International Journal of Control and Automation
Vol.7, No.6 (2014), pp.1-10
http://dx.doi.org/10.14257/ijca.2014.7.6.01

Generation of Odd-Periodic Sequences of Even Length for Optimal Frame Synchronization

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Abstract

Synchronization sequences with special correlation properties can be inserted into a bit stream to correlate the timing at the transmitter and the receiver. On the basis of the ideal auto-correlation property of sequences with an odd length, a new method of constructing odd-periodic sequences of even length is proposed. The auto-correlation of a sequence has double maximum values equal in magnitude and opposite in polarity at zero and middle shifts with the lowest out-of-phase values excluding those at the middle shift. Such sequences are particularly useful for frame synchronization, since the special correlation property can be used to double-check the synchronization timing. As a special case, we can also generate these sequences using binary maximal length sequences.

Keywords: *Odd-periodic sequence, synchronization timing, lowest out-of-phase, maximal length sequence*

1. Introduction

In digital and mobile communication systems, a sequence with good correlation property is used as a synchronization sequence. Such sequences usually have extremely low out-of-phase correlation values for better synchronization performance [1-5]. If the out-of-phase auto-correlation values of a sequence are all zero, the sequence is said to have perfect auto-correlation. However, (0,1,1,1) is the only binary sequence with perfect auto-correlation [6].

In this paper, we propose a general method of generating odd-periodic sequences of length N=2n=4l+2 that possess the optimal synchronization property using a sequence (length n=2l+1, where l is a positive integer) with ideal auto-correlation. A sequence is said to possess the "ideal auto-correlation property" if the out-of-phase auto-correlation value of the sequence consists only of values "-1" and "1. The proposed sequence has the lowest out-of-phase values except for the case of middle shift. This sequence satisfies auto-correlation of the double maximum values, equal in magnitude and opposite in polarity, at zero and middle shifts. Thus, the sequences with this special correlation property can be inserted periodically into a bit stream to check the correlation between times at the transmitter and receiver [7-10]. Furthermore, this property can be used to confirm synchronization and improve the synchronization performance [11]. As a special case, we can construct optimal odd-periodic synchronization sequences, because all of the binary maximal length sequence of odd period $n = 2^r - 1,\ r \geq 2$, possess ideal auto-correlation.

ISSN: 2005-4297 IJCA

2. Definitions

Let $\mathbf{S}=(s_i)$ be a binary sequence of period *n* and let *T* denote a cyclic shift left operator such that $T\mathbf{S}=(s_1, s_2, \cdots, s_0)$. For two integers *i* and *j*, $T^i\mathbf{S}=T^j\mathbf{S}$ if $i \equiv j(\mathrm{mod} n)$. The periodic auto-correlation function is defined by [1]

$$R(\tau)=\sum_{t=0}^{n-1}(-1)^{s_t+s_{(t+\tau)\bmod n}}, \quad 0 \le \tau \le n-1, \tag{1}$$

, where $\mathrm{mod}\, n$ denotes modulo *n* and $s_t + s_{(t+\tau)\bmod n}$ is computed modulo 2. We call $R(0)$ the in-phase auto-correlation value and $R(\tau)$ $(\tau \neq 0)$ the out-of-phase auto-correlation values. The out-of-phase auto-correlation values are also called sidelobes.

Sequence **S** is said to possess the “ideal auto-correlation property” if its periodic auto-correlation function has the lowest out-of-phase auto-correlation value of “1” or “-1,” which is given by

$$R_s(\tau)=\begin{cases} n, & \tau \equiv 0(\mathrm{mod} n) \\ -1, & \tau \neq 0(\mathrm{mod} n) \end{cases} \tag{2}$$

or

$$R_s(\tau)=\begin{cases} n, & \tau \equiv 0(\mathrm{mod} n) \\ 1, & \tau \neq 0(\mathrm{mod} n) \end{cases} \tag{3}$$

Let $\mathbf{S}_1=(s_{1,i})$ and $\mathbf{S}_2=(s_{2,i})$ be binary sequences of length *n*. Two sequences are said to be “cyclically equivalent” if there exists an integer *i* such that $\mathbf{S}_1 = T^i\mathbf{S}_2$; otherwise, they are said to be “cyclically distinct.” The periodic cross-correlation function between the two sequences is defined as

$$R_{1,2}(\tau)=\sum_{t=0}^{n-1}(-1)^{s_{1,t}+s_{2,(t+\tau)\bmod n}} . \tag{4}$$

Let $\mathbf{S}=(s_i)$ be a binary sequence of period *n*. Periodic repletion, but with the reversal of the signs of alternate periods, gives the odd-periodic sequence $\mathbf{A}=(a_i)$ [12]-[13]. This means that

$$a_i=\begin{cases} s_i, & 0 \le i \le n-1 \\ \bar{s}_i, & n \le i \le 2n-1 \end{cases} \tag{5}$$

, where $\bar{s}_i=(s_i+1)\,\mathrm{mod}\,2$. The odd-periodic sequence has double maximum correlation values that are equal in magnitude and opposite in polarity at the zero and middle shifts. In this paper, the sidelobe of an odd-periodic sequence is defined as the out-of-phase auto-correlation value except at the middle shift.

3. Basic Properties

We note that $R_{1,2}(\tau)$ is simply expressed as the number of agreements minus the number of disagreements between $\mathbf{S}_1$ and $\mathbf{S}_2$ for each value of τ; it is given by

$$\begin{aligned} R_{1,2}(\tau) &= (\text{number of agreements}) - (\text{number of disagreements}) \\ &= n - 2\left\|\mathbf{S}_1 + T^{\tau}\mathbf{S}_2\right\| \end{aligned} \tag{6}$$

, where $\|\mathbf{S}\|$ is the number of "1" values in the n-tuple binary sequence $\mathbf{S}$, and $\mathbf{S}_1 + T^{\tau}\mathbf{S}_2$ is the element-by-element modulo-2 addition between two vectors $\mathbf{S}_1$ and $T^{\tau}\mathbf{S}_2$. The periodic auto-correlation function of $\mathbf{S}$ is $R(\tau) = n - 2\left\|\mathbf{S} + T^{\tau}\mathbf{S}\right\|$. The number of disagreements between sequence $\mathbf{S}$ and its shifted version $T^{\tau}\mathbf{S}$ is $\left\|\mathbf{S} + T^{\tau}\mathbf{S}\right\| = 2K$, where K is given by

$$K = \|\mathbf{S}\| - (\text{number of coincident 1's between } \mathbf{S} \text{ and } T^{\tau}\mathbf{S}). \tag{7}$$

Then, the auto-correlation function becomes [14]

$$R_s(\tau) = n - 4 \cdot K \tag{8}$$

Now, consider a sequence $\mathbf{S}$ of period $n = 2l + 1$ for positive odd integers $l = 2m + 1$. Then,

$$R_s(\tau) = n - 4K = 4(m - K + 1) - 1 \tag{9}$$

and we see that the lowest out-of-phase auto-correlation value is "-1". In the case of even integers $l = 2m$, we have

$$R_s(\tau) = n - 4K = 4(m - K) + 1 \tag{10}$$

Thus, a sequence of period $n = 2l + 1$ possessing the ideal auto-correlation property of period has the following out-of-phase auto-correlation value for positive integer l:

$$R_s(\tau) = \begin{cases} -1, \text{ for positive odd } l \\ +1, \text{ for positive even } l \end{cases} , \tau \neq 0. \tag{11}$$

4. Generation Method

In this section we derive important equations used to construct odd-periodic sequences of even length with the lowest sidelobes. From the definition and basic properties of sequences, we obtain the following lemmas:

Lemma 1: Let $\mathbf{A} = (a_i)$ be a sequence of even period. Further, let $\mathbf{S}_1 = (s_{1,i})$ and

$\mathbf{S}_2 = (s_{2,i})$ be sequences obtained by decimating $\mathbf{A}$ and $T\mathbf{A}$ by 2, respectively. Then, we have

$$R_a(2\tau) = R_1(\tau) + R_2(\tau) \tag{12}$$

$$R_a(2\tau+1) = R_{1,2}(\tau) + R_{2,1}(\tau+1) \tag{13}$$

, where $R_1(\tau)$ and $R_2(\tau)$ are periodic auto-correlation functions of $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively.

Proof: As two sequences $\mathbf{S}_1 = (s_{1,i})$ and $\mathbf{S}_2 = (s_{2,i})$ are obtained by decimating $\mathbf{A}$ and $T\mathbf{A}$ by 2 and the even period of sequence $\mathbf{A}$ is $N = 2n$, we see that $s_{1,z(\mathrm{mod}\, n)} = a_{2z(\mathrm{mod}\, N)}$ and $s_{2,z(\mathrm{mod}\, n)} = a_{(2z+1)(\mathrm{mod}\, N)}$, where z is an integer. Thus, the auto-correlation functions of the sequences become

$$R_1(\tau) = \sum_{z=0}^{n-1} (-1)^{s_{1,z} + s_{1,(z+\tau)(\mathrm{mod} n)}} = \sum_{z=0}^{n-1} (-1)^{a_{2z} + a_{2(z+\tau)(\mathrm{mod} N)}}$$

$$R_2(\tau) = \sum_{z=0}^{n-1} (-1)^{s_{2,z} + s_{2,(z+\tau)(\mathrm{mod} n)}} = \sum_{z=0}^{n-1} (-1)^{a_{(2z+1)} + a_{(2z+1+2\tau)(\mathrm{mod} N)}}$$

, and

$$R_1(\tau) + R_2(\tau) = \sum_{i=0}^{N-1} (-1)^{a_i + (i+2\tau)(\mathrm{mod}\, N)} = R_a(2\tau).$$

Using the definition of the periodic cross-correlation function, we obtain

$$R_{1,2}(\tau) = \sum_{z=0}^{n-1} (-1)^{s_{1,z} + s_{2,(z+\tau)(\mathrm{mod} n)}} = \sum_{z=0}^{n-1} (-1)^{a_{2z} + a_{(2z+2\tau+1)(\mathrm{mod} N)}}$$

$$R_{2,1}(\tau+1) = \sum_{z=0}^{n-1} (-1)^{s_{2,z} + s_{1,(z+\tau+1)(\mathrm{mod} n)}} = \sum_{z=0}^{n-1} (-1)^{a_{(2z+1)} + a_{(2z+1+2\tau+1)(\mathrm{mod} N)}}$$

and by summing two functions, we obtain

$$R_{1,2}(\tau) + R_{2,1}(\tau+1) = \sum_{i=0}^{N-1} (-1)^{a_i + a_{(i+2\tau+1)(\mathrm{mod} N)}} = R_a(2\tau+1)$$

□

Lemma 1 explains the relationships between the auto-correlation function of a sequence $\mathbf{A}$ with an even period and correlation functions of two sequences given by decimating sequences $\mathbf{A}$ and $T\mathbf{A}$ by 2.

Lemma 2: Let $\mathbf{A}=(a_i)$ be a sequence of even period *N*. Let $\mathbf{S}_1=(s_{1,i})$ and $\mathbf{S}_2=(s_{2,i})$ be sequences obtained by decimating $\mathbf{A}$ and $T\mathbf{A}$ by 2. If $\mathbf{S}_2$ is a complemented version of $T^j\mathbf{S}_1$, then the following relationship between the elements of sequence $\mathbf{A}$ holds.

$$a_i=\begin{cases}\overline{a_{(i-2j+1)(\bmod N)}}, & \text{for odd } i\\ \overline{a_{(i+2j-1)(\bmod N)}}, & \text{for even } i\end{cases} \tag{14}$$

Proof: As two sequences $\mathbf{S}_1=(s_{1,i})$ and $\mathbf{S}_2=(s_{2,i})$ are obtained by decimating $\mathbf{A}$ and $T\mathbf{A}$ by 2 (the even period of sequence $\mathbf{A}$ is given by $N=2n$), we find that $s_{1,z(\bmod n)}=a_{2z(\bmod N)}$ and $s_{2,z(\bmod n)}=a_{(2z+1)(\bmod N)}$, where *n* and *z* are integers. Now, assume that $\mathbf{S}_2$ is a complemented version of $T^j\mathbf{S}_1$, then $s_{2,z}=\overline{s_{1,(z+j)(\bmod n)}}$ and $s_{1,z}=\overline{s_{2,(z-j)(\bmod n)}}$. Hence we have the following equations:

$$a_{2z}=s_{1,z}=\overline{s_{2,(z-j)(\bmod n)}}=\overline{a_{(2z-2j+1)(\bmod N)}}$$

$$a_{2z+1}=s_{2,z}=\overline{s_{1,(z+j)(\bmod n)}}=\overline{a_{(2z+2j)(\bmod N)}}=\overline{a_{(2z+1+2j-1)(\bmod N)}}$$

If $i=2z$ or $i=2z+1$, we obtain (14). □

Now, assume that *n* is a positive odd integer and the period of sequence $\mathbf{A}$ is $N=2n$. As a special case, if the number of cyclic shifts is given by $j=(n+1)/2$, then $2j-1=n=N/2$ and (14) becomes

$$a_i=\overline{a_{(i+n)(\bmod N)}},\quad \text{for all } i \tag{15}$$

From the definition in (5), we find that a sequence satisfying (15) is an odd-periodic sequence.

Theorem 1: Let $\mathbf{S}=(s_i)$ be an *n*-tuple binary sequence over GF(2) with an ideal auto-correlation function, where $n=2l+1$, $l=1,2,3,\cdots$. If a sequence $\mathbf{A}=(a_i)$ of length $N=2n$ is constructed using the following relationships,

$$a_{2z}=s_z \tag{16}$$

$$a_{(2z+1)}=\overline{s_{(z+(n+1)/2)(\bmod n)}} \tag{17}$$

, $z=0,1,\cdots,n-1$, then sequence $\mathbf{A}$ has the following optimal synchronization property:

$$R_a(\tau) = \begin{cases} N, \tau \equiv 0 (\mathrm{mod} N) \\ -N, \tau \equiv n (\mathrm{mod} N) \\ 2, \tau = \mathrm{odd}, \tau \neq n (\mathrm{mod} N) \\ -2, \tau = \mathrm{even}, \tau \neq 0 (\mathrm{mod} N) \end{cases} \quad \text{for positive odd } l \tag{18}$$

$$R_a(\tau) = \begin{cases} N, \tau \equiv 0 (\mathrm{mod} N) \\ -N, \tau \equiv n (\mathrm{mod} N) \\ -2, \tau = \mathrm{odd}, \tau \neq n (\mathrm{mod} N) \\ 2, \tau = \mathrm{even}, \tau \neq 0 (\mathrm{mod} N) \end{cases} \quad \text{for positive even } l \tag{19}$$

Proof: Let $\mathbf{S}_1 = (s_{1,i})$ and $\mathbf{S}_2 = (s_{2,i})$ be sequences obtained by decimating $\mathbf{A}$ and $T\mathbf{A}$ by 2. Then, we find that $s_{1,z(\mathrm{mod}\, n)} = a_{2z(\mathrm{mod}\, N)}$ and $s_{2,z(\mathrm{mod}\, n)} = a_{(2z+1)(\mathrm{mod}\, N)}$. From the conditions in (16) and (17), we see that $\mathbf{S} = \mathbf{S}_1$ and sequence $\mathbf{S}_2$ is a complemented version of $T^{(n+1)/2}\mathbf{S}_1$. Therefore, using lemma 2 and (15), we obtain

$$\begin{aligned} R_a(j+n) &= \sum_{i=0}^{N-1} (-1)^{a_i + a_{(i+j+n)(\mathrm{mod} N)}} \\ &= -\sum_{i=0}^{N-1} (-1)^{a_i + a_{(i+j)(\mathrm{mod} N)}} \\ &= -R_a(j) \end{aligned} \tag{20}$$

Further, using lemma 1 and (20), we obtain

$$\begin{aligned} R_{1,2}(j) &= \sum_{i=0}^{n-1} (-1)^{s_{1,i} + s_{2,(i+j)(\mathrm{mod} n)}} \\ &= -\sum_{i=0}^{n-1} (-1)^{s_{1,i} + s_{1,(i+j+l+1)(\mathrm{mod} n)}} \\ &= -R_s(j+l+1) = -\tfrac{1}{2} R_a(2j+n+1) \\ &= \tfrac{1}{2} R_a(2j+1) \end{aligned} \tag{21}$$

From (21) and lemma 1, we see that

$$R_{1,2}(j) = R_{2,1}(j+1) = -R_s(j+l-1) \tag{22}$$

and thus, we obtain the following more general equations:

$$R_a(2j) = 2R_s(j) \tag{23}$$

$$R_a(2j+1) = -2R_s(j+l+1) \tag{24}$$

The auto-correlation function values of $\mathbf{A}$ at even or odd time shifts are can be obtained from the auto-correlation function of $\mathbf{S}$. For positive odd values of l, we see that sequence **A** possesses the optimal odd-periodic synchronization property in (18), because sequence **S** has the ideal auto-correlation property in (2). Similarly, for positive even values of l, we obtain (19). □

As the primitive polynomial exists for every degree, and each binary maximal length sequence generated by a primitive polynomial of degree r has the period $n = 2^r - 1$ [15], we can produce the following corollary.

Corollary 1: Let **S** be a maximal length sequence over GF(2) of period $n = 2^r - 1$, $r \geq 2$. Then, the auto-correlation function of **A** with period $N = 2n$, which is generated by (16) and (17), is given by (18).

Example: The results of an exhaustive computer search for all of the cyclically distinct odd-periodic sequences possessing the optimal synchronization property are given in Table 1, for $N = 2n = 4l + 2$ and $n = 2^r - 1$ to N = 38. As a special case, we can generate the odd-periodic sequence $\mathbf{A} = (1,0,1,1,0,1,1,1,0,0,1,1,1,1,1,0,1,0,0,1,0,0,0,1,1,0,0,0,0,0)$ using the maximal length sequence $\mathbf{S} = (1,1,0,1,0,1,1,1,1,0,0,0,1,0,0)$ with the primitive polynomial $g(x) = x^4 + x + 1$. □

Table 1. Odd-periodic Sequences of Length from 6 to 34 with Lowest Out-of-Phase Values

l	n	N	r	**S**	**A**
1	3	6	2	101 010	100011 011100
2	5	10		00001 11110	0100010111 1011101000
3	7	14	3	0100111 1011000 0100011 1011100	00100001101111 11011110010000 01100001001111 10011110110000
4	9	18		Not found	Not found
5	11	22		00010010111 11101101000 01110001011 10001110100	0001001000011101101111 1110110111100010010000 0110111000010010001111 1001000111101101110000
6	13	26		0100000100011 1011111011100 0001000001011 1110111110100 0101100111111 1010011000000 0010100000011 1101011111100	00110101000001100101011111 11001010111110011010100000 01010011000001010110011111 10101100111110101001100000 00100010100001101110101111 11011101011110010001010000 01011101100001010001001111 10100010011110101110110000
7	15	30	4	011100001010011 100011110101100 001010000111011 110101111000100	001110110100000110001001011111 110001001011111001110110100000 010010001100000101101110011111 101101110011111010010001100000
8	17	34		Not found	Not found
9	19	38		0010011000010101111 1101100111101010000 0011011000010101111 1100100111101010000	01001100011010000001011001110010111111 10110011100101111110100110001101000000 01001110011010000001011000110010111111 10110001100101111110100111001101000000

5. Conclusions

A sequence with extremely low out-of-phase correlation is inserted into a bit stream to check the time of the transmitter and receiver using correlations. If the out-of-phase auto-correlation values of a sequence are all zero, the sequence is said to have perfect auto-correlation. However, (0,1,1,1) is the only binary sequence with perfect auto-correlation. In this paper we have proposed a general method of generating odd-periodic sequences of length $N=2n=4l+2$ with lowest out-of-phase values using a sequence (length $n=2l+1$, where l is a positive integer) with ideal auto-correlation. For example, the results of an exhaustive computer search for all of the cyclically distinct odd-periodic sequences possessing the optimal synchronization property are given in table 1, for $N=2n=4l+2$ and $n=2^r-1$ to N = 38. The sequences can be particularly useful in designing a synchronization device of mobile communication and digital communication systems.

Acknowledgements

This paper was supported by Research Fund, Kumoh National Institute of Technology.

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