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Transfer Function Approximation Via Rationalized Haar Transform in Frequency Domain

Joon-Hoon Park

Korea National University of Transportation, Republic of Korea
jhpark@ut.ac.kr

Abstract

For system analysis and design purposes, it is meaningful discussion to define control system pole status whether a pole is significant or not. If a pole is less important, it can be canceled from the transfer function of system and system order is reduced. In this paper a method for system order reduction of transfer function using Rationalized Haar functions based on approximation and transform algorithm is presented. The Haar function set forms a complete set of orthogonal rectangular functions such as Walsh and block pulse functions. But the Haar functions have some disadvantages of calculation because of including irrational numbers such as $\pm\sqrt{2^p}, (p = 1, 2, ..)$. *The Rationalized Haar functions were introduced by M. Ohkita to overcome these disadvantages. The Rationalized Haar functions constitute of rational numbers only. The applied method to solve the system order reduction of transfer function problem is superior to conventional numerical methods.*

Keywords: *Rationalized Haar functions, transform, transfer function approximation, order reduction, frequency domain*

1. Introduction

The classical method of modeling linear control system is to use transfer function to represent input-output relations between system variables. As well as mathematical model in the form of linear ordinary differential equations, the transfer function is of great importance, particularly in control system analysis and design. A high order control system often contains insignificant or less important poles relative to dominant poles that have little effect on the operating characteristics and responses of the system. Therefore, we can consider neglecting of insignificant or less important poles of transfer function and reducing the system order. And thus, it is desirable to find a low order approximating system from given a high order system. Approximating system could reduce efforts to analysis and design than the high order system case. We can neglect insignificant pole and reduce system order to analysis and design system more easily. So that, there have been numerous attempts at approximating control system by low order system. The Rationalized Haar functions that were suggested by M. Ohkita are based on the Haar functions. The Haar functions were introduced by the Hungarian mathematician Alfred Haar and were established rather earlier than the Walsh functions that were completed from the incomplete orthogonal function of Rademacher in 1923 [1]. Alfred Haar described a set of orthogonal functions, each taking essentially only two values and providing a simple convergence and expansion of system. The Rationalized Haar functions form an orthogonal and orthonormal system of periodic square waves. If we consider the time base to be defined as $0 \leq t \leq 1$ then, the Rationalized Haar functions is described as complete orthogonal functions. The Rationalized Haar functions that are used in this paper are very valuable to approximation of transfer function for simple system analysis

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and design. The waveform of first eight Haar and Rationalized Haar functions are shown in Figure 1 and 2 respectively.

2. Haar and Rationalized Haar Functions

Since 1970s, orthogonal functions such as Walsh functions, block pulse functions and Haar functions have been developed and used for solving analysis and optimization problems of control systems [2]. The Haar functions form an orthogonal and orthonormal system of periodic square waves. The amplitude value of these square waves do not have uniform values, but assume a limited set of values, $0, \pm 1, \pm\sqrt{2}$ etc. Thus the Haar functions include irrational numbers [3]. If we consider the time base to be defined as $0 \le t \le 1$ then, the set of Haar functions is described in equation (2.1) ~ (2.4).

$$Har(0,t) = 1 \quad for \;\; 0 \le t \le 1 \tag{2.1}$$

$$Har(1,t) = \begin{cases} 1 & for \;\; 0 \le t \le \frac{1}{2} \\ -1 & for \;\; \frac{1}{2} \le t \le 1 \end{cases} \tag{2.2}$$

$$Har(2,t) = \begin{cases} \sqrt{2} & for \;\; 0 \le t \le \frac{1}{4} \\ -\sqrt{2} & for \;\; \frac{1}{4} \le t \le \frac{1}{2} \\ 0 & for \;\; \frac{1}{2} \le t \le 1 \end{cases} \tag{2.3}$$

$$Har(2^p + n, t) = \begin{cases} \sqrt{2^p} & for \;\; \frac{n}{2^p} \le t \le \frac{n+\frac{1}{2}}{2^p} \\ -\sqrt{2^p} & for \;\; (n+\frac{1}{2})/2^p \le t \le (n+1)/2^p \\ 0 & for \;\; elsewhere \end{cases} \tag{2.4}$$

where $p = 0, 1, 2 \ldots . \log_2 \frac{m}{2}, n = 0, 1 \ldots 2^p - 1$

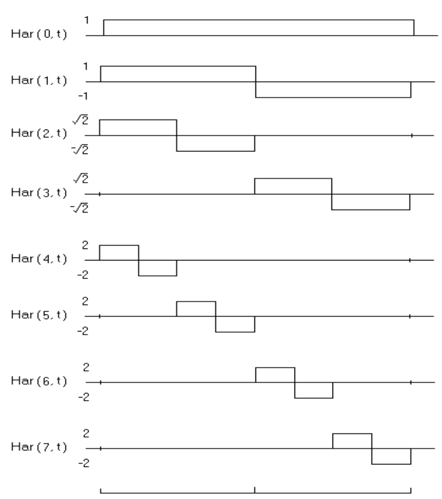

Figure 1. The first eight Haar functions

The Rationalized Haar functions were introduced by M. Ohkita to overcome the disadvantages of original Haar functions that include irrational numbers [4]. On the other hand, the Rationalized Haar functions constitute of rational numbers only.

$$RHar(0,t) = 1 \quad for \;\; 0 \le t \;\le 1 \tag{2.5}$$

$$RHar(k,t) = \begin{cases} +1 \quad for \quad \frac{n}{2^p} \le t \le \frac{n+\frac{1}{2}}{2^p} \\ -1 \;\; for \;\; (n+\frac{1}{2})/2^p \le t \le (n=1)/2^p \\ 0 \;\; for \;\; elsewhere \end{cases} \tag{2.6}$$

where $k=2^p+n$, $p=0, 1, 2 \ldots \log_2 \frac{m}{2}$, $n=0, 1 \ldots 2^p-1$

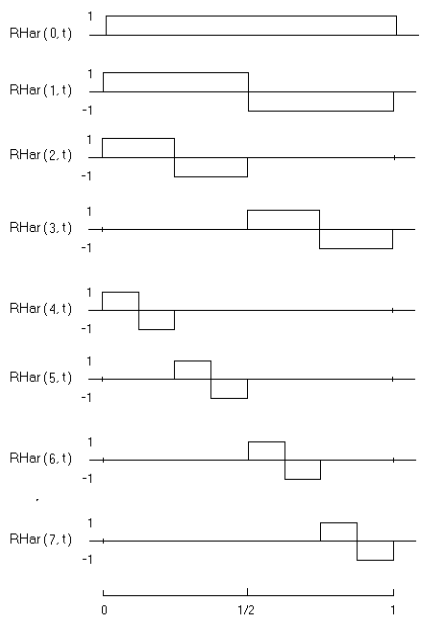

Figure 2. The first eight Rationalized Haar functions

3. Rationalized Haar Transform

The Rationalized Haar functions have orthogonality as follows,

$$\int_0^1 RHar(i,t)RHar(j,t)dt = \begin{cases} 1, \; i=j=0 \\ 0, \; i \ne j \end{cases} \tag{3.1}$$

$$\int_0^1 RHar(i,t)RHar(j,t)dt = \begin{cases} 2^{-p}, & i=j \\ 0, & i \ne j \end{cases} \tag{3.2}$$

where $i=2^p+n$, $p=0, 1, 2 \ldots \log_2 \frac{m}{2}$, $n=0, 1 \ldots 2^p-1$

The Rationalized Haar functions *RHar(t)* is closed set. Thus, every signal *f(t)* which is absolutely integral in $t \in [0,\ 1)$ can be expanded in an infinite series of the Rationalized Haar functions and transform.

$$f(t) = \sum_{k=0}^{\infty} f_k RHar(k,t) \tag{3.3}$$

where f_k is the *k*th sequentially ordered coefficient of the Rationalized Haar functions expansion of function *f(t)* and *RHar(k,t)* is the *k*th ordered Rationalized Haar functions. Now, we can get the approximation of *f(t)* as a finite series of the Rationalized Haar transform and matrix.

$$f(t) = \sum_{k=0}^{m-1} f_k RHar(k,t) = F^T \emptyset_R(t) \tag{3.4}$$

where F is coefficient vector of *f(t)*, $\emptyset_R(t)$ is its Rationalized Haar functions vector and T denotes transposition.

$$F = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{m-1} \end{bmatrix} \qquad \emptyset_R(t) = \begin{bmatrix} RHar(0,t) \\ RHar(1,t) \\ \vdots \\ RHar(m-1,t) \end{bmatrix} \tag{3.5}$$

For example, let apply the Rationalized Haar transform to a function *f(t)=t* and obtain its coefficients F. In this case, *i*th coefficient F_i is defined as follow and the result is shown in equation (3.7) and Figure 3.

$$F_i = 2^p \int_0^1 tRHar(i,t), \begin{cases} p = 0,\ 0 \le i \le 1 \\ p = 1,\ 2 \le i \le 3 \\ p = 2,\ 4 \le i \le 8 \end{cases} \tag{3.6}$$

$$f(t) \cong 0.5RHar(0,t) - 0.25RHar(1,t) - 0.125RHar(2,t) - 0.125RHar(3,t)$$

$$-0.0625RHar(4,t) - 0.0625RHar(5,t) - 0.0625RHar(6,t) - 0.06125RHar(7,t) \tag{3.7}$$

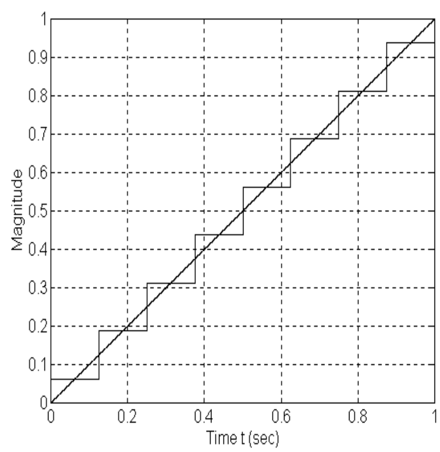

Figure 3. Rationalized Haar transform of *f(t)=t* with *i=8*

If the function is given in the form of data, discrete transform method of the Rationalized Haar functions is applied as shown in equation (3.8)[5].

$$f_i^* = \sum_{k=0}^{m-1} RHar(k,i)F_i \tag{3.8}$$

where *i=0, 1, 2…m-1*

Equation (3.9) is matrix expressions of equation (3.8). *T* denotes transpose of a matrix.

$$f_i^* = \emptyset_{Ri}^T F_i \tag{3.9}$$

$$f_i^* = \begin{bmatrix} f_0^* \\ f_1^* \\ \vdots \\ f_{m-1}^* \end{bmatrix}$$

$$\emptyset_{Ri} = \begin{bmatrix} RHar(0,0) & RHar(0,1) & & RHar(0,m-1) \\ RHar(1,0) & RHar(1,1) & \cdots & RHar(1,m-1) \\ \vdots & \vdots & & \vdots \\ RHar(m-1,0) & RHar(m-1,1) & & RHar(m-1,m-1) \end{bmatrix}$$

$$F_i = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{m-1} \end{bmatrix} \tag{3.10}$$

4. Approximation of Transfer Function using the Rationalized Haar Transform

Given a high-order transfer function $M_H(s)$, we can find a low-order transfer function $M_L(s)$ as an approximation. Let the high-order system transfer function be written by equation (4.1) and the transfer function of the approximating low-order system be represented by equation (4.2). A method of approximating high-order system by low-order system is based on one in the sense that the frequency responses of two systems are similar [6].

$$M_H(\mathrm{s}) = \mathrm{K}\frac{1 + a_1 s + a_2 s^2 + \cdots + a_m s^m}{1 + \mathrm{b}s + \mathrm{b}s^2 + \cdots + b_n s^n} \tag{4.1}$$

$$M_L(\mathrm{s}) = \mathrm{K}\frac{1 + c_1 s + c_2 s^2 + \cdots + c_q s^q}{1 + d_1 + d_2 s^2 + \cdots + d_p s^p} \tag{4.2}$$

where $n \geq m$, $n \geq p \geq q$. And $s=jw$ is applied to the above equations, thus we can obtain equation (4.3) and (4.4) respectively.

$$M_H(jw) = k\frac{(1 - a_2 w^2 + a_4 w^4 - \cdots) + jw(a_1 - a_3 w^2 + a_5 w^4 - \cdots)}{(1 - b_2 w^2 + b_4 w^4 - \cdots) + jw(b_1 - b_3 w^2 + b_5 w^4 - \cdots)}$$

$$= k\frac{\alpha(w) + jw\beta(w)}{\gamma(w) + jw\delta(w)} \tag{4.3}$$

$$M_L(jw) = \ k\frac{(1 - c_2w^2 + c_4w^4 - \cdots) + jw(c_1 - c_3w^2 + c_5w^4 - \cdots)}{(1 - dw^2 + d_4w^4 - \cdots) + jw(d_1 - d_3w^2 + d_5w^4 - \cdots)}$$

$$= k\frac{\varepsilon(w) + jw\mu(w)}{\rho(w) + jw\tau(w)} \tag{4.4}$$

$$\varepsilon(w) = 1 - c_2w^2 + c_4w^4 - \cdots$$

$$\mu(w) = c_1 - c_3w^2 + c_5w^4 - \cdots$$

$$\rho(w) = 1 - d_2w^2 + d_4w^4 - \cdots$$

$$\tau(w) = d_1 - d_3w^2 + d_5w^4 - \cdots \tag{4.5}$$

In this case, the zero frequency gain k of the two transfer functions is the same. Thus, we can obtain the criterion of finding the low-order $M_L(s)$, given $M_H(s)$, is that the following relation of equation (4.6) should be satisfied as closely as possible.[7]

$$\frac{|M_H(jw)|^2}{|M_L(jw)|^2} = 1 \tag{4.6}$$

Equating both side of equation (4.5) and satisfying the condition, similar relationships can be obtained for coefficients of equation (4.3) and (4.4). Now, we can expand equation (4.5) in a series form of equation (4.7) using the Rationalized Haar functions and its transform.[8] Thus, coefficients of equation (4.7) can be written in equation (4.8).

$$\varepsilon(\mathrm{w}) = E_0RHar_0(w) + E_1RHar_1(w) + \cdots + E_nRHar_n(w)$$

$$\mu(\mathrm{w}) = M_0RHar_0(w) + M_1RHar_1(w) + \cdots + M_nRHar_n(w)$$

$$\rho(\mathrm{w}) = P_0RHar_0(w) + P_1RHar_1(w) + \cdots + P_nRHar_n(w)$$

$$\tau(\mathrm{w}) = T_0RHar_0(w) + T_1RHar_1(w) + \cdots + T_nRHar_n(w) \tag{4.7}$$

$$E_i = 2^p\int_0^1 \alpha(w)RHar_i(w)dw$$

$$M_i = 2^p\int_0^1 \beta(w)RHar_i(w)dw$$

$$P_i = 2^p\int_0^1 \gamma(w)RHar_i(w)dw$$

$$T_i = 2^p\int_0^1 \delta(w)RHar_i(w)dw \tag{4.8}$$

Applying the Rationalized Haar transform and similar relationships, we can get the coefficients of low order transfer function $M_L(jw)$ from high order transfer function $M_H(jw)$ conveniently. This method is useful to approximate high-order transfer function as low-order system.

5. Examples

Consider the following closed loop transfer function with unity feedback control system is given as,

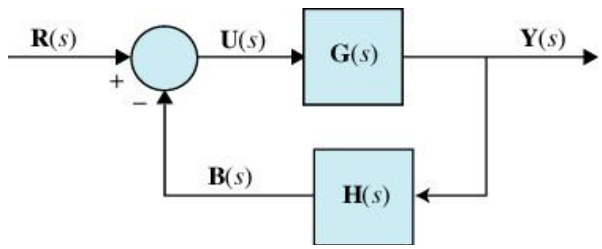

Figure 4. Block diagram of basic feedback control system

$$\mathrm{G(s)} = \frac{1}{0.05s^3 + 0.6s^2 + 1.1s}\ , H(s) = 1 \tag{5.1}$$

Thus, the system transfer function is written as third order system in equation (5.2).

$$M_H(s) = \frac{1}{1 + 1.1s + 0.6s^2 + 0.05s^3} \tag{5.2}$$

From equation (5.2), approximated low-order system $M_L(s)$ is the second order system. We can compare the results of proposed method in this paper with the results of numerical method. Equation (5.3) is the second order system that is obtained using a numerical method.

$$M_{LN}(s) = \frac{1}{1 + m_1 s + m_2 s^2} = \frac{1}{1 + 1.01s + 0.5s^2} \tag{5.3}$$

Now, we apply the Rationalized Haar functions and its transform that is suggested in this paper to get the low-order transfer function $M_L(s)$ from the original high-order transfer function M_H(s). For this purpose, we can get $M_H(jw)$ in equation (5.4) and $M_L(jw)$ in equation (5.5) using equation (4.3) and (4.4).

$$M_H(jw) = \frac{1}{1 + 1.1jw + 0.6(jw)^2 + 0.05(jw)^3}$$

$$= \frac{1}{(1 - 0.6w^2) + jw(1.1 - 0.05w^2)} \tag{5.4}$$

$$M_{LR}(jw) = \frac{1}{1 + m_1 jw + m_2 (jw)^2}$$

$$= \frac{1}{(1 - m_2 w^2) + jwm_1} \tag{5.5}$$

From similar relationships of equation (4.6), m_1 and m_2 are represented as follows.

$$m_1 = 1.1 - 0.05w^2, \ \ m_2 = 0.6 \tag{5.6}$$

And also, equation (5.7) can be obtained by equation (4.4) and (5.6).

$$\varepsilon(w) = 1, \mu(w) = 0, \ \ \rho(w) = 0.6$$

$$\tau(w) = 1.1 - 0.05w^2 \tag{5.7}$$

In order to determine the coefficients of $\tau(w)$, we can apply the Rationalized Haar functions and transform to $\tau(w)$, then the coefficient of equation (5.7) is obtained by equation (5.8).

$$T_i = 2^p \int_0^1 RHar_i(w)(1.1 - 0.05w^2)dw \tag{5.8}$$

The results of the coefficients of $\tau(w)$ based on equation (5.8) is shown in Table 1. In this case, the term of functions is set to 8.

Table 1. Coefficients of $\tau(w)$ by the Rationalized Haar transform with n=8

coefficients of $\tau(w)$	Results
T_0	1.0834
T_1	0.0124
T_2	0.0032
T_3	0.0094
T_4	0.0000
T_5	1.6524
T_6	0.5540
T_7	0.0056

Thus, we can define the discrete values of coefficients using the Rationalized Haar operational matrix and equation (3.9). It is shown in Table 2 and approximated low-order system $M_L(s)$ is obtained in equation (5.9).

$$M_{LR}(s) = \frac{1}{1 + 1.11s + 0.6s^2} \tag{5.9}$$

Table 2. Discrete coefficients with n=8

Discrete coefficients	Results
f_{T0}^{*}	3.3204
f_{T1}^{*}	1.1036
f_{T2}^{*}	1.0832
f_{T3}^{*}	1.0928
f_{T4}^{*}	1.0710
f_{T5}^{*}	-1.0062
f_{T6}^{*}	1.6524
f_{T7}^{*}	0.5484

The result analyses in frequency response are shown in Figure 5 to 7. Transfer function (TF) analysis between the original third-order system $M_H(s)$ and the numerical second-order system $M_{LN}(s)$ is shown in Figure 5. The dashed graph shows the original third-order system and the dotted graph displays the numerical second-order system. Figure 6 shows analysis of between the original third-order system $M_H(s)$ and the proposed second-order system $M_{LR}(s)$. In the figure, solid line stands for the proposed second-order system. And in Figure 7, frequency response analysis of original, numerical and proposed method is shown. Figure 8 is the analysis of proposed method for magnitude and phase.

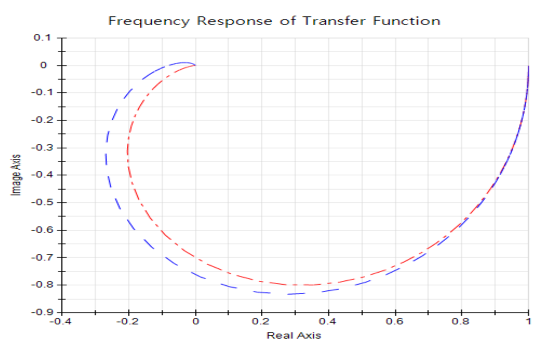

Figure 5. TF analysis between $M_H(s)$ and $M_{LN}(s)$

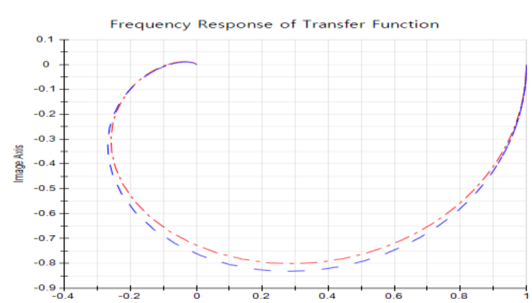

Figure 6. TF analysis between $M_H(s)$ and $M_{LR}(s)$

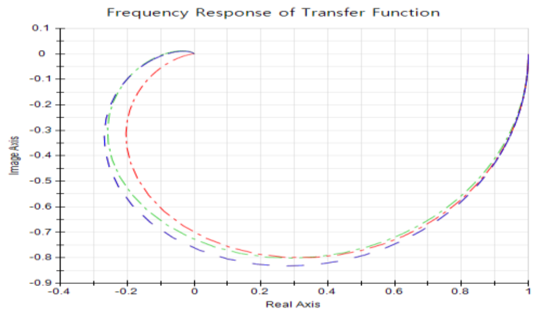

Figure 7. TF analysis between $M_H(s)$, $M_{LN}(s)$ and $M_{LR}(s)$

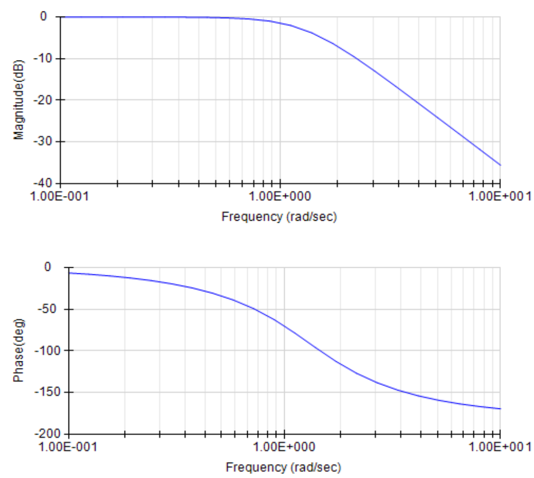

Figure 8. Magnitude and phase analysis of $M_{LR}(s)$

6. Conclusions

We have seen that the Rationalized Haar functions and its transform can be used to approximate system order reduction of transfer functions. System order reduction is useful for system analysis and design because insignificant pole can be neglected with regard to the transient response. In frequency domain analysis, the transfer function by the proposed method is almost the same as the original third-order system. Moreover, the result of proposed method is better than the result of numerical method in terms of accuracy. Thus, the presented method for approximating transfer function in this paper is valuable. The proposed method is quite simple and accurate to implement and has, therefore, obvious advantages in practical situations. And we can obtain more accurate results by increasing the term of transform and expansion. As a result, this method is useful to design and analysis control systems.

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Authors

Joon-Hoon Park received the B.S. degree in electrical engineering from SungKyunKwan University in 1981 and the M.S. and the Ph.D. degree in control engineering from SKK Univ., in 1983 and 1992, respectively. He was a senior researcher of the R&D Center, Samsung Electronics Co. from 1982 to 1990. And he was a Post-Doc. Fellow of the Leicester University in UK and the University of Western Ontario, Canada. Since 1991, he is currently professor of the Department of Control and Instrumentation Eng., Korea National University of Transportation. He is interested in optimal control theory and applications