International Journal of Control and Automation
Vol.7, No.9 (2014), pp.257-270
http://dx.doi.org/10.14257/ijca.2014.7.9.22

Construction of the Kalman Filter Algorithm on the Model Reduction

Didik Khusnul Arif [*,a], Widodo[b], Salmah and Erna Apriliani[a]

[*] *Post Graduate Student in Department of Mathematics, Universitas Gadjah Mada (UGM), Yogyakarta, Indonesia.*
[a] *Department of Mathematics, Institut Teknologi Sepuluh Nopember (ITS), Surabaya, Indonesia.*
[b] *Department of Mathematics, Universitas Gadjah Mada (UGM), Yogyakarta, Indonesia.*
didik@matematika.its.ac.id, widodo_mathugm@yahoo.com, syalmah@yahoo.com, april@matematika.its.ac.id

Abstract

In this paper we derive a state variable estimation method of discrete stochastic dynamical systems. It aims to obtain accurate estimation with short computing time. Therefore, the point of this paper is to discuss a construction of Kalman filter algorithm on the reduced model. First, we construct a reduced model by using balanced truncation method. Further, we apply state variable estimation steps of discrete stochastic dynamical systems by using Kalman filter on the reduced model. Thus, Kalman filter algorithm will be constructed on the reduced model.

Keywords: *estimation, Kalman filter, balanced system, reduced model*

1. Introduction

Estimation of state variable in a system is quite important. One method for estimating the state variable in a system is Kalman filter [1]. Kalman filter is a recursive algorithm for estimating a state variable in stochastic dynamical systems. Kalman filter estimation is applied by assessing a state variable based on the dynamical systems. Further, based on measured data, the result will be improved [2].

Kalman filter was applied in many problems, such as estimation of sea level [3], estimation of heat distribution [4, 5], on the hydrodynamical model problem [6], and many others. On the application of Kalman filter, there are some weaknesses due to numerical stability problems or due to inaccurate modeling system [7].

In 1979, R. Anderson [8], developed the square root covariance filter algorithm to avoid instability numerical problem. The algorithm of square root covariance filter successfully overcame numerical stability problems, but the order of square root matrix become larger. Furthermore, reduce rank is applied on the square root covariance matrix to reduce the order of matrix so that the computing time become faster [9].

In generally, the construction of estimation method aims to obtain accurate result with short computing time. The computing time is influenced by order of the system. To reduce the computing time can be done by reduce model [10, 11].

Based on the above description, so it is important to construct a modified Kalman filter to obtain an accurate estimation with less computational time. In this paper, we combine the Kalman filter and reduction model method to construct Kalman filter algorithm on the model

ISSN: 2005-4297 IJCA

reduction as the extended of our results [5, 12]. Kalman filter algorithm aims to estimate a state variable in the system, whereas the model reduction aims to construct a simple model which has smaller order. In other words, we will construct Kalman filter algorithm on the reduced model. The first step, we discuss about the construction of Kalman filter algorithm in discrete stochastic dynamical systems. It aims to analyze the steps of estimation process. Second, we construct the reduced model from discrete system by using balanced truncation method. It aims to analyze the construction and characteristic of reduced model. Finally, we construct Kalman filter algorithm on the reduced model from discrete stochastic dynamical system.

2. The Algorithm of Kalman Filter on the Discrete System

A Before constructing Kalman filter algorithm for reduced system, firstly we analyze an estimation process of state variable in discrete system. Here, we analyze estimation process and construct the best estimator for a discrete system based on the Kalman filter algorithm by F. L. Lewis [2]. Further, we use this process to apply Kalman filter on reduce model.

Let be given a dynamic system

$$x_{k+1} = Ax_k + Bu_k \ ,$$
$$z_k = Cx_k + Du_k \ , \tag{1}$$

where $x_k \in R^n$ is a state vector at the time k, $u_k \in R^m$ is an input vector at the time k and $z_k \in R^p$ is a measurement vector or output vector at the time k. While A, B, C and D are appropriately dimensioned real constant matrices. Furthermore, we build a dynamic stochastic system

$$x_{k+1} = Ax_k + Bu_k + Gw_k \ ,$$
$$z_k = Cx_k + Du_k + v_k \ , \tag{2}$$

where w_k is system noise and v_k is measurement noise. w_k and v_k are stochastic scale and assumed by $w_k \sim (0, Q)$ and $v_k \sim (0, R)$.

Estimation of state variables in the system (2) can be doing by using Kalman filter. First, we estimate the state variables based on a dynamic system and then we corrected based on the measurement data to improve the estimations results. Estimation process of state variables based on the dynamics system are called the prediction step, while the correction process that involving the measurement data is called correction step. Kalman filter algorithm for discrete stochastic dynamic system can be written as follows [2].

a. Initialization

$$P_0 = P_{x_0} \ ; \ \hat{x}_0 = \bar{x}_0 \qquad , w_k \sim (0, Q) \ ; \ \ v_k \sim (0, R).$$

b. Prediction step describes the effect of the dynamical systems

Error covariance : $P_{k+1}^{-} = AP_kA^T + +GQG^T$.

Estimation : $\hat{x}_{k+1}^{-} = A\hat{x}_k + Bu_k$.

c. Correction step describes the effect of measurement

Error covariance : $P_{k+1} = P_{k+1}^{-} - P_{k+1}^{-}C^T(CP_{k+1}^{-}C^T + R)^{-1}CP_{k+1}^{-}$.

Estimation : $\hat{x}_{k+1} = \hat{x}_{k+1}^{-} + P_{k+1}C^TR^{-1}(z_{k+1} - C\hat{x}_{k+1}^{-})$.

If we use Kalman gain $K_{k+1} = P_{k+1}^{-}C^T(CP_{k+1}^{-}C^T + R)^{-1}$, then we rewrite

Error covariance : $P_{k+1} = (I - K_{k+1}C)P_{k+1}^{-}$.

Estimation : $\hat{x}_{k+1} = \hat{x}_{k+1}^{-} + K_{k+1}(z_{k+1} - C\hat{x}_{k+1}^{-})$.

d. To estimate the state at time $k + 2$, we are back again in step b and so on.

Prediction step and correction step are processed repeatedly until the time K is given or until we find a best estimation $\hat{x}_k$ that minimizes error.

3. Reduced Model Construction on the Discrete Systems

In this section, we describe the formation of the model reduction by balanced truncation [10, 13, 14]. Suppose the state space representation (1). We define two matrices, controllability gramian, M and observability gramian, N. The controllability gramian associated with the system (1) is matrix

$M \coloneqq \sum_{k=0}^{\infty} A^k B B^T (A^T)^k,$

and the observability gramian associated with the system (1) is matrix

$N \coloneqq \sum_{k=0}^{\infty} (A^T)^k C^T C A^k.$

The gramian M and N are each symmetric positive definite matrix and is the unique solution of the Lyapunov equation

$AMA^T + BB^T - M = 0,$

$A^T N A + C^T C - N = 0.$

Furthermore, the system (1) is called balanced if controllability and observability gramian are equal and diagonal, i.e., if

$$M = N = diag(\sigma_1, \sigma_2, \cdots, \sigma_n) = \Sigma \ ; \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0 \ , \tag{3}$$

with σ_i is a positive real number which is ordered Hankel singular values of the system (1) and Σ is called balanced gramian. Based on its Hankel singular value, then the balanced gramian Σ can be partitioned into

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \tag{4}$$

where $\Sigma_1 = diag\ (\sigma_1, \sigma_2, \cdots, \sigma_r)$ and $\Sigma_2 = diag\ (\sigma_{r+1}, \sigma_{r+2}, \cdots, \sigma_n)$ with $\sigma_r \gg \sigma_{r+1}$.

Next, we can construct the balanced system

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}u_k \ , \\ \tilde{z}_k &= \tilde{C}\tilde{x}_k + Du_k \ , \end{aligned} \tag{5}$$

with $\tilde{x}_k \in R^n$ is a state variable in a balanced system at time k, $u_k \in R^m$ is the input vector in a balanced system at time k and $\tilde{z}_k \in R^p$ is the output vector in a balanced system at time k. While the $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and D are appropriately dimensioned matrices of a balanced system, respectively. Based on (4), then the system (5) can be partitioned into

$$G = \left(\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2 \\ \hline \tilde{C}_1 & \tilde{C}_2 & D \end{array} \right). \tag{6}$$

with

$$\tilde{x}_k = \begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix} \tag{7}$$

and

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}, \tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2) \ . \tag{8}$$

$\tilde{x}_{1_k}$ is a state variable that corresponds to the large Hankel singular value and $\tilde{x}_{2_k}$ is a state variable that corresponds to the small Hankel singular value. If we assume $\tilde{x}_{1_k}$ has dimension r with $r < n$ then $\tilde{A}_{11} \in R^{r \times r}$, $\tilde{B}_1 \in R^{r \times m}$, $\tilde{C}_1 \in R^{p \times r}$, and $D \in R^{p \times m}$.

Furthermore the reduced models of the system (1) can be obtained by discarding the state variables in a balanced system (5) corresponding to small Hankel singular values. So generally reduced system formed order with respect to r with $r < n$ and can be modeled in the form of [14]

$\tilde{x}_{r_{k+1}} = \tilde{A}_{11}\tilde{x}_{r_k} + \tilde{B}_1 u_k \ ,$

$$\tilde{z}_{r_k} = \tilde{C}_1 \tilde{x}_{r_k} + Du_k \,. \tag{9}$$

4. The Algorithm of Kalman Filter on the Reduced Model

In this section, we analyze about the construction of Kalman filter algorithm on the reduced model. As explained in Section 3 that the construction of reduced model is begun with constructing the balanced system. This balanced system is transformation result of original system with sorted state variable position based on Hankel singular value. Further, we can get reduced model by discarding the state variable on the balanced system that corresponding to small Hankel singular values. The algorithm of the state variables estimation on the reduced model, can be initiated by applying Kalman filter on the balanced system.

Suppose there is a dynamic stochastic balanced system

$$\begin{aligned} \tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}u_k + Gw_k \\ \tilde{z}_k &= \tilde{C}\tilde{x}_k + Du_k + v_k \,, \end{aligned} \tag{10}$$

where $w_k \sim (0, Q)$ and $v_k \sim (0, R)$ are system noise and measurement noise, respectively. Furthermore, we given the initial state $\tilde{x}_0$ and initial estimation $E[\tilde{x}_0] = \hat{\tilde{x}}_0$ with error covariance $E\left[\left(\tilde{x}_0 - \hat{\tilde{x}}_0\right)\left(\tilde{x}_0 - \hat{\tilde{x}}_0\right)^T\right] = \tilde{P}_0$ and we denote $\tilde{P}_k^-$ and $\tilde{P}_k$ as error covariance on the prediction step and on the correction step at the time k, respectively.

Next, we estimate state variables on the balanced system based on the dynamic system, as called the prediction step. Estimation for the state variable $\tilde{x}_k$ on the balanced system (10) based on the dynamic system can be described as follows:

$$\begin{aligned} \hat{\tilde{x}}_{k+1}^- &= E(\tilde{x}_{k+1}) \\ &= E\left(\tilde{A}\tilde{x}_k + \tilde{B}u_k + Gw_k\right) \\ &= \tilde{A}\hat{\tilde{x}}_k + \tilde{B}u_k. \end{aligned} \tag{11}$$

Based on the partition which occurs in balanced system in equation (6), (7), (8), then we can write the prediction step as follow

$$\begin{aligned} \begin{pmatrix} \hat{\tilde{x}}_{1_{k+1}}^- \\ \hat{\tilde{x}}_{2_{k+1}}^- \end{pmatrix} &= E\left(\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}\begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix} + \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix} u_k + Gw_k\right) \\ &= \begin{pmatrix} \tilde{A}_{11}E[\tilde{x}_{1_k}] + \tilde{A}_{12}E[\tilde{x}_{2_k}] + \tilde{B}_1 u_k \\ \tilde{A}_{21}E[\tilde{x}_{1_k}] + \tilde{A}_{22}E[\tilde{x}_{2_k}] + \tilde{B}_2 u_k \end{pmatrix}, \end{aligned} \tag{12}$$

where $\hat{\tilde{x}}_{1_{k+1}}^-$ and $\hat{\tilde{x}}_{2_{k+1}}^-$ are the estimation for state variable corresponding to the large Hankel singular value and the small Hankel singular value, respectively.

Based on the minimum realization lemma for continuous system [10], then can be derived a minimum realization lemma for discrete system as following:

Lemma 1: Minimum Realization for the Discrete System

Given the dynamic system (1) and consider $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *as state space realization of transfer function* G. *Assume there is a symmetry matrices* $P = P^T = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}$, *where* P_1 *nonsingular, such that* $APA^T + BB^T - P = 0$. *Then, partition a realization* (A, B, C, D) *corresponding to* P, *i.e.* $\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right)$, *such that* $\left(\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right)$ *is also a realization of* G.

Proof. Based on the assumption that the matrix P can be partitioned into $P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}$, then we can write the dynamic system (1) becomes

$$x_k = \begin{pmatrix} x_{1k} \\ x_{2k} \end{pmatrix},\ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},\ C = (C_1 \quad C_2).$$

Furthermore, the Lyapunov equation $APA^T + BB^T - P = 0$ based on partition of P and (A, B, C, D) becomes

$$\begin{pmatrix} A_{11}P_1A_{11}^T + B_1B_1^T - P_1 & A_{11}P_1A_{21}^T + B_1B_2^T \\ A_{21}P_1A_{11}^T + B_2B_1^T & A_{21}P_1A_{21}^T + B_2B_2^T \end{pmatrix} = 0. \tag{13}$$

Equation (13) satisfies

$$A_{11}P_1A_{11}^T + B_1B_1^T - P_1 = 0, \tag{14}$$

$$A_{11}P_1A_{21}^T + B_1B_2^T = 0, \tag{15}$$

$$A_{21}P_1A_{11}^T + B_2B_1^T = 0, \tag{16}$$

$$A_{21}P_1A_{21}^T + B_2B_2^T = 0. \tag{17}$$

Equation (14) is a Lyapunov equation for P_1, then

P_1 definite positive (or $P_1 > 0$). (18)

While from equation (17), we get

$$A_{21}P_1A_{21}^T = -B_2B_2^T\ . \tag{19}$$

Multiplying (19) from the left by x^T and from the right by x, yield

$$x^TA_{21}P_1A_{21}^Tx = -x^TB_2B_2^Tx \tag{20}$$

where x is any vector that is not zero.

We can write equation (20) in the form of quadratic norm applicable to any non-zero vector x

$$\left\|{P_1}^{1/2}A_{21}^Tx\right\|^2 = -\left\|B_2^Tx\right\|^2 \tag{21}$$

and for $P_1 > 0$, then equation (21) is valid only if

$\mathrm{A}_{21} = 0$ and $B_2 = 0$. (22)

If $A_{21} = 0$ and $B_2 = 0$, then equation (15) and (16) is also satisfied.

By equation (22), then partition of the realization (A, B, C, D) is $\left(\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right)$ and the transfer function of (1) becomes:

$$\begin{aligned} G(z) &= C(zI - A)^{-1}B + D \\ &= (C_1 \quad C_2)\left(zI - \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\right)^{-1}\begin{pmatrix} B_1 \\ 0 \end{pmatrix} + D \\ &= C_1(zI - A_{11})^{-1}B_1 + D\ . \end{aligned}$$

Hence, we have $(z) = \left(\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right)$. □

Now, we consider the balanced system (10) having an equilibrium gramian as in equation (4). Furthermore, if we assume σ_{r+1} is very small ($\sigma_{r+1} \to 0$), then the balanced gramian Σ can be written as $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$, where $\Sigma_1 = diag\ (\sigma_1, \sigma_2, \cdots, \sigma_r)$. Because Σ definite positive and satisfy Lyapunov equation $\tilde{A}\Sigma\tilde{A}^T + \tilde{B}\tilde{B}^T - \Sigma = 0$, then we can apply Lemma 1 on the balanced system (10), so the balanced system (10) applies

$$\tilde{A}_{21} = 0,\ \tilde{B}_2 = 0. \tag{23}$$

By Lemma 1, we can see that the state variable $\tilde{x}_{2_k}$ does not affect the system so that it can be ignored. Therefore, we can assume that the expectations of the state variable $\tilde{x}_{2_k}$ close to zero, i.e.

$$E\left[\tilde{x}_{2_k}\right] \approx 0. \tag{24}$$

Furthermore, by substituting equation (23) and (24) into the equation (12) is obtained

$$\begin{aligned} \hat{\tilde{x}}_{1_{k+1}}^{-} &= \tilde{A}_{11}\hat{\tilde{x}}_{1_k} + \tilde{B}_1 u_k , \\ \hat{\tilde{x}}_{2_{k+1}}^{-} &= 0 . \end{aligned} \tag{25}$$

Based on the equation (25), we can estimate the state variables of the balanced system at the time $k+1$ based on the dynamics of the system by the formula

$$\hat{\tilde{x}}_{k+1}^{-} = \tilde{A}_{11}\hat{\tilde{x}}_{1_k} + \tilde{B}_1 u_k \tag{26}$$

Next, we will determine error covariance at the time $k+1$ for the state variable on a balanced system based on its dynamic system, i.e.

$$\begin{aligned} \tilde{P}_{k+1}^{-} &= E\left[\left(\tilde{x}_{k+1} - \hat{\tilde{x}}_{k+1}^{-}\right)\left(\tilde{x}_{k+1} - \hat{\tilde{x}}_{k+1}^{-}\right)^T\right] \\ &= E\left[\tilde{A}\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)^T \tilde{A}^T\right] + GQG^T \\ &= E\left[\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}\left(\begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix} - \begin{pmatrix} \hat{\tilde{x}}_{1_k} \\ \hat{\tilde{x}}_{2_k} \end{pmatrix}\right)\left(\begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix} - \begin{pmatrix} \hat{\tilde{x}}_{1_k} \\ \hat{\tilde{x}}_{2_k} \end{pmatrix}\right)^T \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}^T\right] + GQG^T . \end{aligned} \tag{27}$$

Based on equations (24) and because $\tilde{x}_{1_k}$ and $\tilde{x}_{2_k}$ are independent, then we get

$$\begin{aligned} & E\left[\hat{\tilde{x}}_{2_k}\right] \approx 0, \\ & E\left[\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right] \approx 0, \\ & E\left[\left(\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right)^T\right] \approx 0, \\ & E\left[\left(\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right)\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)^T\right] \approx 0, \\ & E\left[\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)\left(\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right)^T\right] \approx 0, \\ & E\left[\left(\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right)\left(\tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\right)^T\right] \approx 0. \end{aligned} \tag{28}$$

Therefore we can write equation (27) becomes

$$\begin{aligned} \tilde{P}_{k+1}^{-} &= \tilde{A}_{11} E\left[\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)^T\right] \tilde{A}_{11}{}^T + GQG^T \\ &= \tilde{A}_{11}\tilde{P}_{\tilde{x}_{1_k}}\tilde{A}_{11}{}^T + GQG^T . \end{aligned} \tag{29}$$

According to equation (29), we obtain that the result of error covariance in the balanced system based on the dynamical system is only affected by error covariance on the state variable $\tilde{x}_{1_k}$.

Then we consider measurement factor in balanced system (10). Measurement estimation $\tilde{z}_k$ of balanced system (10) is

$$\hat{\tilde{z}}_k = E\left[\tilde{C}\tilde{x}_k\right] + E\left[v_k\right] . \tag{30}$$

By using partition the balanced system, then we obtain

$$\begin{aligned} \hat{\tilde{z}}_k &= E\left[\begin{pmatrix} \tilde{C}_1 & \tilde{C}_2 \end{pmatrix}\begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix}\right] + E\left[v_k\right] \\ &= \tilde{C}_1 E\left[\tilde{x}_{1_k}\right] + \tilde{C}_2 E\left[\tilde{x}_{2_k}\right] + E\left[v_k\right] \\ &= \tilde{C}_1 \hat{\tilde{x}}_{1_k} . \end{aligned} \tag{31}$$

Equation (31) shows that the result of estimating measurement in balanced system is only affected by the state variable $\tilde{x}_{1_k}$, while the state variable $\tilde{x}_{2_k}$can be ignored.

Furthermore, we determine error covariance of measurement factor $\tilde{z}_k$ in the balanced system (10) is

$$\begin{aligned}\tilde{P}_{\tilde{z}_k} &= E\left[\left(\tilde{z}_k - \hat{\tilde{z}}_k\right)\left(\tilde{z}_k - \hat{\tilde{z}}_k\right)^T\right] \\ &= E\left[\tilde{C}\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)^T \tilde{C}^T\right] + R \\ &= E\left[\begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}\begin{pmatrix}\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k} \\ \tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\end{pmatrix}\begin{pmatrix}\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k} \\ \tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\end{pmatrix}^T \begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}^T\right] + R. \end{aligned} \tag{32}$$

Based on equations (28), then we get

$$\tilde{P}_{\tilde{z}_k} = \tilde{C}_1 \tilde{P}^-_{\tilde{x}_{1_k}} \tilde{C}_1^{\,T} + R\ . \tag{33}$$

Equation (33) shows that error covariance of measurement on the balanced system is only affected by error covariance of the state variable $\tilde{x}_{1_k}$.

Furthermore, based on the combined effect of the dynamical system and the measurement factor, then we can be determined a joint error covariance between $\tilde{x}_k$ and $\tilde{z}_k$

$$\begin{aligned}\tilde{P}_{\tilde{x}_k\tilde{z}_k} &= E\left[\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)\left(\tilde{z}_k - \hat{\tilde{z}}_k\right)^T\right] \\ &= E\left[\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)^T \tilde{C}^T\right] + E\left[\left(\tilde{x}_k - \hat{\tilde{x}}_k\right)\right]E\left[v_k^{\,T}\right] \\ &= E\left[\begin{pmatrix}\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k} \\ \tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\end{pmatrix}\begin{pmatrix}\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k} \\ \tilde{x}_{2_k} - \hat{\tilde{x}}_{2_k}\end{pmatrix}^T \begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}^T\right] \\ &= E\left[\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)\left(\tilde{x}_{1_k} - \hat{\tilde{x}}_{1_k}\right)^T \tilde{C}_1^{\,T}\right] \\ &= \tilde{P}^-_{\tilde{x}_{1_k}} \tilde{C}_1^{\,T}. \end{aligned} \tag{34}$$

Equation (34) shows that the only state variable $\tilde{x}_{1_k}$ that affect the result of joint error covariance between $\tilde{x}_k$ and $\tilde{z}_k$.

In the same way as in equation (34), then we can formulate

$$\tilde{P}_{\tilde{z}_k\tilde{x}_k} = \tilde{C}_1 \tilde{P}^-_{\tilde{x}_{1_k}}\ . \tag{35}$$

Based on the result that has been obtained in equation (33), (34) and (35), then error covariance of balanced system (9) based on the effect of dynamical system involving the effect of measurement factor at the time k can be expressed as

$$\begin{aligned}\tilde{P}_k &= \tilde{P}_{\tilde{x}_k/\tilde{z}_k} \\ &= \tilde{P}^-_k - \tilde{P}_{\tilde{x}_k\tilde{z}_k}\left(\tilde{P}_{\tilde{z}_k}\right)^{-1}\tilde{P}_{\tilde{z}_k\tilde{x}_k} \\ &= \tilde{P}^-_{\tilde{x}_{1_k}} - \tilde{P}^-_{\tilde{x}_{1_k}}\tilde{C}_1^{\,T}\left(\tilde{C}_1\tilde{P}^-_{\tilde{x}_{1_k}}\tilde{C}_1^{\,T} + R\right)^{-1}\tilde{C}_1\tilde{P}^-_{\tilde{x}_{1_k}}\ . \end{aligned} \tag{36}$$

Thus, refers to equation (36), can be determined error covariance of balanced system (9) based on the effect of dynamical system involving the effect of measurement factor at the time $k+1$ can be expressed as

$$\tilde{P}_{k+1} = \tilde{P}^-_{\tilde{x}_{1_{k+1}}} - \tilde{P}^-_{\tilde{x}_{1_{k+1}}}\tilde{C}_1^{\,T}\left(\tilde{C}_1\tilde{P}^-_{\tilde{x}_{1_{k+1}}}\tilde{C}_1^{\,T} + R\right)^{-1}\tilde{C}_1\tilde{P}^-_{\tilde{x}_{1_{k+1}}}, \tag{37}$$

If we define Kalman gain as follows:

$$K_{k+1} = \tilde{P}_{k+1}\tilde{C}_1^{\,T}R^{-1} = \tilde{P}^-_{\tilde{x}_{1_{k+1}}}\tilde{C}_1^{\,T}\left(\tilde{C}_1\tilde{P}^-_{\tilde{x}_{1_{k+1}}}\tilde{C}_1^{\,T} + R\right)^{-1},$$

then equation (37) can be written as

$$\tilde{P}_{k+1} = \left(I - K_{k+1}\tilde{C}_1\right)\tilde{P}^-_{\tilde{x}_{1_{k+1}}}. \tag{38}$$

The estimation of state variable at time k which obtained based on $\tilde{P}_{\tilde{x}_k\tilde{z}_k}$, $\tilde{P}_{\tilde{z}_k}$ and $\hat{\tilde{z}}_k$, that is called the best linear estimator given by measurement $\tilde{z}_k$, $\hat{\tilde{x}}_k$ and $\tilde{P}_k$, can be formulated as

$$\hat{\tilde{x}}_k = E\left[\tilde{x}_k/\tilde{z}_k\right]$$

$$= \hat{\tilde{x}}_k^- + \tilde{P}_{\tilde{x}_k \tilde{z}_k} \tilde{P}_{\tilde{z}_k}^{-1} (\tilde{z}_k - \hat{\tilde{z}}_k)$$
$$= \hat{\tilde{x}}_k^- + \tilde{P}_{\tilde{x}_{1_k}}^- \tilde{C}_1{}^T \left(\tilde{C}_1 \tilde{P}_{\tilde{x}_{1_k}}^- \tilde{C}_1{}^T + R \right)^{-1} \left(\tilde{z}_k - \tilde{C}_1 \hat{\tilde{x}}_k^- \right)$$
$$= \hat{\tilde{x}}_{1_k}^- + \tilde{P}_k \tilde{C}_1{}^T R^{-1} \left(\tilde{z}_k - \tilde{C}_1 \hat{\tilde{x}}_{1_k}^- \right) . \quad (39)$$

Based on equation (39), the state variable estimation at the time $k+1$ can be formulated into

$$\hat{\tilde{x}}_{k+1} = \hat{\tilde{x}}_{1_{k+1}}^- + \tilde{P}_{k+1} \tilde{C}_1{}^T R^{-1} \left(\tilde{z}_{k+1} - \tilde{C}_1 \hat{\tilde{x}}_{1_{k+1}}^- \right), \quad (40)$$

or

$$\hat{\tilde{x}}_{k+1} = \hat{\tilde{x}}_{1_{k+1}}^- + K_{k+1} \left(\tilde{z}_{k+1} - \tilde{C}_1 \hat{\tilde{x}}_{1_{k+1}}^- \right) . \quad (41)$$

Based on the above explanation, we can construct an algorithm for estimating the state variables in the reduced system by using Kalman filter. This algorithm is called the algorithm of Kalman filter on the reduced system.

The Algorithm of Kalman Filter on the Reduced System is

a. Give an original system in the form of discrete system as denoted in equation (1)
b. Determine a balanced gramian of the system (1) as stated in equation (3).
c. Construct a stochastic balanced system as expressed in equation (10), i.e.
   - the dynamical system model: $\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k + Gw_k$,
   - the measurement model: $\tilde{z}_k = \tilde{C}\tilde{x}_k + Du_k + v_k$.
d. Based on the partition balanced gramian as stated in equation (4), we can partition balanced system (10), i.e.
   - partition of balanced gramian: $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$,
   - the partition of the system: $\tilde{x}_k = \begin{pmatrix} \tilde{x}_{1_k} \\ \tilde{x}_{2_k} \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{pmatrix}$, $\tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2)$
e. Determine the estimation of the state variable on system (10) by this following steps :
   i. Initialization

      We give initial conditions on the stochastic system (9), i.e.
      - initial state: $\tilde{x}_0 = \begin{pmatrix} \tilde{x}_{1_0} \\ \tilde{x}_{2_0} \end{pmatrix}$,
      - initial estimation: $E[\tilde{x}_0] = \hat{\tilde{x}}_0 = \begin{pmatrix} \hat{\tilde{x}}_{1_0} \\ \hat{\tilde{x}}_{2_0} \end{pmatrix}$,
      - initial the error covariance: $E\left[(\tilde{x}_0 - \hat{\tilde{x}}_0)(\tilde{x}_0 - \hat{\tilde{x}}_0)^T \right] = \tilde{P}_0$
   ii. Prediction step

      We determine the estimation and the error covariance of state variable on the system (10) at the time $k+1$ based on the dynamical system, i.e.
      - Error covariance: $\tilde{P}_{k+1}^- = \tilde{A}_{11} \tilde{P}_{\tilde{x}_{1_k}}^- \tilde{A}_{11}{}^T + GQG^T$,
      - Estimation: $\hat{\tilde{x}}_{k+1}^- = \tilde{A}_{11} \hat{\tilde{x}}_{1_k} + \tilde{B}_1 u_k$,
   iii. Correction step

      By using the measurement data, we do a correction to the estimation and error covariance results that have been obtained in the prediction step. So that we get
      - Error covariance: $\tilde{P}_{k+1} = \tilde{P}_{\tilde{x}_{1_{k+1}}}^- - \tilde{P}_{\tilde{x}_{1_{k+1}}}^- \tilde{C}_1{}^T \left(\tilde{C}_1 \tilde{P}_{\tilde{x}_{1_{k+1}}}^- \tilde{C}_1{}^T + R \right)^{-1} \tilde{C}_1 \tilde{P}_{\tilde{x}_{1_{k+1}}}^-$,
      - Estimation: $\hat{\tilde{x}}_{k+1} = \hat{\tilde{x}}_{1_{k+1}}^- + \tilde{P}_{k+1} \tilde{C}_1{}^T R^{-1} \left(\tilde{z}_{k+1} - \tilde{C}_1 \hat{\tilde{x}}_{1_{k+1}}^- \right)$.

      And with Kalman gain $K_{k+1} = \tilde{P}_{k+1} \tilde{C}_1{}^T R^{-1} = \tilde{P}_{\tilde{x}_{1_{k+1}}}^- \tilde{C}_1{}^T \left(\tilde{C}_1 \tilde{P}_{\tilde{x}_{1_{k+1}}}^- \tilde{C}_1{}^T + R \right)^{-1}$ then

- Error covariance: $\tilde{P}_{k+1} = \left(I - K_{k+1}\tilde{C}_1\right)\tilde{P}^-_{\tilde{x}_{1_{k+1}}}$,
- Estimation: $\hat{\tilde{x}}_{k+1} = \hat{\tilde{x}}^-_{1_{k+1}} + K_{k+1}\left(\tilde{z}_{k+1} - \tilde{C}_1\hat{\tilde{x}}^-_{1_{k+1}}\right)$.

f. To estimate the state at time $\mathrm{k} + 2$, we are back again in step e (ii) and so on.

Prediction step and correction step (in step e) are processed repeatedly until the time k is given or until we find a best estimation $\hat{x}_k$ that minimizes error.

The steps in the above process can be images in a simple diagram as Figure 1.

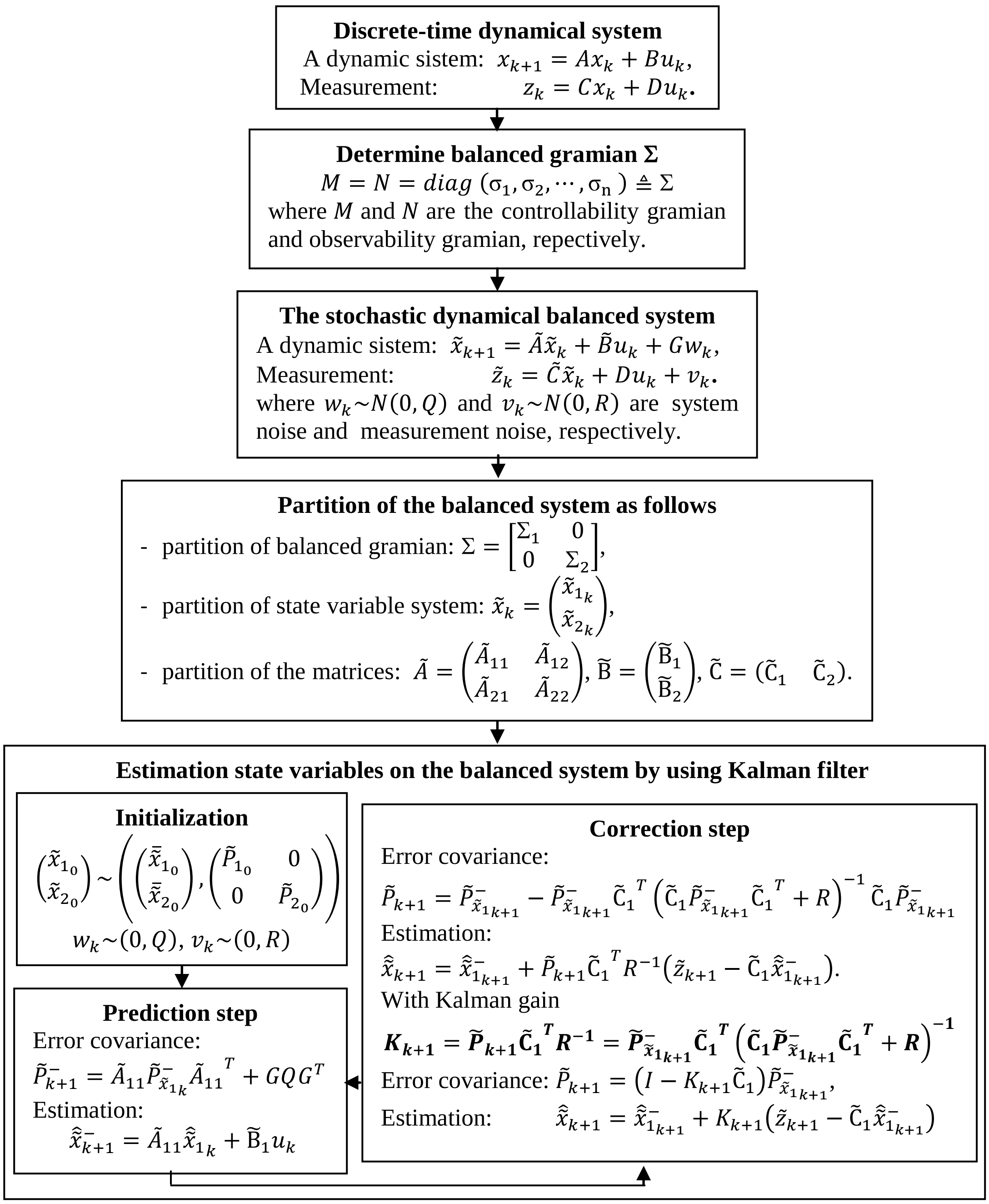

Figure 1. The Algorithm of Kalman Filter on the Reduced System

5. Case study

In this section, we estimate the heat distribution of the wire by using Kalman filter that is applied to the original system and the reduced system. Next, we compare the accuracy of estimation and computational time between the original system with reduced system.

First, given the problem of heat distribution on the wire. Suppose the length of the wire is l and the heat conduction coefficient is α. We assume that the sides of the wire is insulated perfectly, it means that there is no heat which able to penetrate the wire. Besides that, heat flows along the wire just depends on the position and time. Furthermore, we denote the temperature as U, the position denoted as x, and time is denoted as t. So U is a function of x and t, or can be written as $U(x,t)$. Further, at one of the end of wire insulated, $U_x(0,t)=0$, while at the other end of temperature was kept on U_N constantly for all $t>0$, where N is the number of wire parts. The heat conduction equation can be expressed as

$$\alpha^2 U_{xx} = U_t$$

with the boundary condition and initial condition as follows:

$$U_x(0,t)=0,$$
$$U(l,t)=U_N,$$
$$U(x,0)=f(x),$$

where $f(x)$ is a function of x and $0<x<l,\ 0<t<\infty$.

We discretize by forward different methods for time t and central difference approach for position x, then we get

$$U_{i,k+1}=\gamma\left(U_{i-1,k}-2U_{i,k}+U_{i+1,k}\right)+U_{i,k},$$

where $\gamma=\frac{\alpha^2}{\Delta x^2}\Delta t$.

To keep the stability discretization explicitly, then we have to define $\gamma \le 0{,}5$. While the boundary conditions and initial conditions are:

$$U_x(0,t)\cong\frac{U_{1,k}-U_{0,k}}{\Delta x}=0,$$
$$U(l,t)=U_{N,k}=\overline{U},$$
$$U(x,0)=f(x)\cong f_{i,0}.$$

The system including noise, so we can write

$$U_{i,k+1}=\gamma\left(U_{i-1,k}-2U_{i,k}+U_{i+1,k}\right)+U_{i,k}+W_{i,k} \tag{42}$$

where $W_{i,k}$ is a system noise.

Then we define the measurement equation as follows:

$$z_{k+1}=CU_{i,k+1}+V_{i,k+1} \tag{43}$$

where $V_{i,k}$ is a measurement noise.

The system (42) and measurement (43) can be written in the form of state space systems:

$$x_{k+1}=Ax_k+Bu_k+GW_k, \tag{44}$$
$$z_{k+1}=Cx_{k+1}+V_{k+1}, \tag{45}$$

where $x_k=[U_1,U_2,\cdots,U_{N-1}]_k^t$ is state vector, $z_k=\left[z_1,z_2,\cdots,z_p\right]_k^t$ is measurement vector. $W_k=[w_1,w_2,\cdots,w_m]_k^t$ is system noise and we assumed $W_k\sim N(0,Q)$. $V_k=\left[v_1,v_2,\cdots,v_p\right]_k^t$ is measurement noise and we assumed $V_k\sim N(0,R)$. While matrices A, B, C and G are appropriately dimensioned matrices.

The system (45) has the order N, where N is the number of sections taken or the number of positions along the wire. So that, we have a large scale system, which depends on the amount of N that we take. So we need to modify Kalman filter for estimating state variables. In this paper, we reduce the order system before apply the Kalman filter algorithm.

In this simulation, we take the length of wire is $l = 100$, the heat conduction coefficient is $\alpha^2 = 10$, the boundary condition are $U_0 = 0$, $U_N = 100$. In this research, we divide the length of wire $l = 100$ to 10, 20, 50 grids and we do simulation during 50 time step. The initial estimates $\hat{x}_0 = 0$ and initial estimation of error covariance $P_0 = 0{,}3I$ with I is the identity matrix. In this simulation, we compare the estimation result by Kalman filter algorithm for the original system (44) with the reduced system (9).

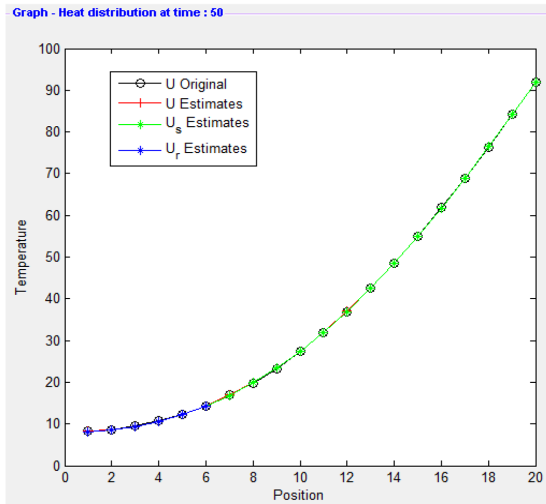

Figure 2. Estimated Heat Distribution by using Kalman Filter

Figure 2 describes the temperature at all positions at the 50th time for the original system with $N = 20$. It shows that the estimation of heat distribution on the original system, $U\_estimates$, and the estimation of heat distribution on the reduced system, $U_r\_estimates$, respectively has a performance that is not much different from the original heat distribution as $U\_original$. The comparison of estimation error on the original system and the reduced system can be shown in Figure 3.

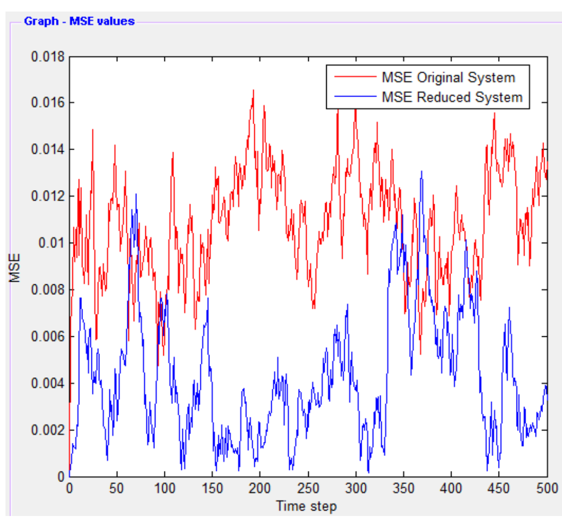

Figure 3. MSE Estimation of Heat Distribution by using Kalman Filter

Figure 3 describes the comparison between average estimation error on the original system ($N = 20$) with on the reduced system ($r = 6$) at the 50th time, while the calculation results of average error is listed in Table 1. Table 1 shows that the error estimation of heat distribution on the reduced system less than on the original system. It means that the results of state estimation by using the Kalman filter which is applied to the reduced system is more accurate than the results of state estimation on the original system.

Table 1. MSE Estimation of Heat Distribution by using Kalman Filter

Original Sistem		Reduced System	
#State	MSE	#State	MSE
10	0.10259442	5	0.04569062
20	0.010861	6	0.00424763
50	0.01081432	7	0.006846894

Furthermore, the computational time can be shown in Table 2. The the computational time is calculated based on the calculation of software MATLAB with Intel i5 2.50GHz, RAM 4.00 GB, operating system win7 64-bit. The computational time is a total calculation time of the state estimation by using Kalman filter algorithm. Table 2 shows that the computational time on the reduced system is less than on the original system.

Table 2. Computational Time to Estimate the Distribution of the Heat by Kalman Filter

Original System		Reduced System	
#State	Time	#State	Time
10	0.0154224	5	0.00914874
20	0.048102	6	0.0317882
50	1.25008	7	0.239162

6. Conclusions

Based on the above description, it can be concluded that Kalman filter algorithm on the reduced model can be constructed through the balanced system by applying the Kalman filter algorithm in discrete system. In the balanced system, we get $\tilde{x}_{k+1}$. By applying the Kalman filter algorithm in balanced system, we get $\hat{\tilde{x}}_{k+1}$ and $\tilde{P}_{k+1}$. Based on the calculation above, we can conclude that the state estimation $\hat{\tilde{x}}_{k+1}$ is only influenced by the estimation result of state variable which has a great effect to the system, i.e. $\hat{\tilde{x}}^{-}_{1_{k+1}}$, and error covariance $\tilde{P}_{k+1}$, is only affected by the error covariance of the state variables that have a large effect on the system, i.e. $\tilde{P}^{-}_{\tilde{x}_{1_{k+1}}}$.

Based on case study in the estimation of heat distribution on wire by using Kalman filter algorithm, we can conclude that the state estimation in the reduced system is more accurate and the computational time is less than the state estimation on the original system.

Acknowledgements

We give a great appreciation to the Postgraduate Program particularly for doctoral programme of Gadjah Mada University (UGM), Yogyakarta-Indonesia. And we give a great appreciation to the Central Research Institution of Institut Teknologi Sepuluh Nopember (ITS) Surabaya-Indonesia, which has provided us a research grant with contract number 2184.92/IT2.7/PN.01/2012 November 27th, 2012. The first author would like to thank Fatmawati of Universitas Airlangga for helpful discussions.

References

[1] H. Kwakernaak and R. Sivan, “Linear Optimal Control Systems”, Wiley-Interscience, Inc, New York, **(1972)**.
[2] F. L. Lewis, “Optimal Estimation”, John Wiley and Sons, **(1986)**.
[3] E. Apriliani, “The Estimation Of The Water Level By The Reduced Rank Square Root Information Filter”, Proceedings of the Asia – Pacific Vibration Conference, **(2001)**; China.
[4] E. Apriliani, “The Estimation Of The One Dimensional Temperature Distribution By The Modification Of The RRSQRT Filter”, Proceedings of the national mathematics seminar, Departement of Mathematics Institut Teknologi Sepuluh Nopember (ITS), **(2000)**; Surabaya, Indonesia.
[5] D. K. Arif, Widodo, Salmah and E. Apriliani, “Distribution Estimation of Heat Conduction using Kalman filtering which Implemented on Reduction Model. Proceedings of the 3rd International Conference and Workshop on Basic and Applied Sciences 2011 (ICOWOBAS 2011), **(2011)** September 21-23; Surabaya, Indonesia.
[6] M. Verlaan, Efficient Kalman filtering for Hydrodynamic Models, PhD Thesis, **(1998)**; Delft University of Technology, Netherland.
[7] P. S. Maybeck, “Stochastic Models”, Estimation and Control, vol. 1. Academic Press, **(1979)**.
[8] R. Anderson, “Optimal Filtering”, Prentice-Hall, **(1979)**.
[9] M. Verlaan and A. W. Heemink, “Convergence of the RRSQRT Algorithm for Large Scale Kalman Filtering Problems”, Technical report 97-19, TU-Delft, **(1997)**.
[10] K. Zhou, “Essentials of Robust Control”, Prentice-Hall, **(1998)**.
[11] S. Sigurd, “Multivariable Feedback Control: Analysis and Design”, John-Willey and Sons, **(2001)**.
[12] D. K. Arif, Widodo, Salmah and E. Apriliani, “Implementation of The Kalman Filter On Reduction Model”, Proceedings of the 8th International Conference on Mathematics, Statistics and Its Applications 2012 (ICMSA 2012), **(2012)** November 19-21; Sanur Paradise Plaza Hotel, Bali-Indonesia.
[13] K. Zhou , J. C. Doyle and K. Glover, “Robust and Optimal Control”, Prentice-Hall, Englewood Cliff, New Jersey 07632, **(1996).**
[14] S. Weiland, “Balancing and Hankel Norm Approximation of Dynamical Systems”, Course Model Reduction, **(2009)** February 4, http://w3.tue.nl/fileadmin/ele/MBS/CS/Files/Courses /Model_Reduction/modred2.pdf.

Authors

Didik Khusnul Arif, Post Graduate Student in Department of Mathematics, Universitas Gadjah Mada (UGM), Yogyakarta, Indonesia and Lecturer of Department of Mathematics, Institut Teknologi Sepuluh Nopember (ITS), Surabaya, Indonesia.

Widodo, Professor on Department of Mathematics, Universitas Gadjah Mada (UGM), Yogyakarta, Indonesia.

Salmah, Department of Mathematics, Universitas Gadjah Mada (UGM), Yogyakarta, Indonesia.

Erna Apriliani, Professor on Department of Mathematics, Institut Teknologi Sepuluh Nopember (ITS), Surabaya, Indonesia.

