix}\begin{bmatrix}C & -(1+2C) & 2+C & -1\\ -S & 2S & -S & 0\\ -1 & 1+2C & -(1+2C) & 1\\ \alpha & -2\alpha C & \alpha & 0\end{bmatrix}\begin{bmatrix}b_i\\ b_{i+1}\\ b_{i+2}\\ b_{i+3}\end{bmatrix} \quad (1)$$

And $0<\alpha\le\pi, 0<t\le\alpha, S=\sin\alpha, C=\cos\alpha, i=0,1,\cdots,n-3; b_i, b_{i+1}, b_{i+2}, b_{i+3}$ is the control point of C-B spline

$$B_0(t)=\frac{(\alpha-t)-\sin(\alpha-t)}{2\alpha(1-C)} \qquad B_3(t)=\frac{t-\sin t}{2\alpha(1-C)}$$

$$B_1(t)=B_3(t)-2B_0(t)+\frac{\alpha-t}{\alpha} \qquad B_2(t)=B_0(t)-2B_3(t)+\frac{t}{\alpha}$$

B0 (t), B1 (t), B2 (t), B3 (t) is called C-B spline's basis function.

The endpoint characteristic of C-B spline is: [7]

$$P_i(0)=\frac{1}{2\alpha(1-C)}((\alpha-S)b_i+2(S-\alpha C)b_{i+1}+(\alpha-S)b_{i+2})$$

$$P_i(\alpha)=\frac{1}{2\alpha(1-C)}((\alpha-S)b_{i+1}+2(S-\alpha C)b_{i+2}+(\alpha-S)b_{i+3})$$

$$P_i^{'}(0)=\frac{1}{2\alpha}(b_{i+2}-b_i), P_i^{'}(\alpha)=\frac{1}{2\alpha}(b_{i+3}-b_{i+1})$$

$$P_i^{''}(0)=\frac{S}{2\alpha(1-C)}(b_i-2b_{i+1}+b_{i+2})$$

$$P_i^{''}(\alpha)=\frac{S}{2\alpha(1-C)}(b_{i+1}-2b_{i+2}+b_{i+3})$$

$$i=0,1,\cdots,n-1 \quad (2)$$

4. The Definition and Properties of C-Bézier Curve

The C-Bézier curve is formed by substituting basis $\begin{bmatrix}\sin t & \cos t & t & 1\end{bmatrix}$ for the basis of triple Bézier curve equations $\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$. Its definition can be represented by a matrix form as [8]:

$$B_\alpha(t) = Z_0(t)q_0 + Z_1(t)q_1 + Z_2(t)q_2 + Z_3(t)q_3 =$$

$$\frac{1}{\alpha - S}\begin{pmatrix} \sin t & \cos t & t & 1 \end{pmatrix} \begin{bmatrix} C & 1-C-M & M & -1 \\ -S & (\alpha-K)M & -KM & 0 \\ -1 & M & -M & 1 \\ \alpha & -(\alpha-K)M & KM & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \quad (3)$$

Among, $0 < \alpha \le \pi, 0 \le t \le \alpha, S = \sin a, C = \cos a$

$$K = \frac{\alpha - S}{1-C}, M = \begin{cases} 1 & \alpha = \pi \\ \frac{S}{\alpha - 2K} = \frac{S(1-C)}{2S - \alpha - \alpha S} & 0 < \alpha < \pi \end{cases}$$

q0, q1, q2, q3 is the control point of the C-Bézier curve.

$$Z_0(t) = \frac{(\alpha - t) - \sin(\alpha - t)}{\alpha - \sin a}$$

$$Z_1(t) = M\left[\frac{1-\cos(\alpha - t)}{1-\cos\alpha} - Z_0(t)\right]$$

$$Z_2(t) = M\left[\frac{1-\cos t}{1-\cos\alpha} - Z_3(t)\right]$$

$$Z_3(t) = \frac{t - \sin t}{\alpha - \sin\alpha}$$

is called C-Bézier curve's basis function.
The endpoint characteristic of C-Bézier curve is:

$$\begin{aligned} & B_\alpha(0) = q_0, B_a(\alpha) = q_3 \\ & B_\alpha^{'}(0) = \frac{1}{K}(q_1 - q_0), B_\alpha^{'}(\alpha) = \frac{1}{K}(q_3 - q_2) \\ & B_\alpha^{''}(0) = \frac{1}{(\alpha - S)}(Sq_0 - (\alpha - K)Mq_1 + KMq_2) \\ & B_\alpha^{''}(\alpha) = \frac{1}{(\alpha - S)}(Sq_3 - (\alpha - K)Mq_2 + KMq_1) \end{aligned} \quad (4)$$

5. The C-Bézier Curve Representation of Elliptical Arcs and Circular Arcs

The No.9 in bibliography provides the conditions for presenting elliptical arcs and circle arcs by C-Bézier curve. To set the polygon as an isosceles trapezoid $q_0 q_1 q_2 q_3$ with the bottom edge as $|q_3 - q_0| = d$, the base angle as $\theta(0 < \theta \le \pi/2)$, and the height as h , and the conditions: [9]

$$\frac{h}{d}=\frac{|q_1-q_0|}{|q_3-q_0|}=\frac{K}{2\sin\theta} \tag{5}$$

The C-Bézier curve whose control vertices apply to the condition (4.5) is handled as an arc with the parameter as $\alpha=2\theta$ and the central angle as 2θ . If θ = π/2, it represents a semi-circle arc.

If make the polygon meet the condition of:

$$|q_3-q_0|=|q_2-q_1|=2a,|q_3-q_2|=|q_1-q_0|=Kb=\frac{\pi}{2}b \tag{6}$$

The C-Bézier curves at this time is a semi-elliptical arc with the parameter $\alpha=\pi$ and the radius as a,b ; Figure 4.1 and Figure 4.2 have given the C-Bézier curve representations of semi- circular arcs and semi-elliptical arcs respectively.

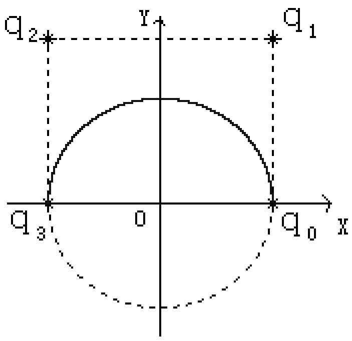


Figure 1. The C-Bézier Representation of Semi-Circle Arcs

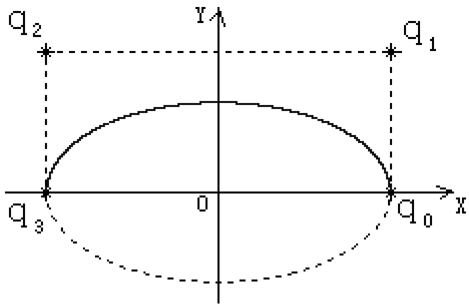


Figure 2. The C-Bézier Representations of Semi-Elliptical Arcs

6. The Condition of G1 Splicing Method for C-B Spline and C-Bézier Curve

The C-B spline can not precisely describe the semi-circle and semi-elliptical arcs. Therefore, when designing complicated free curves, it should be with the help of the C-Bézier curve. In order to utilize them effectively, we should consider the splicing method for C-B spline and C-Bézier curve. Their splicing should meet the continuity requirement of G^1 .

In order to gain the G^1 splicing of the two, C-B spline should pass through the first and last vertices and gain tangent with the first and end sides, that is to say, requiring b_0 and b_n as the starting and end point of curves, and the tangent at b_0 and b_n should be $b_1 - b_0$ and $b_n - b_{n-1}$ respectively [10]. And it needs to add two control points --- $b_{-1} = 2b_0 - b_1$ and $b_{n+1} = 2b_n - b_{n-1}$ to achieve the requirements. Substituting b_{-1} and b_{n+1} into the equation (2.2) respectively, it can obtain that:

$$P(0) = b_0, P(\alpha) = b_n$$
$$P^{'}(0) = \frac{1}{\alpha}(b_1 - b_0)$$
$$P^{'}(\alpha) = \frac{1}{\alpha}(b_n - b_{n-1}) \tag{7}$$

Provided that $P(u)$ and $Q(w)$ are C-B spline and C-Bézier curve respectively, defining $P(u)$ by $b_0, b_1, \cdots, b_n$ and $Q(w)$ by q_0, q_1, q_2, q_3 [11], it can get the following representations of $P(u)$ and $Q(w)$ from the formula (2.1) and (3.1):

$$P_i(u) = \sum_{j=0}^{3} B_j(u) b_{i+j} \tag{8}$$

$$Q(w) = \sum_{i=0}^{3} Z_i(w) q_i \tag{9}$$

From formula (5.1), we can know that, after increasing control points, the first vector at b_n of the curve $P(u)$ is:

$$P^{'}(\alpha) = \frac{1}{\alpha}(b_n - b_{n-1}) \tag{10}$$

And from formula (3.2), we can know that the first vector at q_0 of the curve $Q(w)$ is:

$$Q^{'}(0) = \frac{1}{K}(q_1 - q_0) \tag{11}$$

The G^1 continuity of the two curves will firstly be achieved by the continuity of $P(u)$'s end and Q (w)'s head, i.e.

$$b_n = q_0 \tag{12}$$

Secondly, the tangent direction at their junction should be same, i.e.

$$Q^{'}(0) = \lambda P^{'}(\alpha), \lambda > 0 \tag{13}$$

Substituting formula (5.4) and (5.5) into formula (5.7), we can get:

$$\frac{1}{K}(q_1 - q_0) = \frac{\lambda}{\alpha}(b_n - b_{n-1})$$

It can be changed to：

$$q_1 = q_0 + \frac{\lambda K}{\alpha}(b_n - b_{n-1}) = p + \frac{\lambda K}{\alpha} q \qquad (14)$$

In the formula $p = b_n = q_0, q = b_n - b_{n-1}, \lambda, K, \alpha$ are the normal numbers.

The C-B spline and C-Bézier curve will achieve their continuity at G^1 when they apply to formula (5.6) and (5.8) simultaneously. Its geometric meaning is: The control point must be collinear and ordered when C-B spline and C-Bézier curve are spliced. The line is the common tangent [12] at the common connection point. Figure 5.1 shows an example of their splicing:

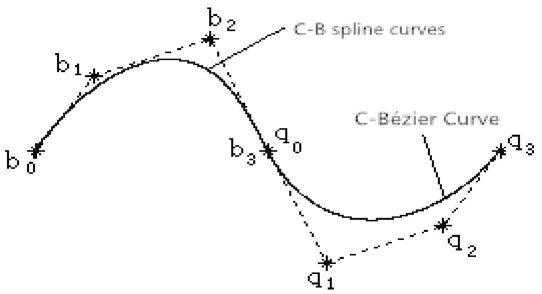
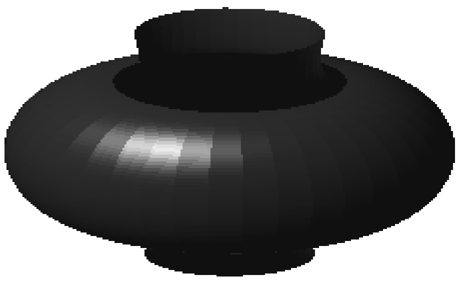
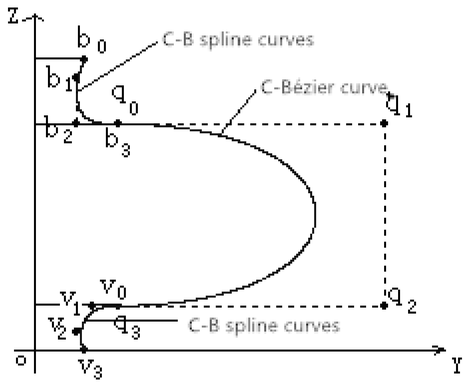


Figure 3. The G^1 Splicing Of C-B Spline $\alpha = \pi/2$ and C-Bézier Curve $\alpha = \pi/3$



(a) Lantern surface of revolution



(b) Lantern rotation surface of bus bar

Figure 4. Lanterns Surface Of Revolution

By the G^1 splicing method in C-B spline curve surface modeling, the representation of semi-circle and semi-elliptical arcs can be easily achieved. Figure 5.2 shows an example of C-B spline curve surface modeling [13]. Figure 4.4 is a lantern shape and Figure 5.2 (b) is the lantern's generating line which is formed by the splicing of two C-B splines and one C-Bézier curve which represents a half elliptical arc.

7. Summaries

The condition for splicing C-B spline and C-Bézier curve presented in this paper is simple, intuitive and has a clear geometric meaning. This method can be a good solution to the problem of representing semi-circle and semi-elliptical arcs in C-B spline curve surface modeling. Moreover, it can be further applied to the condition of splicing C-B spline curve surfaces and C-Bézier curve surfaces.

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References

[1] W. Wang and G. Wang, "Uniform B-spline with Shape Parameter. Journal of Computer-Aided Design & Computer Graphics", vol. 16, (**2004**), pp. 783-788
[2] X. Wu and X. Han, "Extension of Cubic Bezier Curve. Journal of Engineering Graphics, vol. 6, (**2005)**, pp. 98-102
[3] X. Wu and X. Han, "Two Different Extensions of Quartic Bézier Curve", Journal of Engineering Graphics, vol. 5, (**2006)**, pp. 59-64
[4] X. Han, Q. Hu and F. Peng, "Generalized Bézier Curves, Journal of Computer-Aided Design & Computer Graphics", vol. 3, (**2006**), pp. 406-409.
[5] X. Zhu, "Free Curves and Surfaces Modeling Technology, Science Press", (**2000).**
[6] B. Su and D. Liu, "Computation Geometry, Shanghai Science and Technology Press, **(1981).**
[7] G. Wang, G. Wang and J. Zheng, "Computer Aided Geometric Design, Higher Education Press", (**2001).**
[8] A. H. Barr, "Global and local Deformation of Solid Primitives", SIGGRAPH' 84, ACM Comp. Graph, vol. 18, no. 3, (**1984**), pp. 21-30.
[9] T. W. Sederberg and S. R. Parry, "Free-form deformation of solid geometric models", Computer Graphics (SIGGRAPH), vol. 20, no. 4, (**1986**), pp. 151-160.
[10] L. Piegl and W. Tiller, "Curve and surface construction using B-splines", Computer Aided Design, vol. 19, (**1987**), pp. 487-498.
[11] G. Farin, "Rational curves and surfaces, Mathematical Methods in Computer Aided Geomet-ric Design", Boston: Academic Press, (**1989**).
[12] L. Piegl, "On NURBS: A survey [J]", IEEE Computers Graphics and Application, vol. 11, no. 1, (**1991**), pp. 55-71.
[13] G. Farin, "From conics to NURBS: A tutorial and survey", IEEE Computers Graphics and Application, vol. 1, no. 5, (**1992),** pp. 78-86.

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