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Backstepping based Trajectory Tracking Control for a Quadrotor Aircraft with Nonlinear Disturbance Observer

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Abstract

In this paper, we investigate the trajectory tracking problem for the 6-DOF quadrotor aircraft with both internal uncertainties and external disturbances. The nonlinear system model is established based on Modified Rodrigues Parameters and the error model is then obtained based on the trajectory tracking task. To deal with the cascade property of the controlled objective, the system is divided into the translational and rotational subsystem, respectively. Hence, a hierarchical technique is introduced to implement the control system. Then, nonlinear disturbance observer based backstepping control strategy is proposed based on each subsystem. Global stability of the closed-loop system is analyzed based on Lyapunov theorem. Simulation results verify the effectiveness of the proposed control scheme.

Keywords: *Quadrotor aircraft; trajectory tracking; nonlinear disturbance observe; cascade system*

1. Introduction

In recent years, researches and applications of quadrotor aircraft have been widely concerned. Quadrotor is a kind of nonholonomic cascade system with complicated restrictions. From the necessity of Brockett, there is no gloss or time invariant controller that can stabilize the underactuated system to the equilibrium point[1]. Considering that the quadrotor is an underactuated system with four inputs and six outputs, it is very hard for the traditional control techniques to be used directly. Meanwhile, the inevitable system uncertainties, such as un-modeled dynamics, aerodynamic disturbances and parameters perturbation will affect the control accuracy or even make the control system unstable. Thus, the implementation of the control system for the quadrotor aircraft is a changing work.

Aiming at the problems mentioned above, several approaches have been proposed, such as sliding mode control [2-3], neutral networks [4], fuzzy systems [5], adaptive backstepping control [6]. However, the system model used in these researches is usually simplified based on Euler angles. This simplification is meaningful only if the aircraft rotates in one direction at a time and the pitch/roll angle changes while the roll/pitch angle is equal to zero degrees. Thus, the usage of controller designed based on simplified model is limited, especially for the trajectory tracking problem. Meanwhile, we need to face the issues, for example, the system uncertainties, chattering of sliding mode control, convergence rate of weights in neural networks and fuzzy systems, Consequently, advanced control methodology should be explored for the quadrotor aircraft.

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In this paper, we focus on the trajectory tracking control problem of the 6-DOF quadrotor aircraft with system uncertainties. Firstly, the Modified Rodrigues Parameters (MRPs) is applied to represent the attitude and thus, non-redundant error model is established based on the trajectory tracking problem. Then, the system error model is divided into two parts, translational subsystem and rotational subsystem, based on which the hierarchical control structure is applied to implement. The backstepping control scheme combine with nonlinear disturbance observer (NDOB) is proposed for each subsystem. The NDOB is used to compensate for the system uncertainties, whereas the backstepping controller is applied for desired tracking performance. At last, according to the Rodrigues formula and the character of the MRPs, it's proved that the tracking errors of the aircraft are bounded through global stability analysis.

This paper is organized as follows. In Section 2, the system model of quadrotor aircraft is expressed based on MRPs. In Section 3, the trajectory tracking problem is descripted to establish the system error model, and the hierarchical control structure is applied based on the cascade property. In Section 4, the controller is proposed and stability is analyzed strictly based on Lyapunov theorem. Simulation in Section 5 is carried out to show the effectiveness of the proposed method, followed by the Conclusions in Section 6.

2. System Model

The quadrotor aircraft, as depicted in Figure 1, is a kind of aircrafts with the appearance of dish. It consists of four independent motor diver systems which are binded on a rigid criss-cross structure. The four rotors locate at the tips of the rigid body, and the directions of rotation of the diagonal rotors are clockwise and counterclockwise, respectively. The rates of rotation of the rotors are identical during hovering. By varying the rotor speed, one can change the lift force and create motion. Increasing or decreasing the four propellers' speeds together generates vertical motion. Changing the speeds of two propellers which are diagonal conversely produces pitch/roll motion. Yaw motion results from the difference in the counter-torque between each pair of propellers, which is caused by changing the speeds of two pair of propellers conversely.

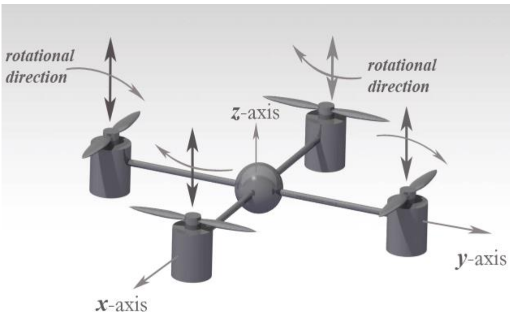

Figure 1. The Principle of the Quadrotor UAV

We define an earth frame, whose x-axis points northward, y-axis eastward and z-axis downward, namely a North-East-Down frame. Then the system model is as follows:

$$\begin{cases}\dot{\boldsymbol{\xi}}=\boldsymbol{v}\\ m\dot{\boldsymbol{v}}=\boldsymbol{e}_3 mg-R\boldsymbol{e}_3 T\\ \dot{\boldsymbol{\sigma}}=G(\boldsymbol{\sigma})\boldsymbol{\omega}\\ J\dot{\boldsymbol{\omega}}=-\boldsymbol{\omega}\times J\boldsymbol{\omega}+\boldsymbol{\tau}\end{cases} \tag{1}$$

where $\boldsymbol{\xi},\boldsymbol{v}\in\mathbf{R}^3$ are the position and velocity vectors under the earth frame. $\boldsymbol{\omega}\in\mathbf{R}^3$ is the angle velocity of the rigid body. $R\in\mathbf{R}^{3\times3}$ is attitude transition matrix. $\boldsymbol{e}_3=\begin{bmatrix}0 & 0 & 1\end{bmatrix}^{\mathrm{T}}$ is a unit vector. $\boldsymbol{\sigma}$ is the MRPs which is defined as $\boldsymbol{\sigma}=\boldsymbol{r}\tan\alpha/4$. $\boldsymbol{r}$ and α represent a rotation axis that does not coincide with one of the original coordinate axes and angle about the axis, respectively. m and $J\in\mathbf{R}^{3\times3}$ are the mass and inertia matrix of the rigid body, where J is considered to be a diagonal, positive definite matrix. J_x, J_y, J_z are moments of inertia about the their coordinate axes. g is the gravity acceleration. T and $\boldsymbol{\tau}\in\mathbf{R}^3$ are the throttle and moment on the rigid body. The expressions of $R(\boldsymbol{\sigma})$ and $G(\boldsymbol{\sigma})$ are given in [7].

Assume that there is no hysteresis in the motors, such that the motor speed can achieve desired speed immediately. Hence, the trajectory tracking control of the quadrotor aircraft equals to design the controller T and $\boldsymbol{\tau}$ to stabilize the system.

3. Problem Formulation

The trajectory tracking problem of the quadrotor aircraft is considered in this work. The control objective is to design the control thrust T_d and $\boldsymbol{\tau}_d$ to stabilize the quadrotor to a desired trajectory quickly and accurately. The vector $\begin{bmatrix}\boldsymbol{x}_{\mathrm{d}} & \dot{\boldsymbol{x}}_{\mathrm{d}} & \ddot{\boldsymbol{x}}_{\mathrm{d}}\end{bmatrix}$ is the desired trajectory which contains position, velocity and acceleration. The desired acceleration $\ddot{\boldsymbol{x}}_{\mathrm{d}}$ satisfies the following **Assumption 1**.

Assumption 1. The desired acceleration is bounded with $\|\ddot{\boldsymbol{x}}_{\mathrm{d}}\|\le\nu$, where ν is a positive constant.

Define the system errors as follows: $\tilde{\boldsymbol{\xi}}=\boldsymbol{\xi}-\boldsymbol{x}_{\mathrm{d}}$ is position error, $\tilde{\boldsymbol{v}}=\boldsymbol{v}-\dot{\boldsymbol{x}}_{\mathrm{d}}$ is velocity error, $\tilde{R}=RR_{\mathrm{d}}^{\mathrm{T}}$ is the error of attitude matrix, $\tilde{\boldsymbol{\sigma}}$ is the error of MRPs, $\tilde{\boldsymbol{\omega}}=\boldsymbol{\omega}-\tilde{R}\boldsymbol{\omega}_{\mathrm{d}}$ is the error of angle velocity. Considering the parameters perturbation as: $m=m_0+\Delta m$, $J=J_0+\Delta J$. Then the system error model based on trajectory tracking problem is given as:

$$\begin{cases}\dot{\tilde{\boldsymbol{\xi}}}=\tilde{\boldsymbol{v}}\\ \dot{\tilde{\boldsymbol{v}}}=\boldsymbol{e}_3 g-\dfrac{1}{m_0}\left(R\boldsymbol{e}_3 T_{\mathrm{d}}+d_1'\right)-\ddot{\boldsymbol{x}}_{\mathrm{d}}\\ \dot{\tilde{\boldsymbol{\sigma}}}=G(\tilde{\boldsymbol{\sigma}})\tilde{\boldsymbol{\omega}}\\ \dot{\tilde{\boldsymbol{\omega}}}=J_0^{-1}\left(-L(\boldsymbol{\omega})J_0^*+\boldsymbol{\tau}_{\mathrm{d}}+d_2'\right)-\left(\tilde{R}\dot{\boldsymbol{\omega}}_{\mathrm{d}}-[\tilde{\boldsymbol{\omega}}\times]\tilde{R}\boldsymbol{\omega}_{\mathrm{d}}\right)\end{cases} \tag{2}$$

where d_1' and d_2' are the composite disturbances that contains both parameters perturbation and external disturbances that satisfies **Assumption 2**. Operator $L(\cdot)$ and $[\cdot\times]$ are defined as:

$$L(\boldsymbol{\omega})=\begin{bmatrix}0 & \omega_2\omega_3 & -\omega_2\omega_3\\ -\omega_3\omega_1 & 0 & \omega_3\omega_1\\ \omega_1\omega_2 & -\omega_1\omega_2 & 0\end{bmatrix},[\boldsymbol{\omega}\times]=\begin{bmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}.$$

Assumption 2. The first order derivative of composite disturbances d_1' and d_2' are bounded as $\|\dot{d}_1'\| \le \bar{d}_1'$, $\|\dot{d}_2'\| \le \bar{d}_2'$.

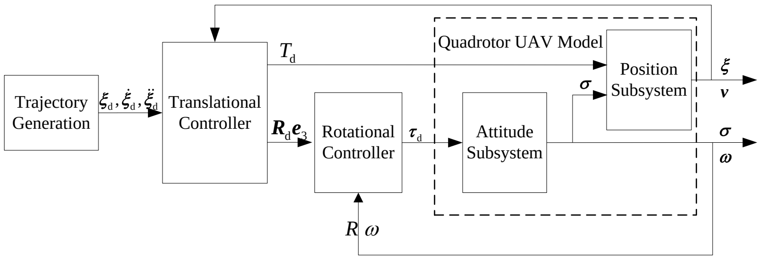

Figure 2. Hierarchical Control Structure

From the principle of the quadrotor aircraft, we know that the position error will not converge until the attitude errors converge. Here, we divided the whole system into translational and rotational subsystems, respectively. Then, a practical hierarchical strategy is applied to implement the control system. The hierarchical control structure is shown in Figure 2. Consequently, translational and rotational controllers can be designed separately. The translational controller is firstly proposed to get the desired thrust T_d and attitude transition matrix R_d. The desired attitude transition matrix R_d is applied for the input of the rotational subsystem. Thereafter, the desired torque $\boldsymbol{\tau}_d$ can be acquired based on rotational controller.

The two subsystems in given as:

$$\begin{cases} \dot{\tilde{\boldsymbol{\xi}}} = \tilde{\boldsymbol{v}} \\ \dot{\tilde{\boldsymbol{v}}} = \boldsymbol{e}_3 g - \dfrac{1}{m_0}\left(R_\mathrm{d}\boldsymbol{e}_3 T_\mathrm{d} + d_1'\right) - \ddot{\boldsymbol{x}}_\mathrm{d} \end{cases} \tag{3}$$

$$\begin{cases} \dot{\tilde{\boldsymbol{\sigma}}} = G\left(\tilde{\boldsymbol{\sigma}}\right)\tilde{\boldsymbol{\omega}} \\ \dot{\tilde{\boldsymbol{\omega}}} = J_0^{-1}\left(-L(\boldsymbol{\omega}) J_0^* + \boldsymbol{\tau}_\mathrm{d} + d_2'\right) - \left(\tilde{R}\dot{\boldsymbol{\omega}}_\mathrm{d} - [\tilde{\boldsymbol{\omega}}\times]\tilde{R}\boldsymbol{\omega}_\mathrm{d}\right) \end{cases} \tag{4}$$

Notice that in the controller design procedure, the translational controller is designed under the assumption that the rotational subsystem is already convergent. However, the stability should be analyzed based on the original closed-loop control system, and this assumption is only used in the procedure of controller design.

4. Controller Design and Stability Analysis

In this section, we first design the controller based on backstepping technique combined with NDOB. Then, the Lyapunov theorem is applied to analyze the global stability of the closed-loop system according to some previous researches.

4.1. Controller Design

STEP 1. Controller design for the translational subsystem.

The purpose of controller design of the position error nominal subsystem is to acquire the controller T_d and virtual controller R_d, which make the states of translational subsystem uniformly ultimately bounded. First, a NDOB is applied to estimate the composite disturbances d_1', the NDOB is expressed as:

$$\begin{cases} \hat{d}_1' = \rho_1 \tilde{\boldsymbol{v}} + \boldsymbol{z}_1 \\ \dot{\boldsymbol{z}}_1 = -\rho_1 \left[\boldsymbol{e}_3 g - \frac{1}{m_0} \left(R\boldsymbol{e}_3 T_{\mathrm{d}} + \hat{d}_1' \right) - \ddot{\boldsymbol{x}}_{\mathrm{d}} \right] \end{cases} \tag{5}$$

Define the estimation error of the NDOB as $\tilde{d}_1' = \hat{d}_1' - d_1'$, and we get $\dot{\tilde{d}}_1' = -\rho_1 \tilde{d}_1' - \dot{d}_1'$ from Eq. (5). From **Assumption 2**, we find that the estimation error of the NDOB is bounded as: $\|\tilde{d}_1'\| \le \bar{d}_1'/\rho_1$. It is clear that the gain of NDOB ρ_1 determines the estimation accuracy of the observer. If the gain of NDOB is selected arbitrary large, the estimation error can converge to 0. However, in the practical system, the high gain of NDOB may lead the system less robustness or even unstable. We must make a tradeoff between the robustness and performance in a practical system.

By compensating the estimation of NDOB in the control system, the backstepping controller can thus be proposed. We first design the error between the system states and virtual controller as:

$$\begin{cases} \boldsymbol{\varepsilon}_1 = \tilde{\boldsymbol{\xi}} \\ \boldsymbol{\varepsilon}_2 = \tilde{\boldsymbol{v}} - \alpha_1 (\boldsymbol{\varepsilon}_1) \end{cases} \tag{6}$$

where $\alpha_1(\cdot)$ is the virtual controller to be designed. By defining a candidate Lyapunov function $V_1 = \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_1 / 2$, and choose the virtual controller as: $\alpha_1(\boldsymbol{\varepsilon}_1) = -k_1 \boldsymbol{\varepsilon}_1$, k_1 is a positive constant. Then we have:

$$\dot{V}_1 = \boldsymbol{\varepsilon}_1^{\mathrm{T}} \dot{\boldsymbol{\varepsilon}}_1 = \boldsymbol{\varepsilon}_1^{\mathrm{T}} \left(\alpha_1(\boldsymbol{\varepsilon}_1) + \boldsymbol{\varepsilon}_2 \right) = -k_1 \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_2 \tag{7}$$

It is clear that if $\boldsymbol{\varepsilon}_2 = 0$, then the error $\boldsymbol{\varepsilon}_1$ is exponentially stable.

Define the second candidate Lyapunov function $V_2 = \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_1 / 2 + \boldsymbol{\varepsilon}_2^{\mathrm{T}} \boldsymbol{\varepsilon}_2 / 2$, its first order derivative is given as:

$$\dot{V}_2 = -k_1 \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_2 + \boldsymbol{\varepsilon}_2^{\mathrm{T}} \left(\boldsymbol{e}_3 g - \frac{1}{m_0} \left(R_{\mathrm{d}} \boldsymbol{e}_3 T_{\mathrm{d}} + d_1' \right) - \ddot{\boldsymbol{x}}_{\mathrm{d}} + k_1 \left(\boldsymbol{\varepsilon}_2 - k_1 \boldsymbol{\varepsilon}_1 \right) \right) \tag{8}$$

Consider that $\|R_{\mathrm{d}} \boldsymbol{e}_3\| = 1$, we get the virtual controller R_{d} and controller T_{d} as:

$$\begin{cases} T_{\mathrm{d}} = \left\| m_0 \left[\boldsymbol{e}_3 g - \ddot{\boldsymbol{x}}_{\mathrm{d}} + k_1 \left(\boldsymbol{\varepsilon}_2 - k_1 \boldsymbol{\varepsilon}_1 \right) \right] - \hat{d}_1' + \boldsymbol{\varepsilon}_1 + k_2 \boldsymbol{\varepsilon}_1 \right\| \\ R_{\mathrm{d}} \boldsymbol{e}_3 = \dfrac{m_0 \left[\boldsymbol{e}_3 g - \ddot{\boldsymbol{x}}_{\mathrm{d}} + k_1 \left(\boldsymbol{\varepsilon}_2 - k_1 \boldsymbol{\varepsilon}_1 \right) \right] - \hat{d}_1' + \boldsymbol{\varepsilon}_1 + k_2 \boldsymbol{\varepsilon}_1}{T_{\mathrm{d}}} \end{cases} \tag{9}$$

where k_2 is a positive constant.

By substituting Eq. (9) into Eq. (8), we finally have:

$$\dot{V}_2 = -k_1 \boldsymbol{\varepsilon}_1^{\mathrm{T}} \boldsymbol{\varepsilon}_1 - k_2 \boldsymbol{\varepsilon}_2^{\mathrm{T}} \boldsymbol{\varepsilon}_2 - \frac{1}{m_0} \boldsymbol{\varepsilon}_2^{\mathrm{T}} \tilde{d}_1' \tag{10}$$

STEP2. Desired attitude information extraction.

In Eq. (9), $R_{\mathrm{d}} \boldsymbol{e}_3$ contains the desired attitude information of the control system. However, it cannot be used directly in the controller design for rotational subsystem. We should extract the desired MPRs according to $R_{\mathrm{d}} \boldsymbol{e}_3$. It is clear that:

$$R_{\mathrm{d}}\boldsymbol{e}_3 T_{\mathrm{d}} = \frac{T_{\mathrm{d}}}{\left(1+\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)^2}\begin{bmatrix} 8\boldsymbol{\sigma}_{\mathrm{d}1}\boldsymbol{\sigma}_{\mathrm{d}3} - 4\boldsymbol{\sigma}_{\mathrm{d}2}\left(1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right) \\ 8\boldsymbol{\sigma}_{\mathrm{d}2}\boldsymbol{\sigma}_{\mathrm{d}3} - 4\boldsymbol{\sigma}_{\mathrm{d}1}\left(1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right) \\ 4\left(-\boldsymbol{\sigma}_{\mathrm{d}1}^2 - \boldsymbol{\sigma}_{\mathrm{d}2}^2 + \boldsymbol{\sigma}_{\mathrm{d}3}^2\right) + \left(1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)^2 \end{bmatrix} \tag{11}$$

We define $\boldsymbol{\delta} =: R_{\mathrm{d}}\boldsymbol{e}_3 T_{\mathrm{d}}$. Notice that $\|R_{\mathrm{d}}\boldsymbol{e}_3\| = 1$, that is $R_{\mathrm{d}}\boldsymbol{e}_3$ only have two degrees of freedom. Here, we combine the desired attitude information with $\boldsymbol{\sigma}_{\mathrm{d}3} = 0$, that is we assume the orientation of the quadrotor aircraft keeps the original direction. Then we define:

$$\eta_{\mathrm{d}} =: \frac{1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}}{1+\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}} = \sqrt{\frac{\delta_3}{2T_{\mathrm{d}}} + \frac{1}{2}} \tag{12}$$

It can be obtained that

$$\begin{cases} \delta_1 = \dfrac{-4T_{\mathrm{d}}\boldsymbol{\sigma}_{\mathrm{d}2}\left(1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)}{\left(1+\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)^2} \\ \delta_2 = \dfrac{4T_{\mathrm{d}}\boldsymbol{\sigma}_{\mathrm{d}1}\left(1-\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)}{\left(1+\boldsymbol{\sigma}_{\mathrm{d}}^{\mathrm{T}}\boldsymbol{\sigma}_{\mathrm{d}}\right)^2} \end{cases} \tag{13}$$

Consequently, we can get the desired MRPs as

$$\boldsymbol{\sigma}_{\mathrm{d}} = \frac{1}{2T_{\mathrm{d}}\eta_{\mathrm{d}}\left(1+\eta_{\mathrm{d}}\right)}\begin{bmatrix} \delta_2 \\ -\delta_1 \\ 0 \end{bmatrix} \tag{14}$$

The derivative information of $\boldsymbol{\sigma}_{\mathrm{d}}$ can be acquired by the tracking differentiator.

STEP3. Controller design for the rotational subsystem.

The purpose of controller design of the position error nominal subsystem is to acquire the torque controller $\boldsymbol{\tau}_{\mathrm{d}}$, which make the states of rotational subsystem uniformly ultimately bounded.

Similar with **STEP 1**, a NDOB is applied to estimate the composite disturbances d_2', the NDOB is expressed as:

$$\begin{cases} \hat{d}_2' = \rho_2\tilde{\boldsymbol{\omega}} + \boldsymbol{z}_2 \\ \dot{\boldsymbol{z}}_2 = -\rho_1\left[J_0^{-1}\left(-L(\boldsymbol{\omega})J_0^* + \boldsymbol{\tau}_{\mathrm{d}} + \hat{d}_2'\right) - \left(\tilde{R}\dot{\boldsymbol{\omega}}_{\mathrm{d}} - [\tilde{\boldsymbol{\omega}}\times]\tilde{R}\boldsymbol{\omega}_{\mathrm{d}}\right)\right] \end{cases} \tag{15}$$

Define the estimation error of the NDOB as $\tilde{d}_2' = \hat{d}_2' - d_2'$, and we get $\dot{\tilde{d}}_2' = -\rho_2\tilde{d}_2' - \dot{d}_2'$ according to Eq. (15). From **Assumption 2**, we find that the estimation error of the NDOB is bounded as: $\|\tilde{d}_2'\| \le \bar{d}_2'/\rho_2$.

By compensating the estimation of NDOB in the control system, the backstepping controller can thus be proposed. The error between the system states and virtual controller are defined as:

$$\begin{cases} \boldsymbol{\varepsilon}_3 = \tilde{\boldsymbol{\sigma}} \\ \boldsymbol{\varepsilon}_4 = \tilde{\boldsymbol{\omega}} - \alpha_2\left(\boldsymbol{\varepsilon}_3\right) \end{cases} \tag{16}$$

where $\alpha_2(\cdot)$ is the virtual controller to be designed. By defining a candidate Lyapunov function $V_3 = 2\ln\left(1+\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3\right)$, and choose the virtual controller as: $\alpha_2(\boldsymbol{\varepsilon}_3) = -k_3\boldsymbol{\varepsilon}_3$, k_2 is a positive constant. Notice that

$$\boldsymbol{\sigma}^{\mathrm{T}}G(\boldsymbol{\sigma}) = \frac{\left(1+\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\sigma}\right)}{4}\boldsymbol{\sigma}^{\mathrm{T}} \tag{17}$$

Then we have:

$$\dot{V}_3 = \boldsymbol{\varepsilon}_3^{\mathrm{T}}\left(\alpha_2(\boldsymbol{\varepsilon}_3)+\boldsymbol{\varepsilon}_4\right) = -k_3\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3 + \boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_4 \tag{18}$$

It is clear that if $\boldsymbol{\varepsilon}_4 = 0$, then the error $\boldsymbol{\varepsilon}_3$ is exponentially stable.

Define the second candidate Lyapunov function $V_4 = 2\ln\left(1+\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3\right)+\boldsymbol{\varepsilon}_4^{\mathrm{T}}\boldsymbol{\varepsilon}_4/2$, its first order derivative is given as:

$$\begin{aligned}\dot{V}_4 = -k_3\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3 + \boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_4 + \boldsymbol{\varepsilon}_4^{\mathrm{T}}\Big[J_0^{-1}\left(-L(\boldsymbol{\omega})J_0^{*} + \boldsymbol{\tau}_{\mathrm{d}} + d_2'\right) \\ -\left(\tilde{R}\dot{\boldsymbol{\omega}}_{\mathrm{d}} - [\tilde{\boldsymbol{\omega}}\times]\tilde{R}\boldsymbol{\omega}_{\mathrm{d}}\right) + k_3 G(\boldsymbol{\varepsilon}_3)(\boldsymbol{\varepsilon}_4 - k_3\boldsymbol{\varepsilon}_3)\Big]\end{aligned} \tag{19}$$

We get the control torque $\boldsymbol{\tau}_{\mathrm{d}}$ as:

$$\boldsymbol{\tau}_{\mathrm{d}} = -\hat{d}_2' + L(\boldsymbol{\omega})J_0^{*} + J\Big[{}_0\left(\tilde{R}\dot{\boldsymbol{\omega}}_{\mathrm{d}} - [\tilde{\boldsymbol{\omega}}\times]\tilde{R}\boldsymbol{\omega}_{\mathrm{d}}\right) - k_3 G(\boldsymbol{\varepsilon}_3)(\boldsymbol{\varepsilon}_4 - k_3\boldsymbol{\varepsilon}_3) - \boldsymbol{\varepsilon}_3 - k_4\boldsymbol{\varepsilon}_4\Big] \tag{20}$$

where k_4 is a positive constant.

By substituting Eq. (20) into Eq. (19), we finally have:

$$\dot{V}_4 = -k_3\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3 - k_4\boldsymbol{\varepsilon}_4^{\mathrm{T}}\boldsymbol{\varepsilon}_4 - \boldsymbol{\varepsilon}_4^{\mathrm{T}}J_0^{-1}\tilde{d}_2' \tag{21}$$

4.1. Stability Analysis

Lemma 1[8] Consider the quadrotor nonlinear error model in Eq. (2). The closed-loop system is globally uniformly bounded if the following three assumptions are satisfied.

A1. The virtual controller R_d and controller T_{d} stabilize the translational subsystem in Eq. (3) asymptotically with a positive definite Lyapunov function V_{trans} such that

$$\dot{V}_{trans}(\boldsymbol{x}_{trans}) \le -a_1\|\boldsymbol{x}_{trans}\|^2 + \mu_1 \tag{22}$$

$$\frac{\partial V_{trans}}{\partial \boldsymbol{x}_{trans}} \le -a_2\|\boldsymbol{x}_{trans}\| \tag{23}$$

where $\boldsymbol{x}_{trans}$ is the generalized system state.

A2. The controller $\boldsymbol{\tau}_{\mathrm{d}}$ can stabilize the rotational subsystem in Eq. (4) asymptotically with a positive definite Lyapunov function V_{trans} such that

$$\dot{V}_{rot}(\boldsymbol{x}_{rot}) \le -a_3\|\boldsymbol{x}_{rot}\|^2 + \mu_2 \tag{24}$$

where $\boldsymbol{x}_{rot}$ is the generalized system state.

A3. The control thrust T_{d} satisfies

$$T_{\mathrm{d}} \le \left(\kappa_1 + \kappa_2\|\boldsymbol{x}_{trans}\|\right) \tag{25}$$

This lemma shows that if the rotational subsystem has an asymptotical property and the thrust controller fulfills the bounded condition, the coupling property between the translational and rotational dynamics will not separate the system state globally.

Theorem 1. For the quadrotor nonlinear error model in Eq. (2).With the controller in Eq. (9) and (20), NDOB in Eq. (5) and (15), the closed-loop system is globally uniformly bounded.

Proof.

For the translational subsystem, we design the generalized system states as: $\boldsymbol{x}_{trans} = \left[\boldsymbol{\varepsilon}_1^{\mathrm{T}} \quad \boldsymbol{\varepsilon}_2^{\mathrm{T}} \quad \tilde{d}_1'^{\mathrm{T}}\right]^{\mathrm{T}}$. By defining the Lyapunov function as:

$$V_{trans} = \frac{1}{2}\boldsymbol{\varepsilon}_1^{\mathrm{T}}\boldsymbol{\varepsilon}_1 + \frac{1}{2}\boldsymbol{\varepsilon}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_2 + \frac{1}{2}\tilde{d}_1'^{\mathrm{T}}\tilde{d}_1' \tag{26}$$

Then, from Eq. (5) and (10), the derivative of V_{trans} is given as:

$$\begin{aligned}\dot{V}_{trans} &= -k_1\boldsymbol{\varepsilon}_1^{\mathrm{T}}\boldsymbol{\varepsilon}_1 - k_2\boldsymbol{\varepsilon}_2^{\mathrm{T}}\boldsymbol{\varepsilon}_2 - \frac{1}{m_0}\boldsymbol{\varepsilon}_2^{\mathrm{T}}\tilde{d}_1' - \rho_1\tilde{d}_1'^{\mathrm{T}}\tilde{d}_1' - \tilde{d}_1'^{\mathrm{T}}\dot{d}_1' \\ &\le -k_1\left\|\boldsymbol{\varepsilon}_1\right\|^2 - \frac{k_2}{2}\left\|\boldsymbol{\varepsilon}_2\right\|^2 - \left(\frac{\rho_1}{2} - \frac{1}{2m_0^2k_2}\right)\left\|\tilde{d}_1'\right\|^2 + \frac{\bar{d}_1'^2}{2\rho_1}\end{aligned} \tag{27}$$

We can also get the following Eq. as:

$$\frac{\partial V_{trans}}{\partial \boldsymbol{x}_{trans}} \le -\frac{1}{2}\left\|\boldsymbol{x}_{trans}\right\| \tag{28}$$

Thus the assumption **A1** of **Lemma 1** is satisfied.

For the rotational subsystem, we design the generalized system states as: $\boldsymbol{x}_{rot} = \left[\boldsymbol{\varepsilon}_3^{\mathrm{T}} \quad \boldsymbol{\varepsilon}_4^{\mathrm{T}} \quad \tilde{d}_2'^{\mathrm{T}}\right]^{\mathrm{T}}$. By defining the Lyapunov function as:

$$V_{rot} = 2\ln\left(1+\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3\right) + \frac{1}{2}\boldsymbol{\varepsilon}_4^{\mathrm{T}}\boldsymbol{\varepsilon}_4 + \frac{1}{2}\tilde{d}_2'^{\mathrm{T}}\tilde{d}_2' \tag{29}$$

Then, from Eq. (15) and (21), the derivative of V_{rot} is given as:

$$\begin{aligned}\dot{V}_{rot} &= -k_3\boldsymbol{\varepsilon}_3^{\mathrm{T}}\boldsymbol{\varepsilon}_3 - k_4\boldsymbol{\varepsilon}_4^{\mathrm{T}}\boldsymbol{\varepsilon}_4 - \boldsymbol{\varepsilon}_4^{\mathrm{T}}J_0^{-1}\tilde{d}_2' - \rho_2\tilde{d}_2'^{\mathrm{T}}\tilde{d}_2' - \tilde{d}_2'^{\mathrm{T}}\dot{d}_2' \\ &\le -k_3\left\|\boldsymbol{\varepsilon}_3\right\|^2 - \frac{k_4}{2}\left\|\boldsymbol{\varepsilon}_4\right\|^2 - \left(\frac{\rho_2}{2} - \frac{1}{2\left[\lambda_{\max}\left\{J_0^{-1}\right\}\right]^2 k_2}\right)\left\|\tilde{d}_1'\right\|^2 + \frac{\bar{d}_2'^2}{2\rho_2}\end{aligned} \tag{30}$$

Then we get the assumption **A2** in **Lemma 1** is satisfied.

From Eq. (9), we get

$$\begin{aligned}T_{\mathrm{d}} &= \left\|m_0\left[\boldsymbol{e}_3 g - \ddot{\boldsymbol{x}}_{\mathrm{d}} + k_1\left(\boldsymbol{\varepsilon}_2 - k_1\boldsymbol{\varepsilon}_1\right)\right] - \hat{d}_1' + \boldsymbol{\varepsilon}_1 + k_2\boldsymbol{\varepsilon}_1\right\| \\ &= \left\|m_0\left(\boldsymbol{e}_3 g - \ddot{\boldsymbol{x}}_{\mathrm{d}}\right) + \left(1 - m_0k_1^2\right)\boldsymbol{\varepsilon}_1 + \left(k_1 + m_0k_2\right)\boldsymbol{\varepsilon}_2 - \hat{d}_1'\right\|\end{aligned} \tag{31}$$

It is clear that for the parameters: $\kappa_1 = m_0\left(\boldsymbol{e}_3 g - \ddot{\boldsymbol{x}}_{\mathrm{d}}\right)$ and $\kappa_2 = \max\left\{\left(1-m_0k_1^2\right), \left(k_1+m_0k_2\right), 1\right\}$, the control thrust satisfies assumption **A3** in **Lemma 1**.

Since assumption **A1**, **A2** and **A3** are satisfied, we can come to a conclusion that the closed-loop system is globally uniformly bounded according to **Lemma 1**.

5. Simulations and Analysis

Simulations are shown to illustrate the effectiveness of the proposed control scheme in this paper. The numerical simulation is carried out in MATLAB/SIMULINK. Parameters of the quadrotor aircraft is shown in Table 1, where ρ is the density of air, r is the blade radius of the airscrew, $A = \pi r^2$ is the area of the radius. C_T and C_Q denote the coefficients of aerodynamic force and moment, respectively.

Table 1. The Simulation Parameters of the Quadrotor Aircraft

variables	values	units
m	0.8	Kg
J_x	0.25	$\mathrm{Kg}\cdot\mathrm{m}^2$
J_y	0.25	$\mathrm{Kg}\cdot\mathrm{m}^2$
J_z	0.05	$\mathrm{Kg}\cdot\mathrm{m}^2$
C_T	0.0047	
C_Q	0.000228	
ρ	1.184	$\mathrm{Kg}\cdot\mathrm{m}^{-3}$
A	0.071	m^2

The aerodynamic force and moment generated from the i 'th rotor are

$$\begin{cases} F_i = C_T \rho A r^2 \omega_i^2 \\ M_i = C_Q \rho A r^3 \omega_i \left|\omega_i\right| \end{cases} \tag{32}$$

Assume that the initial position of the aircraft is $\left[0.5 \quad 0.5 \quad 0\right]^{\mathrm{T}} m$. The desired trajectory expressed as:

$$\left[0.5t \quad 5\sin\left(\pi t/25\right) \quad 5\cos\left(\pi t/25\right)-2\right]^{\mathrm{T}} m \tag{33}$$

Assume that the external disturbances acting on the translational and rotational dynamics are as follows

$$D_1 = \begin{bmatrix} 0.2\sin\left(\pi t/10\right)+1.5\sin\left(\pi t/10\right) \\ 0.1\sin\left(\pi t/10\right)+\sin\left(\pi t/10\right) \\ 0.1\sin\left(\pi t/10\right)+2.5\sin\left(\pi t/10\right) \end{bmatrix} \times 10^{-1} N \tag{34}$$

$$D_2 = \begin{bmatrix} 0.1\sin\left(\pi t\right)+0..1\sin\left(\pi t/10\right) \\ 0.1\sin\left(\pi t\right)+0.1\sin\left(\pi t/10\right) \\ 0.1\sin\left(\pi t\right)+0.1\sin\left(\pi t/10\right) \end{bmatrix} \times 10^{-2} N\cdot m \tag{35}$$

Simulation results are shown in Figure 3 to 8. Fig 3shows the trajectory tracking effect of the quadrotor aircraft. The estimation effect of NDOB is expressed in Figure 4 and 5. The tracking error of position, velocity, MPRs and angular velocity are shown in Figure 6 to 9.

It is shown in Figure 3 that the proposed controller can enable the quadrotor aircraft to track a time-varying trajectory quickly and accurately with the existence of external disturbances and internal uncertainties. We also carry out the adaptive control method in [9] for comparison. The simulations in [9] show that the adaptive law can suppress the constant disturbances effectively. However, in our simulations, if there exists time-varying disturbances and internal uncertainties, the control performance of the adaptive law is not as well as the proposed control scheme in this paper. It is shown in Figure 4 and 5 that the NDOB can estimate the composite disturbances accurately. Figures 6 to 9 also show that the proposed control strategy has good tracking performance.

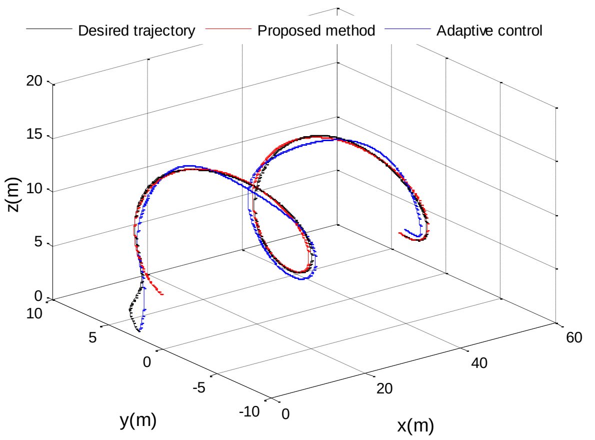

Figure 3. Trajectory Tracking Effect

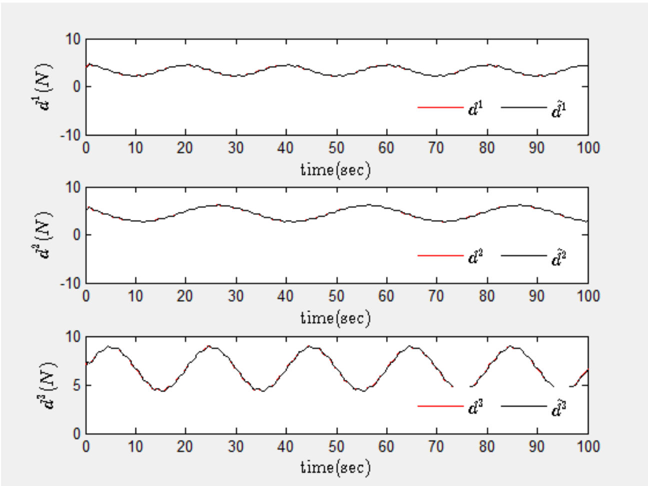

Figure 4. Disturbance Estimation Effect of the Translational Dynamics

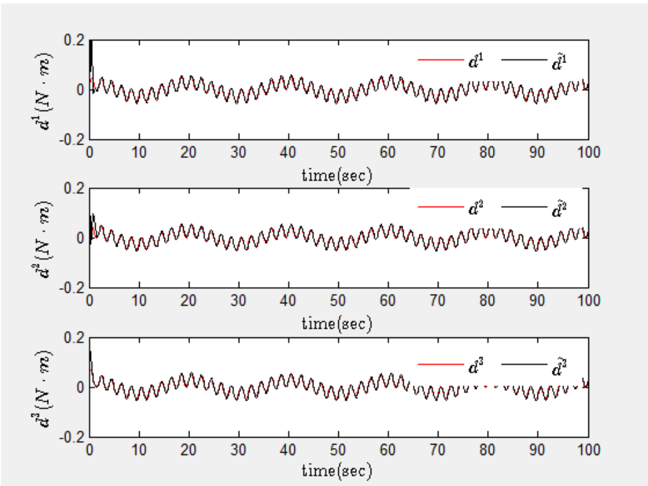

Figure 5. Disturbance Estimation Effect of the Rotational Dynamics

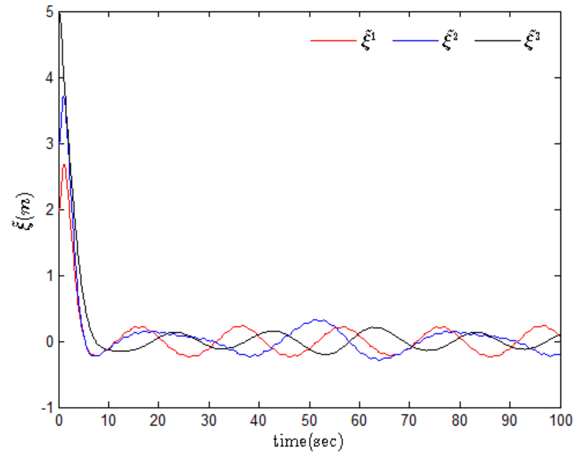

Figure 6. Position Error

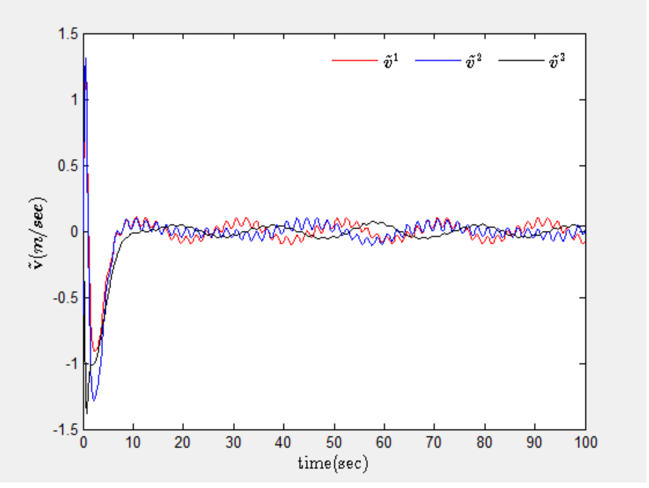

Figure 7. Velocity Error

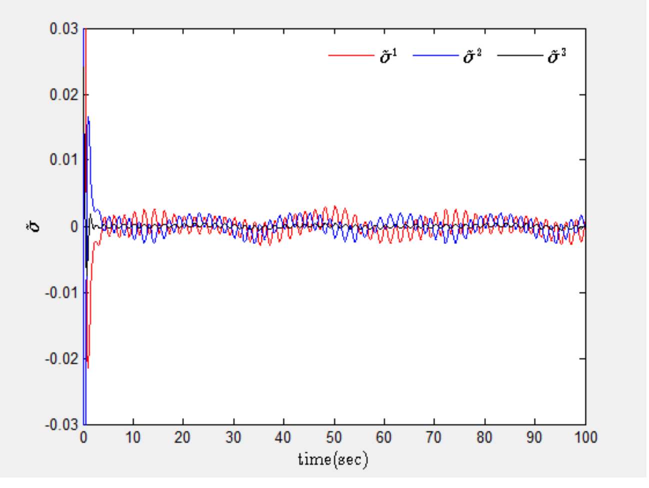

Figure 8. Mrps Error

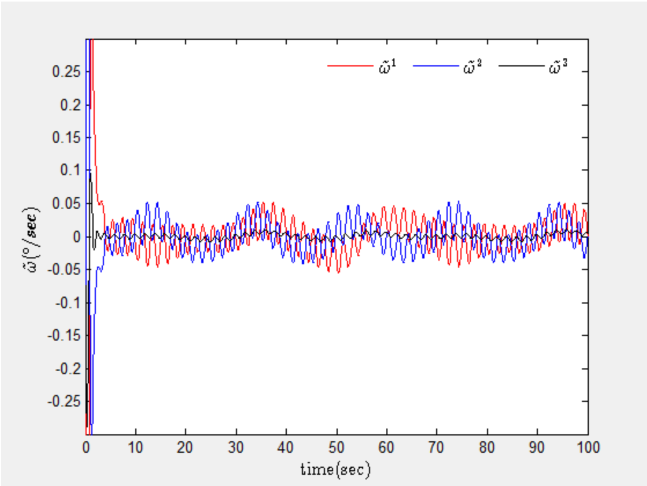

Figure 9. Angular Velocity Error

6. Conclusions

In this paper, the trajectory tracking control of a quadrotor aircraft with both parameters perturbation and external disturbances is considered. The system error model is established based on MPRs, based on which a hierarchical control structure is applied for controller implementation. The practical NDOB based backstepping control strategy is proposed to stabilize the translational and rotational subsystems, respectively. Stability of the closed-loop system is analyzed based on Lyapunov theory. Simulations show that the proposed controller can enable the quadrotor aircraft to track a desired trajectory effectively. The NDOB can estimate the composite disturbances accurately.

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