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Two Case Studies of Robust Multi-parametric Model Predictive Control Algorithm

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Abstract

This paper presents case studies of robust algorithm of multi-parametric model predictive control (mpMPC). For discrete-time linear parameter-varying (LPV) systems, the robust algorithm is investigated based on the solution of the mpMPC problem for discrete-time linear time-invariant (LTI) systems. The algorithm provides a controller design method, adaptable to parameter changes of a LPV system. By this controller the performance of the controller LPV system is improved in terms of robustness particularly for slowly varying parameters in the system. Thus the design method is applicable to systems suffering from slow parameter variations due, for example, to aging or degradation. The paper illustrates two control problems, one of which is disturbance rejection problem while the other is reference tracking problem. Each control problem is studied by computer simulation. Particularly for the reference tracking problem the robust mpMPC technique not only reduces the tracking error but renders a system stable while the nominal mpMPC could not.

Keywords: *Robust control, mathematical programming, model predictive control, discrete-time systems, linear time-varying systems, disturbance rejection, reference tracking*

1. Introduction

In this paper we design a robust mpMPC controller for discrete-time LPV systems. A robust stabilisation problem for discrete-time linear parameter-varying (LPV) systems is investigated and slow variation of parameter of the mathematical model is considered, due for example to system degradation or aging, based on the solution of the multi-parametric model predictive control (mpMPC) for linear time-invariant (LTI) systems.

Some parameters are assumed to be time-varying but unknown how they evolve. The designed controller can be adapted to treat time-varying parameters. Accordingly the design method suggested in this paper can be applied to processes with time-varying conditions. Although the parameter variation may cause instability in the mpMPC implementation for LTI systems, the proposed compensation procedure is able to prevent the state divergence, as shown in this paper.

The chapter is organised as follows. Section II gives a brief description of the result of [1, 5] in Subsection II-A and a summary of the suggested robust controller design method in Subsection

II-B. Section III presents case studies for low dimensional systems to show the performance improvement by the robust mpMPC technique. Then we conclude the paper with Section IV.

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2. Multi-parametric Model Predictive Control and Robust Technique

A. Multi-parametric Model Predictive Control

We provide a summary of the result of the mpMPC for LTI systems (see [1, 5]). We consider the discrete-time LTI system

$$x(t+1) = \hat{A}x(t) + Bu(t), \quad (1)$$
$$y(t) = Cx(t),$$

subject to

$$y_{min} \leq y(t) \leq y_{max}, \quad (2)$$
$$u_{min} \leq u(t) \leq u_{max}, \quad (3)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^q$, and $y(t) \in \mathbb{R}^p$ are the state, input, and output vectors, respectively. Equations (2) and (3) denote component-wise inequalities. In these two inequalities the subscripts min and max denote the lower and upper bounds for the constraints, respectively. The pair $(\hat{A}, B)$ is stabilisable.

We formulate the MPC for the regulation of the state of the LTI system (1) to the origin as the constrained optimisation problem:

$$\begin{aligned}
&min_U J(U, x(t)) = min_U \{x'(t+N_y|t)Px(t+N_y|t) + \\
&\sum_{k=0}^{N_y-1} [x'(t+k|t)Qx(t+k|t) + u'(t+k)Ru(t+k)]\}, \\
&\textbf{s.t.} \\
&y_{min} \leq y(t+k|t) \leq y_{max}, \quad k = 1, \dots, N_c, \\
&u_{min} \leq u(t+k) \leq u_{max}, \quad k = 0,1, \dots, N_c \\
&x(t|t) = x(t), \\
&x(t+k+1|t) = \hat{A}x(t+k|t) + Bu(t+k), \quad k \geq 0, \\
&y(t+k|t) = Cx(t+k|t), \quad k \geq 0, \\
&u(t+k) = Kx(t+k|t), \quad N_u \leq k \leq N_y,
\end{aligned} \quad (4)$$

where $U = \{u(t), u(t+1), \dots, u(t+N_u-1)\}, Q = Q' \geq 0, R = R' > 0, P \geq 0, (Q^{1/2}, A)$ detectable, and K is a stabilising state feedback gain. $N_u, N_y,$ and N_c are the input, output, and constraint horizons, respectively, and are such that $N_u \leq N_y$ and $N_c \leq N_y - 1$. $x(t+k|t)$ denotes the state vector of the system (1) at time $t + k$ with the initial condition $x(t)$ and the input sequence $u(t), \dots, u(t+k-1)$.

The input $u(t)$ of the constrained QP problem (4) is described as continuous piecewise affine functions of $x(t)$([1]). Thus the input $u(t)$ used in the MPC implementation is obtained as explicit functions of $x(t)$

$$u(t) = \begin{cases} K_1 x(t) + c_1 & if\ H_1 x(t) \leq b_1, \\ \vdots & \\ K_{n_C} x(t) + c_{n_C} & if\ H_{n_C} x(t) \leq b_{n_C}, \end{cases} \quad (5)$$

where for $i \in \{1, \dots, n_C\}, K_i x(t) + c_i$ is the optimal solution in the corresponding critical region $H_i x(t) \leq b_i$, and n_C is the number of critical regions of the mpQP problem. Note that we can employ parametric optimization software, such as the POP toolbox ([3]), to obtain the explicit control laws (5).

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B. Robust mpMPC Technique for Discrete-time LPV Systems

In this section we discuss a new control strategy for mpMPC, implementable on-line for a class of LPV systems.

We consider discrete-time LPV systems described by the equations

$$x(t+1) = A(t)x(t) + Bu(t),$$
$$y(t) = Cx(t). \quad (6)$$

The matrix $A(t)$is assumed to be time-varying. Similar considerations can be performed for the cases with time-varying B and C.

The matrix $A(t)$can be rewritten as

$$A(t) = \hat{A} + \sum_{i=1}^{m} \Delta_i a_i(t), \quad (7)$$

where$\Delta_i \in \mathbb{R}^{n\times n}$and $\alpha_i(t) \in \mathbb{R}$.

In this research it is assumed that we know only that $A(t)$is a time-varying matrix with "sufficiently slow" parameters, but we do not know how the matrix evolves. Therefore $\hat{A}$is assumed to be known and also we assume that the matrices $\Delta_i$are known, while the functions $\alpha_i(t)$are not known.

The main idea of this robust technique is derived from the following proposition.

Proposition 1: Consider the two equations

$$\hat{x}(t+1) = \hat{A}\hat{x}(t) + Bu(t), \quad (8)$$
$$x(t+1) = A(t)x(t) + Bu(t). \quad (9)$$

Assume that $\hat{A}$is invertible. Then

$$\hat{x}(t+1) = x(t+1)$$

if$\hat{x}(t)$in (8) is replaced by$L(t)x(t)$, where

$$L(t) = \hat{A}^{-1}A(t). \quad (10)$$

Based on the record of the system operation and the state measurement, we obtain $A(t-1)$. Then we obtain $L(t-1)$by Proposition 1 and we assume that

$$L(t) \cong L(t-1). \quad (11)$$

Table 1. Parametric Solution *u* for the Example in Subsection III-A [2]

Input u	Region
$[-6.8355 \quad -6.8585]x$	$\begin{bmatrix} 0.7059 & 0.7083 \\ -0.7059 & -0.7083 \end{bmatrix} x \leq \begin{bmatrix} 0.2065 \\ 0.2065 \end{bmatrix}$
-2	$[-0.7059 \quad -0.7083]x \leq -0.2065$
2	$[0.7059 \quad 0.7083]x \leq -0.2065$

This implies that we may have an error in$L(t)$. Nonetheless Proposition 1 is still useful when we consider the compensation of the error for systems with "sufficiently slow" time-varying parameters, for example associated to aging or degradation.

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The matrix $L(t-1)$is interpreted alternatively as follows. At time t we have the state measurement $x(t)$, the result from (9) with $u(t-1)$ and $x(t-1)$, and the state value $\hat{x}(t)$computed by (8) with $u(t-1)$and$\hat{x}(t-1)$. If

$$\hat{A}\hat{x}(t-1) = A(t-1)x(t-1),$$

then$x(t) = \hat{x}(t)$. This means that at time $t-1$ we should have used the state vector $\hat{x}(t-1)$in (1) and (5), instead of $x(t-1)$, to compensate for the error induced by the time-varying parameters in $A(t)$.

By the approximation (11) we assume that

$$\hat{x}(t) = L(t)x(t) \cong L(t-1)x(t). \tag{12}$$

Note that u in (8) and (9) are the same and that by Proposition 1 we obtain the linear operator L, regardless of the control u.

The on-line implementation of the control scheme is provided in the following steps.

On-line Control Steps

INITIALISATION We consider the system (6).

Set $A(0) = \hat{A}$.Obtain the optimal look-up map (5) for system (1). Set the initial condition $x(0)$of system (6) for $t = 0$.

STEP 1 Measure the state vector $x(t)$of system (6).

STEP 2 If $t = 0$ then$A(-1) = \hat{A}$. Otherwise, calculate$A(t-1)$from the system equation in (6) with $x(t), x(t-1), u(t-1)$.

STEP 3 Obtain $L(t-1)$by Proposition 1. Estimate $\hat{x}(t)$from (12) with $x(t)$and $L(t-1)$. Note that, by Proposition 1, $L(-1) = \hat{A}^{-1}A(-1) = \hat{A}^{-1}\hat{A} = I$.

STEP 4 Evaluate the function $u(t)$from (5) based on$\hat{x}(t)$. Apply $u(t)$to the LPV system (6).

STEP 5 Set $t = t + 1$. Go to STEP 1.

3. Applications with Low Dimensional Systems

A. Simulation Study for Slowly Varying Disturbance

Consider the example of regulation to the origin of the SISO system in [1, 2], namely the discrete-time LTI system

$$x(t+1) = \hat{A}x(t) + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u(t), \tag{13}$$
$$y(t) = [0 \quad 1.4142]x(t)$$

with$-2 \le u(t) \le 2$ and

$$\hat{A} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix}.$$

To regulate the state of the system (13) to the origin an MPC controller is designed based on the optimisation problem

$$\begin{aligned} &\min_{u(t),u(t+1)} x'(t+2|t)Px(t+2|t) + \\ &\sum_{k=0}^{1}[x'(t+k|t)x(t+k|t) + 0.01u(t+k)^2], \\ &\text{s.t} \quad -2 \le u(t+k) \le 2, \qquad k = 0,1, \\ &x(t|t) = x(t), \end{aligned} \tag{14}$$

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where$P$is the solution of the Lyapunov equation$P = \hat{A}'\mathrm{P}\hat{A} + I_{2\times 2}$.

For the implementation of MPC we employ the Matlab toolbox [3] to compute the control law u using the algorithm summarised in Subsection II-A. The result is presented in Table 1.

The goal of the control idea described in this paper is to compensate the error between the computationally expected values via model (1) and the on-line measurement at each sampling time in the MPC implementation. Throughout this paper we assume that the error is due to time-varying elements in the system matrix $A(t)$of the model (6). However, instead of the system equation in (6), we also could assume that the error is caused by disturbances in the LTI system, modelled with $d(t)$and giving

$$x(t+1) = \hat{A}x(t) + d(t) + Bu(t), \tag{15}$$

where$d(t)$is the vector$[d_1(t), \ldots, d_n(t)]'$. If $d(t)$is described as a vector of time-varying functions, rather than white noise, then the methodology of the paper is applicable to compensate the error due to$d(t)$. In such a case $\hat{x}(t)$is considered as$x(t) + \hat{A}^{-1}d(t-1)$. Note that $d(t-1)$ is usedto calculate $\hat{x}(t)$ instead of $d(t)$ because of the same practical implementation issue discussed in the case of the LPV approach.

Figure 1 illustrates the behaviour of the LTI system (15) with disturbance $d(t)$ $(= [d_1(t), 0]')$where $d_1(t)$is modelled by the sum of two logistic functions of time. The main idea for error compensation is the same as in the previous simulation, while the implementation method is briefly described in this section. The top graph in the figure shows the time history of $d_1(t)$.

Note that, in this case, even after the state converges to the origin, the variation in $d_1(t)$is able to perturb the response of the LTI system (15). It is shown that during the simulation the compensated state response (solid line) stays closer to the origin than the state response of the system controlled with the classical method (dotted line).

B. Simulation Study for Reference Tracking Problem

In this subsection we present an example, namely a reference tracking problem for a one-dimensional system, to illustrate the proposed control scheme. Consider the discrete-time LTI system

$$\begin{aligned} x(t+1) &= \hat{a}x(t) + u(t), \\ y(t) &= x(t) \end{aligned} \tag{16}$$

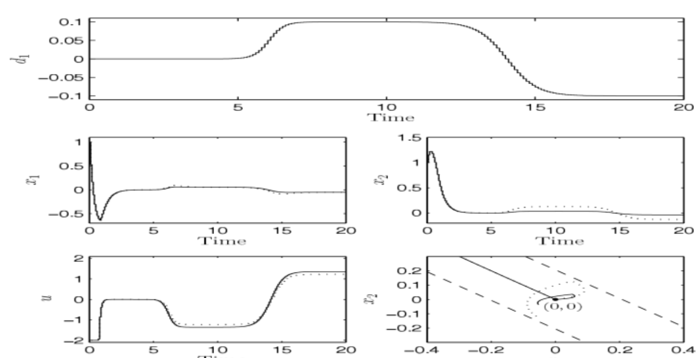

Figure 1. Simulation Results for the LTI System (15) with Disturbance $(d(t) = [d_1(t), 0]')$with $u$as in Table I and the Compensation Method Discussed in Subsection III-A. The Top graph Shows the Time History of$d_1(t)$, which is Modelled by a Sum of Two Logistic Functions of Time

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with $-2 \leq u(t) \leq 2$ and $\hat{a} = 1$.

Note that the model (16) does not have any output/state constraint. For the state of the system (16) to track a given reference, an MPC controller is designed based on the optimisation problem

$$\min_{u(t),u(t+1)} p(x(t+2|t) - x_{ref})^2 + \sum_{k=0}^{1}[(x(t+k|t) - x_{ref})^2 + u(t+k)^2], \quad \text{s.t} \quad -2 \leq u(t+k) \leq 2, \quad k = 0,1, \quad x(t|t) = x(t), \tag{17}$$

where $p = 1.618$ is the solution of the Riccati equation, $p^2 - p - 1 = 0$.

With the help of the POP toolbox of Matlab [3] we obtain the control law u given in Table 2.

Table 2. Parametric Solution *u* for the Example in Subsection III-B. Note that $\Xi := [x\ x_{ref}]$

Input u	Region
$[-0.61803 \quad 0.61803]\Xi$	$\begin{bmatrix} -0.61803 & 0.61803 \\ 0.61803 & -0.61803 \end{bmatrix}\Xi \leq \begin{bmatrix} 2 \\ 2 \end{bmatrix}$
2	$[-0.61803 \quad 0.61803]\Xi \geq 2$
-2	$[0.61803 \quad -0.61803]\Xi \geq 2$

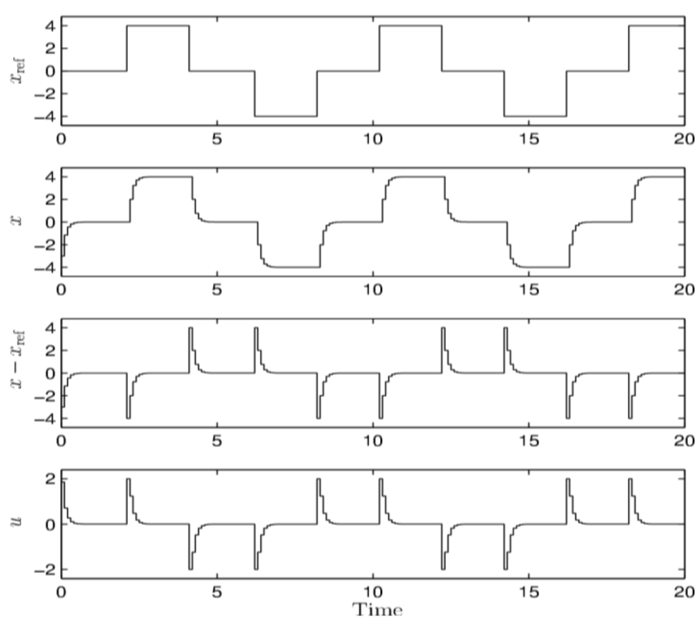

Figure 2. Simulation Results for the Nominal LTI System (1) with the x_{ref} Shown in the Top Graph and u as in Table II. The Initial Condition $x(0)$ is −3, and the Control Goal is State Tracking to the Reference Value x_{ref}

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Figure 2 shows the response of the nominal LTI system (16) with the x_{ref} plotted in the top graph and the mpMPC solution in Table II. The initial condition $x(0)$ is -3 and the control goal is tracking of the state x to the reference x_{ref}, which is successfully demonstrated in this figure.

To illustrate the compensation effect of the proposed control scheme we now assume that $\hat{a}$ in (16) is replaced by a time-varying $a(t)$, which is described by

$$a(t) = \hat{a} + 0.3sin(\pi t/40).$$

Figure 3 presents the simulation results for the time-varying $a(t)$ with the proposed compensation scheme. The reference value $x_{ref}(t)$ is the same as in Figure 2. The top graph shows the time history of $a(t)$. The responses of the controlled system are depicted in the other graphs, in which the dotted lines and the solid lines indicate the responses resulting from the use of the nominal mpMPC implementation and the use of the robust mpMPC implementation, respectively.

Figure 3 shows the improvement achieved by the proposed scheme compared to the LTI mpMPC implementation. Note that the response implemented by the classical mpMPC *diverges*, while the one by the proposed scheme does not.

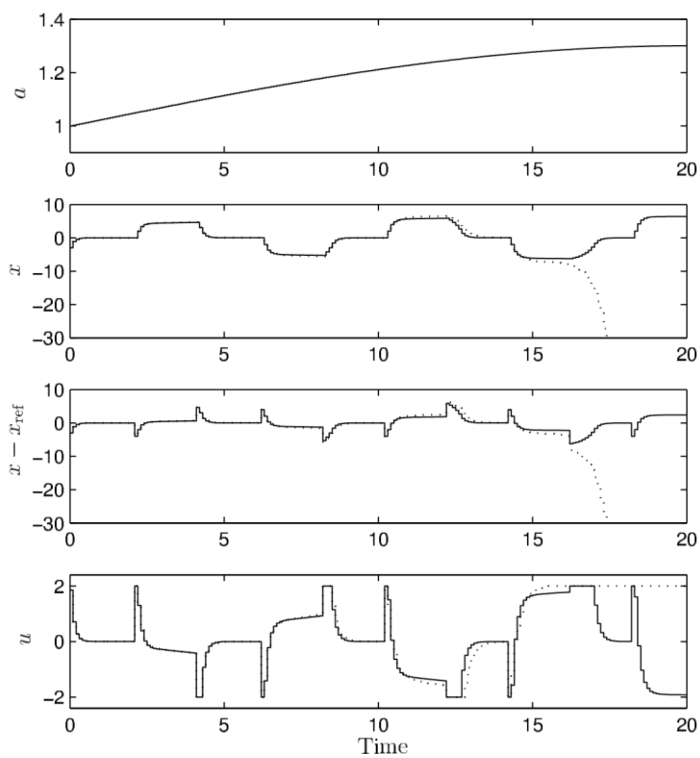

Figure 3. Simulation Results for System (6) with u as in Table II and the Proposed Compensation Method. The Top Graph Shows the Time-Varying $a(t)$, Modelled by a Sinusoidal Function. In the Other Graphs, the Dotted Lines and the Solid Lines Indicate the Responses Resulting from the Use of the Nominal MpMPC Implementation and the Use of the Robust mpMPC Implementation, Respectively. Note the Different Scale of the Two Middle Graphs from Figure 2

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4. Conclusion

The applicability of the robust mpMPC method has been examined by two case studies with low dimensional systems. Throughout the investigation we can see the improvement of the performance in the sense of robustness against time varying parameters. Thus the method can be applied to processes which suffer from slow variation of parameters due to, for example, aging or degradation.

The control method does not require any modification of the mpMPC algorithm for LTI systems [1, 5] and this method can be implemented as an auxiliary routine in the form of the add-on unit. In addition the suggested method requires only a few additional calculation steps. Thus the method is still in line with the concept of 'on-line optimisationvia off-line optimisation' of [4].

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