International Journal of Control and Automation
Vol.8, No.11 (2015), pp.273-278
http://dx.doi.org/10.14257/ijca.2015.8.11.26

A New Reverse Hilbert's Type Inequality

Gaowen Xi[1] and Yue Xi[2]

[1]*College of Mathematics and Physics, Chongqing University of Science and Technology, Chongqing, 401331, P. R. China*
[2]*College of Business Administration, Chongqing University of Science and Technology, Chongqing, 401331, P. R. China*
xigaowen@163.com

Abstract

By introducing a parameter α and using the Euler-Maclaurin expansion, we establish an inequality of a weight coefficient. Using this inequality, we derive a new reverse Hilbert's type inequality. As applications, an equivalent form is obtained.

Keywords: *Hilbert's inequality, weight coefficient, Euler-Maclaurin expansion, everse, Holder's inequality*

1. Introduction

If $p>1$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n \geq 0$, $b_n \geq 0$ and $0<\sum_{n=1}^{\infty} a_n^p<\infty$, $0<\sum_{n=1}^{\infty} b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}\{\sum_{n=1}^{\infty} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} b_n^q\}^{\frac{1}{q}}, \tag{1.1}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,\ n\}}<pq\{\sum_{n=1}^{\infty} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty} b_n^q\}^{\frac{1}{q}}, \tag{2.2}$$

where the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ and pq and is best possible for each inequality respectively. Inequality (1.1) is Hardy-Hilbert's inequality. Inequality (1.2) is a Hilbert's type inequality [1].

In [8], Yang gave a reinforcement of inequality (1.1):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m^2+n^2}<\frac{\pi}{2}\{\sum_{n=1}^{\infty}\frac{1}{n} a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\frac{1}{n} b_n^q\}^{\frac{1}{q}}, \tag{1.3}$$

In [2], [9] and [6], M. Krnic, J. Pecaric and B. Yang gave some generalization and reinforcement of inequality (1.1). In [4], Kuang Jichang and L. Debnath gave a reinforcement of inequality (1.2):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,\ n\}}<\{\sum_{n=1}^{\infty}[pq-G(\ p\ ,\ \ n\ )]a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[pq-G(\ q\ ,\ \ n\ )b_n^q\}^{\frac{1}{q}}, \tag{1.4}$$

ISSN: 2005-4297 IJCA
Copyright ⓒ 2015 SERSC

where $G(r,\ n)=\dfrac{r+\frac{1}{3r}-\frac{4}{3}}{(2n+1)^{\frac{1}{r}}}>0\ (r=p,\ q)$.

In [5], and Yang gave some generalization and reinforcement of inequalities (1.2) and (1.4):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max(m^{\lambda},\ n^{\lambda})}<\{\sum_{n=1}^{\infty}[\kappa(\lambda)-\frac{1}{3qn^{\frac{q+\lambda-2}{q}}}]n^{1-\lambda}a_n^p\}^{\frac{1}{p}}$$

$$\times\{\sum_{n=1}^{\infty}[\kappa(\lambda)-\frac{1}{3pn^{\frac{q+\lambda-2}{p}}}]n^{1-\lambda}b_n^q\}^{\frac{1}{q}}, \tag{1.5}$$

where $\kappa(\lambda)=\dfrac{pq\lambda}{(p+\lambda-2)(q+\lambda-2)}>0\,,2-\min\{p,q\}<\lambda\le 2$.

For the reverse Hardy-Hilbert's inequality, Yang [7] gave a reverse form of inequalities (1.3). In this paper, by introducing a parameter α and using the Euler-Maclaurin expansion, we establish an inequality of a weight coefficient. Using this inequality, we derive a new Hilbert's type inequality. As applications, an equivalent form is obtained.

For this, we need the following expression of the Euler-Maclaurin (see [10]).

$$\sum_{k=n+1}^{m}f(k)=\int_n^m f(x)dx+\frac{1}{2}[f(m)-f(n)]+\int_n^m P_1(x)f'(x)dx\,, \tag{1.6}$$

where $f(x)\in C^1[0,\ \infty), m,n\in N_0(m>n)$, N_0 be the set of non-negative integers, $P_i(x)(i=1,2,\cdots)$ be Bernoulli function $(P_1(x)=x-[x]-\frac{1}{2})$. When $\sum_{k=n}^{\infty}f(k),\ \int_n^{\infty}f(x)dx$ are convergences, we have

$$\sum_{k=n}^{\infty}f(k)=\int_n^{\infty}f(x)dx+\frac{1}{2}f(n)+\int_n^{\infty}P_1(x)f'(x)dx\,, \tag{1.7}$$

and (see [7])

$$\int_n^{\infty}P_1(x)g(x)dx=-\frac{1}{8}g(n)\varepsilon(0<\varepsilon<1), \tag{1.8}$$

where $g(x)\in C^1[0,\ \infty), g'(x)<0(or\ g'(x)>0),\ x\in[n,\ \infty), g(\infty)=0.$

2. A Lemma

Lemma 2.1. Let N be the set of positive integers. The weight coefficient $\omega(n,\alpha)$ is defined by

$$\omega(n,\alpha)=\sum_{k=1}^{\infty}\frac{1}{\max\{k^2,\ n^2\}+\alpha^2},\qquad n\in N, 0<\alpha<1.$$

Then we have

Copyright ⓒ 2015 SERSC

$$\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}<\omega(n,\alpha)<\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}. \tag{2.1}$$

Proof. If $n \in N$, let $f(x)=\sum_{k=1}^{\infty}\frac{1}{\max\{x^2,\ n^2\}+\alpha^2}, \quad x\in[0,\infty)$, we have

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{\max\{x^2,\ n^2\}+\alpha^2}=\begin{cases}\frac{1}{n^2+\alpha^2},\ x<n,\\ \frac{1}{x^2+\alpha^2},\ x\ge n,\end{cases}$$

and

$$f'(x)=\begin{cases}0, & x<n,\\ -\frac{2x}{(x^2+\alpha^2)^2}=-\frac{2}{x(x^2+\alpha^2)}+\frac{2\alpha^2}{x(x^2+\alpha^2)^2}, & x\ge n.\end{cases}$$

By (1.7), we obtain

$$\begin{aligned}\omega(n,\alpha)&=\int_1^{\infty}f(x)dx+\frac{1}{2}f(1)+\int_1^{\infty}P_1(x)f'(x)dx\\ &=\frac{n}{n^2+\alpha^2}-\frac{1}{n^2+\alpha^2}+\frac{1}{\alpha}[\frac{\pi}{2}-\arctan\frac{n}{\alpha}]+\frac{1}{2(n^2+\alpha^2)}\\ &\quad-\int_n^{\infty}P_1(x)\frac{2}{x(n^2+\alpha^2)}dx+\int_n^{\infty}P_1(x)\frac{2\alpha^2}{x(n^2+\alpha^2)^2}dx.\end{aligned}$$

By (1.8), we have

$$-\frac{1}{4n(n^2+\alpha^2)}<\int_n^{\infty}P_1(x)\frac{2}{x(n^2+\alpha^2)}dx<0.$$

$$-\frac{\alpha^2}{4n(n^2+\alpha^2)^2}<\int_n^{\infty}P_1(x)\frac{2\alpha^2}{x(n^2+\alpha^2)^2}dx<0.$$

Since we find

$$\begin{aligned}&\frac{n}{n^2+\alpha^2}-\frac{1}{2(n^2+\alpha^2)}+\frac{1}{\alpha}[\frac{\pi}{2}-\arctan\frac{n}{\alpha}]-\frac{\alpha^2}{4n(n^2+\alpha^2)^2}<\omega(n,\alpha)\\ &<\frac{n}{n^2+\alpha^2}-\frac{1}{2(n^2+\alpha^2)}+\frac{1}{\alpha}[\frac{\pi}{2}-\arctan\frac{n}{\alpha}]+\frac{1}{4n(n^2+\alpha^2)},\end{aligned}$$

$$\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}<\omega(n,\alpha)<\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}.$$

Then we have (2.1). The lemma is proved.

3. Main Results

Theorem 3.1. If $0<p<1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\alpha<1$, $a_n\ge 0$, $b_n\ge 0$, and $0<\sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p<\infty$, $0<\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q<\infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2, n^2\}+\alpha^2} > \frac{1}{4}[\sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p]^{\frac{1}{p}} \times[\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q]^{\frac{1}{q}}. \quad (3.1)$$

Proof. By the reverse Holder 's inequality [3], we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2, n^2\}+\alpha^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\frac{a_m}{(\max\{m^2, n^2\}+\alpha^2)^{\frac{1}{p}}}][\frac{ab_n}{(\max\{m^2, n^2\}+\alpha^2)^{\frac{1}{q}}}]$$

$$\geq \{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\frac{a_m^p}{\max\{m^2, n^2\}+\alpha^2}]\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}[\frac{b_n^q}{\max\{m^2, n^2\}+\alpha^2}]\}^{\frac{1}{q}}$$

$$= \{\sum_{m=1}^{\infty}\omega(m,\alpha)a_m^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}\omega(n,\alpha)b_n^q\}^{\frac{1}{q}}.$$

Since $0 < p < 1$ and $q < 0$, then by (2.1), we obtain (3.1). The theorem is proved.

Theorem 3.2. If $0 < p < 1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\alpha<1$, $a_n \geq 0$, $b_n \geq 0$, and $0<\sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p<\infty$, then

$$\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2, n^2\}+\alpha^2})^p > \sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p. \quad (3.2)$$

Inequalities (3.2) and (3.1) are equivalent.

Proof. Let

$$b_n = [\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2, n^2\}+\alpha^2})^{p-1}, \quad n\in N.$$

By (3.1), we have

$$\{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q\}^p = \{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2, n^2\}+\alpha^2})^p\}^p$$

$$= \{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2, n^2\}+\alpha^2}\}^p$$

$$\geq \sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p\{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q\}^{p-1}.$$

Then we obtain

$$\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q = \sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2, n^2\}+\alpha^2})^p \geq \sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p. \quad (3.3)$$

If $\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q=\infty$ then in view of $0<\sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p<\infty$ and (3.3), we have

 Copyright ⓒ 2015 SERSC

$$\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2,\ n^2\}+\alpha^2})^p > \sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p.$$

If $0<\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q<\infty,$ then by (3.1), we find

$$\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2,\ n^2\}+\alpha^2})^p > \sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p.$$

Hence we obtain (3.2).

On the other-hand, by the reverse Holder's inequality [3], we have

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}+\alpha^2} &= (\sum_{n=1}^{\infty}\{[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{-\frac{1}{q}}\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2,\ n^2\}+\alpha^2}) \\ &\quad \times[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{\frac{1}{q}}b_n\} \\ &\geq \{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]^{1-p}(\sum_{m=1}^{\infty}\frac{a_m}{\max\{m^2,\ n^2\}+\alpha^2})^p\}^{\frac{1}{p}} \\ &\quad \times\{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q\}^{\frac{1}{q}}\end{aligned}$$

Hence by (3.2), it follows

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m^2,n^2\}+\alpha^2} > \{\sum_{n=1}^{\infty}\frac{2n^2+\alpha^2}{4(n^2+\alpha^2)^2}a_n^p\}^{\frac{1}{p}}\{\sum_{n=1}^{\infty}[\frac{\pi}{2\alpha}+\frac{4n+1}{4(n^2+\alpha^2)}]b_n^q\}^{\frac{1}{q}},$$

Then, (3.2}) and (3.1}) are equivalent. The theorem is proved.

Acknowledgements

The referee made several helpful comments.

References

[1] G. H. Hardy, J. E. Littlewood and G. Polya, “Inequalities”, Cambridge University Press **(1934)**.
[2] M. Krnic and J. Pecaric, “General Hilbert's and Hardy's inequalities, Math. Inequal”, Appl. 1, 8 **(2005)**.
[3] J. Kuang, “Applied Inequalities”, Sandong Science and Technology Press **(2004)**.
[4] J. Kuang and L. Debnath, “On new generalizations of Hilbert's inequality and their applications”, J. Math, Anal. Appl. 245 **(2000)**.
[5] G. Xi, “On generalizations and reinforcements of a Hilbert's type inequality, Rocky Mountain Journal of Mathematics”, 1, 42 **(2012)**.
[6] B. Yang, “A new Hilbert-type inequality”, Bull. Belg. Math. Soc. 13**(2006)**.
[7] B. Yang, “A Reverse of the Hardy-Hilbert's Type Inequality”, Mathematics in Practice and Theory, 11, 36 **(2006)**.
[8] B. Yang, “On an Extension of Hardy-Hilbert's Inequality with A Parameter”, Mathematics in Practice and Theory, 4, 36 **(2006)**.
[9] B. Yang, “On best extensions of Hardy-Hilbert's inequality with two parameters”, J. Inequal. Pure Appl. Math. 3, 6 **(2005)**.
[10] B. Yang and J. Zhu, “Inequality on the Hurwitz Zeta-function restricted to the axis of positive reals”, Acta Scientiarum Naturalium Universitatis Sunyatseni, 36.2, 36 **(1997)**.

Copyright ⓒ 2015 SERSC