International Journal of Control and Automation
Vol.8, No.11 (2015), pp.235-244
http://dx.doi.org/10.14257/ijca.2015.8.11.23

An Efficient Algorithm for Constructing all Magic Squares of Order Four

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Abstract

Like the Sudoku puzzle, the magic square involves recreational mathematical contexts and has attracted considerable attention. Although there are many researches available, few of them could provide an efficient solution for constructing all magic squares of order four. In this paper, we firstly formulate constraints of magic squares of order four by a collection of linear equations and provide an intuitive but computationally infeasible solution to these equations. Then, we propose an efficient algorithm for constructing all magic squares of order four. The algorithm transforms solving these equations into finding all possible permutations of seven free variables from sixteen consecutive natural integers based on the Gaussian Elimination method. Furthermore, we show the effectiveness of the proposed algorithm by 48 magic squares in the experimental section.

Keywords: *efficient algorithm, magic square, order four, permutations, Gaussian Elimination*

1. Introduction

As the ancestor of Sudoku, the magic square of order n is an n-by-n matrix of numbers, where the sum of numbers along each row, each column, the forward main diagonal and the backward main diagonal are the same constant μ which is also called the magic sum [1]. A magic square that contains integers from 1 to n^2 is called a classical magic square or natural magic square [3] with $\mu = n(n^2 + 1)/2$.

The research on magic squares has a long rich history. Involving both mathematical contexts and philosophical or religious contexts, magic squares have been firstly found in Chinese literature written about A. D. 1125 [5]. Despite a long rich history, magic squares are still the subject of research projects. For example, in paper [7], magic squares are combined with the historical research, whereas in [9, 11], magic squares are connected with the pure mathematical research, such as the algebraic and combinatorial geometry of polyhedral.

The purpose of the study is to construct all magic squares of order four. Although there are many researches available, few of them could provide an efficient solution for constructing all magic squares of order four. In this paper, we firstly introduce a collection of linear equations to formulate the desired problem. Then, we provide an efficient optimization method for all magic squares of order four by adopting the well

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known Gaussian Elimination algorithm to solve these equations. At last, we show the effectiveness of the algorithm by several experimental results.

2. Magic Squares of Order Four

The magic square of order four is a 4 by 4 matrix, which contains integers $\{1,2,\cdots,16\}$. The task for constructing all magic squares of order four is to find all possible permutations of integers $\{1,2,\cdots,16\}$, so that the magic sum of the square is $\mu = 4(4^2+1)/2 = 34$. Figure 1 shows the fourth order magic square puzzle, where Figure 1(a) shows ten constraints, i.e. all s1 to s10 should be 34, and Figure 1(b) shows such an example.

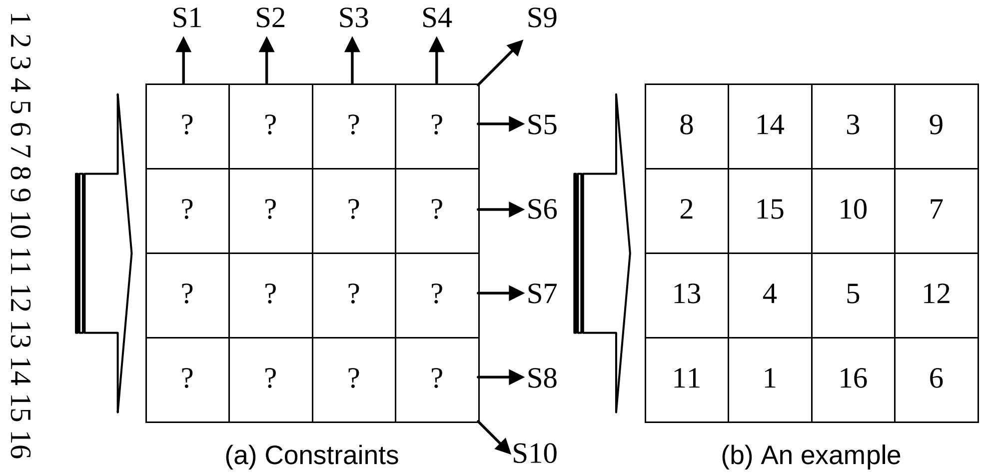

Figure 1. Fourth Order Magic Square Puzzle

Let $\{x_1, x_2, x_3, \cdots, x_{16}\}$ be a permutation from $\{1,2,\cdots,16\}$. The constructing process of all magic squares of order four, as shown in Figure 1, can be formulated in Figure 2.

x_1	x_2	x_3	x_4
x_5	x_6	x_7	x_8
x_9	x_{10}	x_{11}	x_{12}
x_{13}	x_{14}	x_{15}	x_{16}

Figure 2. Problem Formulation

Then, ten constraints of Figure 1(a) can be rewritten by ten equations, as

$$x_1 + x_2 + x_3 + x_4 = 34 \tag{1}$$

$$x_5 + x_6 + x_7 + x_8 = 34 \tag{2}$$

$$x_9 + x_{10} + x_{11} + x_{12} = 34 \tag{3}$$

$$x_{13} + x_{14} + x_{15} + x_{16} = 34 \tag{4}$$

$$x_1 + x_5 + x_9 + x_{13} = 34 \tag{5}$$
$$x_2 + x_6 + x_{10} + x_{14} = 34 \tag{6}$$
$$x_3 + x_7 + x_{11} + x_{15} = 34 \tag{7}$$
$$x_4 + x_8 + x_{12} + x_{16} = 34 \tag{8}$$
$$x_1 + x_6 + x_{11} + x_{16} = 34 \tag{9}$$
$$x_4 + x_7 + x_{10} + x_{13} = 34 \tag{10}$$

Accordingly, the task to construct all magic squares of order four is to find all possible solutions for above ten equations.

3. An Infeasible Solution

One intuitive solution to above equations is to use the exhaustive search method, as shown in Algorithm 1.

Algorithm 1. An exhaustive search method for all magic squares of order four

Input:
None.
Output:
Number of all magic squares: $number\_M\_S$;
Results of all magic squares: $results\_M\_S$.

Set $number\_M\_S = 0$ and $results\_M\_S = []$;
for each permutation $\{x_1, x_2, x_3, \cdots, x_{16}\}$ from $\{1, 2, 3, \cdots, 16\}$
  if Eq. (1-10) are all correct
    $M\_S = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix}$;
    $number\_M\_S = number\_M\_S + 1$;
    $results\_M\_S = [results\_M\_S; M\_S]$;
  end
end
return $number\_M\_S$ and $results\_M\_S$.

However, this is an infeasible solution. Because, for the major loop, the algorithm would compute 16! times, which is computationally expensive.

4. Methodology

In this section, we provide an efficient solution.

4.1. Gaussian Elimination

Firstly, we use the Gaussian Elimination method to reformulate Eq. (1-10). According to Eq. (5-8), we obtain

$$x_{13} = 34 - x_1 - x_5 - x_9 \tag{11}$$

$$x_{14} = 34 - x_2 - x_6 - x_{10} \tag{12}$$

$$x_{15} = 34 - x_3 - x_7 - x_{11} \tag{13}$$

$$x_{16} = 34 - x_4 - x_8 - x_{12} \tag{14}$$

According to Eq. (11-14), we have

$$x_{13} + x_{14} + x_{15} + x_{16} = 4 \times 34 - (x_1 + x_2 + \cdots + x_{12}) \tag{15}$$

Substitute Eq. (1-3) into Eq. (15), we get

$$x_{13} + x_{14} + x_{15} + x_{16} = 34$$

This means that Eq. (4) is redundant. Then, substitute Eq. (14) into Eq. (9) and Eq. (11) into Eq. (10), we acquire

$$x_{12} = x_1 + x_6 + x_{11} - x_4 - x_8 \tag{16}$$

$$x_{10} = x_1 + x_5 + x_9 - x_4 - x_7 \tag{17}$$

Substitute Eq. (16-17) into Eq. (3), we have

$$x_{11} = \frac{1}{2}(2x_4 + x_7 + x_8 - 2x_1 - x_5 - x_6 - 2x_9 + 34) \tag{18}$$

Furthermore, we can derive from Eq. (1) that

$$x_4 = 34 - x_1 - x_2 - x_3 \tag{19}$$

$$x_8 = 34 - x_5 - x_6 - x_7 \tag{20}$$

Thus, variables $\{x_4, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\ x_{16}\}$ can be determined by free variables $\{x_1, x_2, x_3, x_5, x_6, x_7, x_9\}$. Thereby, the task to construct all magic squares of order four is transformed into finding all possible permutations of $\{x_1, x_2, x_3, x_5, x_6, x_7, x_9\}$ from $\{1, 2, 3, \cdots, 16\}$.

4.2. Optimization Algorithm

Secondly, we provide an efficient optimization method for all magic squares of order four, as shown in Algorithm 2.

Algorithm 2. An efficient optimization method for all magic squares of order four

Input:
None.
Output:
Number of all magic squares: $number\_M\_S$;
Results of all magic squares: $results\_M\_S$.

```
1:  Set number_M_S = 0 and results_M_S = [];
2:  for each permutation {x1, x2, x3, x5, x6, x7, x9} from {1, 2, 3, ..., 16}
3:  |  Use Eq. (19) to compute x4;
4:  |  Use Eq. (20) to compute x8;
5:  |  Use Eq. (17) to compute x10;
6:  |  Use Eq. (18) to compute x11;
7:  |  Use Eq. (16) to compute x12;
8.  |  Use Eq. (11) to compute x13;
9:  |  Use Eq. (12) to compute x14;
10: |  Use Eq. (13) to compute x15;
11: |  Use Eq. (14) to compute x16;
12: |  M_S = [x1 x2 x3 x4; x5 x6 x7 x8; x9 x10 x11 x12; x13 x14 x15 x16];
13: |  uni_M_S = unique(M_S);
14: |  if sum(uni_M_S > 0) == 16 and sum(uni_M_S < 17) == 16
15: |  |  number_M_S = number_M_S + 1;
16: |  |  results_M_S = [results_M_S; M_S];
17: |  end
18: end
19: return number_M_S and results_M_S.
```

Line 12 in matrix form:

$$M\_S = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & x_{15} & x_{16} \end{bmatrix};$$

The loop from line 2 to line 18 implements P_{16}^{7} iterations for constructing all magic squares of order four by using the Gaussian Elimination method, where $unique(M\_S)$ returns a vector which contains the same value as in M_S but no repetitions, and $sum(\cdot)$ returns sums along different dimensions of an array. Compared to the Algorithm 1, the proposed Algorithm 2 is 362880 ($16!/P_{16}^{7} = 9!$) times faster, where we take the complexities inside major loops of the two algorithms to be simply all $O(1)$. So, the Algorithm 2 is very efficient.

5. Experiment

We develop the Algorithm 2 in Matlab R2010a. Besides, all experiments are implemented under the Windows 7 Operating System, 4 GB RAM and Intel® Core™ 2 Duo CPU. It takes about 45 minutes to get all the 7040 unique magic squares of order four, which is coincident with results in [13, 15, 17]. Due to the limited space, we show only the first 24 and the last 24 magic squares of order four in Figure 3 and Figure 4 separately.

1	2	15	16
12	14	3	5
13	7	10	4
8	11	6	9

0001

1	2	15	16
13	14	3	4
12	7	10	5
8	11	6	9

0002

1	2	16	15
13	14	4	3
12	7	9	6
8	11	5	10

0003

1	3	14	16
10	13	4	7
15	6	11	2
8	12	5	9

0004

1	3	14	16
12	13	4	5
15	8	9	2
6	10	7	11

0005

1	3	14	16
15	13	4	2
10	6	11	7
8	12	5	9

0006

1	3	14	16
15	13	4	2
12	8	9	5
6	10	7	11

0007

1	3	16	14
8	15	2	9
13	6	11	4
12	10	5	7

0008

1	3	16	14
12	15	2	5
13	10	7	4
8	6	9	11

0009

1	3	16	14
13	15	2	4
8	6	11	9
12	10	5	7

0010

1	3	16	14
13	15	2	4
12	10	7	5
8	6	9	11

0011

1	4	13	16
8	14	3	9
15	5	12	2
10	11	6	7

0012

1	4	13	16
8	15	2	9
14	5	12	3
11	10	7	6

0013

1	4	13	16
12	14	3	5
15	9	8	2
6	7	10	11

0014

1	4	13	16
12	15	2	5
14	9	8	3
7	6	11	10

0015

1	4	13	16
14	15	2	3
8	5	12	9
11	10	7	6

0016

1	4	13	16
14	15	2	3
12	9	8	5
7	6	11	10

0017

1	4	13	16
15	14	3	2
8	5	12	9
10	11	6	7

0018

1	4	13	16
15	14	3	2
12	9	8	5
6	7	10	11

0019

1	4	14	15
9	12	6	7
16	5	11	2
8	13	3	10

0020

1	4	14	15
13	16	2	3
8	5	11	10
12	9	7	6

0021

1	4	14	15
13	16	2	3
12	9	7	6
8	5	11	10

0022

1	4	14	15
16	11	5	2
9	6	12	7
8	13	3	10

0023

1	4	14	15
16	13	3	2
7	6	12	9
10	11	5	8

0024

Figure 3. First 24 Magic Squares of Order Four

16	13	3	2
1	4	14	15
10	11	5	8
7	6	12	9

7017

16	13	3	2
1	6	12	15
8	11	5	10
9	4	14	7

7018

16	13	3	2
4	1	15	14
5	8	10	11
9	12	6	7

7019

16	13	3	2
4	1	15	14
9	12	6	7
5	8	10	11

7020

16	13	3	2
8	5	11	10
1	12	6	15
9	4	14	7

7021

16	13	4	1
2	3	14	15
5	8	9	12
11	10	7	6

7022

16	13	4	1
2	3	14	15
9	12	5	8
7	6	11	10

7023

16	13	4	1
3	2	15	14
5	8	9	12
10	11	6	7

7024

16	13	4	1
3	2	15	14
9	12	5	8
6	7	10	11

7025

16	13	4	1
5	2	15	12
3	8	9	14
10	11	6	7

7026

16	13	4	1
5	3	14	12
2	8	9	15
11	10	7	6

7027

16	13	4	1
9	2	15	8
3	12	5	14
6	7	10	11

7028

16	13	4	1
9	3	14	8
2	12	5	15
7	6	11	10

7029

16	14	1	3
4	2	15	13
5	7	10	12
9	11	8	6

7030

16	14	1	3
4	2	15	13
9	11	6	8
5	7	12	10

7031

16	14	1	3
5	2	15	12
4	7	10	13
9	11	8	6

7032

16	14	1	3
9	2	15	8
4	11	6	13
5	7	12	10

7033

16	14	3	1
2	4	13	15
5	9	8	12
11	7	10	6

7034

16	14	3	1
2	4	13	15
7	11	6	10
9	5	12	8

7035

16	14	3	1
5	4	13	12
2	9	8	15
11	7	10	6

7036

16	14	3	1
7	4	13	10
2	11	6	15
9	5	12	8

7037

16	15	1	2
4	3	13	14
5	10	8	11
9	6	12	7

7038

16	15	2	1
4	3	14	13
5	10	7	12
9	6	11	8

7039

16	15	2	1
5	3	14	12
4	10	7	13
9	6	11	8

7040

Figure 4. Last 24 Magic Squares of Order Four

6. Conclusion

In this paper, we proposed an efficient algorithm, which is much faster than the infeasible exhaustive search method, for constructing all magic squares of order four by using the Gaussian Elimination method to solve a collection of formulated linear equations. In the experiment section, we show the first 24 magic squares and the last 24 magic squares of the algorithm. In the future, we would like to develop a similar efficient algorthm for constructing all magic squares of order five.

Acknowledgments

This paper was supported by the National Natural Science Foundation of China under Grant No. 61402106, the Natural Science Foundation of Guangdong Province, China under Grant No. 2014A030313632, and the Doctor Startup Foundation of Wuyi University under Grant No. 2014BS07. We would like to thank anonymous reviewers for helpful comments.

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