International Journal of Control and Automation
Vol.7, No.3 (2014), pp.195-212
http://dx.doi.org/10.14257/ijca.2014.7.3.19

Actions on a Railway Track, due to an Isolated Defect

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Abstract

The rail running table imposes to the vehicle a forced vibration. It is not smooth but instead it comprises a lot of faults that give to the rail running table a random surface. Furthermore under the primary suspension there are the Unsprung Masses which act without any dumping directly on the track panel. On the contrary the Sprung (Suspended) Masses that are cited above the primary suspension of the vehicle, act through a combination of springs and dumpers on the track. A part of the track mass is also added to the Unsprung Masses, which participates in their motion. The defects with long wavelength, which play a key role, on the dynamic component of the acting loads on the railway track, are analyzed using the second order differential equation of motion. A parametric investigation is performed for the case of an isolated defect

Keywords: *railway track; dynamic stiffness; actions/ loads; deflection; subsidence; eigenperiod; forcing period*

1. Introduction – Loads on the Railway Track

The railway track is usually modeled as a continuous beam on elastic support. Train circulation is a random dynamic phenomenon and, depending on the different frequencies of the loads it imposes, there is a corresponding response of the track superstructure. At the instant when an axle passes from the location of a sleeper, a random dynamic load is applied on the sleeper. The theoretical approach for the estimation of the dynamic loading of a sleeper requires the analysis of the total load acting on the sleeper to individual component loads-actions, which, in general, can be divided into: (a) the static component of the load, and the relevant reaction/action per support point of the rail (sleeper) and (b) the dynamic component of the load, and the relevant reaction/action per support point of the rail (sleeper). The static component of the load on a sleeper, in the classical sense, refers to the load undertaken by the sleeper when a vehicle axle at standstill is situated exactly on top of the sleeper. For dynamic loads with low frequencies the load is essentially static. The static load is further analyzed into individual component loads: the static reaction/action on a sleeper due to wheel load and the semi-static reaction/action due to cant deficiency [1]. The dynamic component of the load of the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track" (Figure 1), and on the excitation caused by the vehicle's motion on the track. The response of the track to the aforementioned excitation results in the increase of the static loads on the superstructure. The dynamic load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. In order to formulate the

ISSN: 2005-4297 IJCA

theoretical equations for the calculation of the dynamic component of the load, the statistical probability of exceeding the calculated load -in real conditions- should be considered, so that the corresponding equations would refer to the standard deviation (variance) of the load [1, 2]. In the present paper the dynamic component of the acting loads is investigated through the second order differential equation of motion of the Non Suspended Masses of the Vehicle and specifically the transient response of the reaction/ action on each support point (sleeper) of the rail.

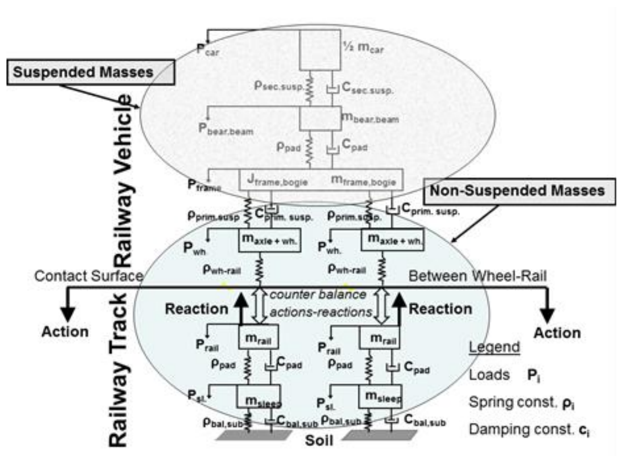

Figure 1. "Railway Vehicle - Railway Track" as an Ensemble of Springs and Dashpots

2. Static and Semi-static Components of the Actions and Reactions

The most widely used theory (referred to as the Zimmermann theory or formula [3]) based on Winkler analysis [4] examines the track as a continuous beam on elastic support whose behavior is governed by the following equation [5]:

$$\frac{d^4 y}{dx^4} = -\frac{1}{E \cdot J} \cdot \frac{d^2 M}{dx^2} \tag{1}$$

where y is the deflection of the rail, M is the bending moment, J is the moment of inertia of the rail, and E is the modulus of elasticity of the rail. From the formula above it is derived that the reaction of a sleeper R_{static} is (since the load is distributed along the track over many sleepers):

$$R_{stat} = \frac{Q_{wheel}}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow \frac{R_{stat}}{Q_{wheel}} = \bar{A} = \bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \tag{2}$$

where Q_{wheel} the static wheel load, ℓ the distance among the sleepers, E and J the modulus of elasticity and the moment of inertia of the rail, R_{stat} the static reaction/action on the sleeper, and ρ reaction coefficient of the sleeper which is defined as: $\rho = R/y$, and is a quasi-coefficient of the track elasticity (stiffness) or a spring constant of the track. $\bar{A} = \bar{A}_{stat}$ equals to R_{stat}/Q_{wheel}, which is the percentage of the acting (static) load of the wheel that the sleeper undertakes as (static) reaction. In reality, the track consists of a sequence of materials –in the vertical axis–

(substructure, ballast, sleeper, elastic pad/ fastening, rail), that are characterized by their individual coefficients of elasticity (static stiffness coefficients) ρ_i (Figure 2).

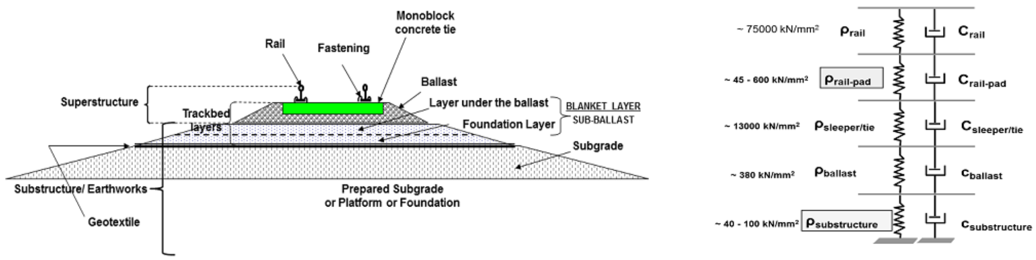

Figure 2. Cross-section of Ballasted Track, Characteristic Values of Static Stiffness Coefficients

$$\rho_i = \frac{R}{y_i} \Rightarrow y_i = \frac{R}{\rho_i} \Rightarrow y_{total} = \sum_{i=1}^{\nu} y_i \Rightarrow y_{total} = \sum_{i=1}^{\nu} \frac{R}{\rho_i} \Rightarrow y_{total} = R \cdot \sum_{i=1}^{\nu} \frac{1}{\rho_i} \Rightarrow \frac{1}{\rho_{total}} = \sum_{i=1}^{\nu} \frac{1}{\rho_i} \qquad (3)$$

where ν is the number of various layers of materials that exist under the rail -including rail– elastic pad, sleeper, ballast, *etc*. The semi-static Load is produced by the centrifugal acceleration exerted on the wheels of a vehicle that is running in a curve with cant deficiency, given by the following equation ([1, 6, see also 7]): $Q_\alpha = \frac{2 \cdot \alpha \cdot h_{CG}}{e^2} \cdot Q_{wheel}$, where α is the cant deficiency, h_{CG} the height of the center of gravity of the vehicle from the rail head and e the track gauge. The semi-static Action/Reaction is derived by the multiplication of Q_α by the $\bar{A}_{stat}$. So equation (2b) is transformed to:

$$\frac{R_{stat}}{Q_{wheel} + Q_\alpha} = \bar{A} = \bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow R_{stat} = \left(Q_{wheel} + Q_\alpha\right) \cdot \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} = \left(Q_{wheel} + Q_\alpha\right) \cdot \bar{A}_{stat} \qquad (2c)$$

3. The Second Order Differential Equation of Motion for the Dynamic Component of the Loads

The dynamic component of the acting load consists of the action due to the Sprung or Suspended Masses (SM) and the action due to the Unsprung or Non Suspended Masses (NSM) of the vehicle. To the latter a section of the track mass is added, that participates in its motion [6]. The Suspended (Sprung) Masses of the vehicle –masses situated above the primary suspension (Figure 1)– apply forces with very small influence on the trajectory of the wheel and on the excitation of the system. This enables the simulation of the track as an elastic media with damping which takes into account the rolling wheel on the rail running table ([7], [8], [9]). Forced oscillation is caused by the irregularities of the rail running table (simulated by an input random signal) –which are represented by n–, in a gravitational field with acceleration g. There are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

We approach the matter considering that the rail running table contains a longitudinal fault/ defect of the rail surface. In the above equation, the oscillation of the axle is damped after its passage over the defect. Viscous damping, due to the ballast, enters the above equation under the condition that it is proportional to the variation of the deflection *dy/dt*. To simplify the investigation, if the track mass (for its calculation see [6], [9]) is ignored -in relation to the much larger Vehicle's Non Suspended Mass- and bearing in mind that *y+n* is the total subsidence of the wheel during its motion (since the y and n are added algebraically), we can approach the problem of the random excitation, based on a cosine defect (V<< $V_{critical}$=500 km/h):

$$\eta = a \cdot \cos \omega t = a \cdot \cos\left(2\pi \cdot \frac{V \cdot t}{\lambda}\right) \qquad (4)$$

The second order differential equation of motion is:

$$m_{NSM} \frac{d^2 z}{dt^2} + \Gamma \cdot \frac{dz}{dt} + h_{TRACK} \cdot z = -m_{NSM} \cdot a \cdot \omega^2 \cdot \cos(\omega t) \qquad (5)$$

The complete solution of which using polar coordinates is ([5], p.199 and ch.3):

$$z = \underbrace{A \cdot e^{-\zeta \omega_n t} \cdot \sin\left(\omega_n t \sqrt{1-\zeta^2} - \varphi\right)}_{transient-part} + \underbrace{a \cdot B \cdot \cos(\omega t - \varphi)}_{steady-state-part} \qquad (6)$$

where, the first term is the *transient part* and the second part is the *steady state*.

4. Calculation Methods of the Actions on Railway Track

The magnitude of the actions on each support point of the rail, (e.g. a concrete sleeper), are calculated using the main four methods of semi-analytic approach and are presented below. The actions are a percentage of the vertical loads, due to their distribution on more than one support point of the rail (sleepers). The track panel, as a continuous beam on elastic foundation, is loaded by the axle of the railway vehicle and this load is distributed to adjacent sleepers (due to the spring constant ρtotal). The sleeper, on which the wheel acts, undertakes its reaction R which in practice is the Design Load/ Action on the sleeper.

Method cited in French Literature: According to the French literature, for the estimation of the total loads acting on track the standard deviation of the dynamic component must be calculated:

$$\left[\sigma\left(\Delta Q_{dynamic}\right)\right] = \sqrt{\sigma^2\left(\Delta Q_{NSM}\right) + \sigma^2\left(\Delta Q_{SM}\right)}$$

Where: $\sigma(\Delta Q_{NSM})$ is the standard deviation of the dynamic component of the total load due to Non Suspended Masses that participates in the increase of the static load ([6], [10]), $\sigma(\Delta Q_{SM})$ is the standard deviation of the dynamic component of the total load due to the Suspended Masses that participates in the increase of the static load ([10]).

$$Q_{total\ \max} = Q_{wh} + Q_{\alpha} + 2 \cdot \sqrt{\left[\sigma^2\left(\Delta Q_{NSM}\right)\right] + \left[\sigma^2\left(\Delta Q_{SM}\right)\right]}$$

where: Q_{wheel} = the static load of the wheel (half the axle load), Q_{α} = load due to superelevation deficiency, the action/ reaction (R) on each support point of the rail, that is for each sleeper sleeper is calculated for the static, semi-static and dynamic components of the acting load [6, 9, 10]:

$$R_{total} = \left(Q_{wheel} + Q_{\alpha} + 2 \cdot \sqrt{\left[\sigma^2\left(\Delta Q_{NSM}\right)\right] + \left[\sigma^2\left(\Delta Q_{SM}\right)\right]}\right) \cdot \bar{A}_{stat} \cdot 1.35 \quad (7)$$

where: R_{total} = the total action/reaction on each sleeper after the distribution of the acting load, the factor of 2 in the equation above covers a 95.5 % probability of occurrence, $\bar{A}stat$ is the static reaction coefficient of the sleeper which is equal to:

$$\bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\rho_{total} \cdot \ell^3}{E \cdot J}}$$

where: ρ_{total} = coefficient of total static stiffness of track in kN/mm, ℓ = distance between the sleepers in mm, E, J = modulus of elasticity and moment of inertia of the rail

Method cited in German Literature: In the German literature, the total load Qtotal (static and dynamic) acting on the track, is equal to the static wheel load multiplied by a factor. After the total load is estimated, the reaction R acting on a sleeper, which is a percentage of the total load Qtotal can be calculated [12, 13]:

$$Q_{total} = Q_{wheel} \cdot (1 + t \cdot \bar{s})$$

where: Q_{wheel} is the static load of the wheel, and: (a) $\bar{s}$ = 0.1 φ for excellent track condition, (b) $\bar{s}$ = 0.2 φ for good track condition and (c) $\bar{s}$ = 0.3 φ for poor track condition, where: φ is determined by the following formulas as a function of the speed: (i) for V < 60 km/h then φ = 1 and (ii) for 60 < V < 200 km/h then: $\varphi = 1 + \frac{V - 60}{140}$, where V the maximum speed on a section of track and t coefficient dependent on the probabilistic certainty P (t=1 for P=68.3%, t=2 for P=95.5% and t=3 for P=99.7%). The reaction R of each sleeper is calculated according to [14]: $R = \frac{Q_{total} \cdot \ell}{2 \cdot L}$, where: ℓ = distance between the sleepers, and: $L = \sqrt[4]{\frac{4 \cdot E \cdot J}{b \cdot C}}$, where: C = ballast modulus [N/mm3] b= a *width of conceptualized longitudinal support* according to [14], that multiplied by ℓ equals to the loaded surface F of the seating surface of the sleeper. Consequently:

$$R_{total} = \frac{Q_{total} \cdot \ell}{2 \cdot L} = \frac{Q_{total} \cdot \ell}{2} \cdot \frac{1}{\left(\sqrt[4]{\frac{4 \cdot E \cdot J}{b \cdot \frac{\rho}{F}}}\right)} = Q_{total} \cdot \underbrace{\frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}}}_{\bar{A}_{stat}} = \bar{A}_{stat} \cdot Q_{total} \quad (9a)$$

The equation (9a) for the action/ reaction (R_{total}) on each support point of the rail, that is each sleeper is transformed, for the total load, static and dynamic as follows:

$$R_{max} = \left(1 + 0.9 \cdot \left(1 + \frac{V_{max} - 60}{140}\right)\right) \cdot \bar{A}_{stat} \cdot Q_{wheel} \quad (9a')$$

for $V_{max} \leq 200$ km/h (124.30 mi/h), with probability of occurrence P=99.7%, where, Q_{wheel} = the static load of the wheel (half the axle load), $\bar{\bar{A}}_{stat}$ is calculated through equation (8). Prof. Eisenmann for speeds above 200 km/h proposed a reduced factor of dynamic component:

$$R_{max} = \left(1 + 0.9 \cdot \left(1 + \frac{V - 60}{380}\right)\right) \cdot \bar{A}_{stat} \cdot Q_{wheel} \quad (9b)$$

Equation (9b) leads to even greater under-estimation of the acting loads on track -than equation (9a)- with possible consequences to the dimensioning of track elements –like, *e.g.*, sleepers- as the literature describe [1, 2, 15], thus equation (9a) should be preferred for the sleepers' dimensioning.

Method cited in American Literature: This method is described in [16] (p. 16-10-26/32 and Chapter 30), [17] (p. 247/273) and it is based on the same theoretical analysis of continuous beam on elastic foundation. The dynamic load is dependent on an impact factor θ:

$$\theta = \frac{D_{33} \cdot V}{D_{wheel} \cdot 100}$$

where: D_{33} in inches a wheel's diameter of 33 inches, D_{wheel} in inches the wheel's diameter of the vehicle examined, V the speed in miles/hour. The total load is: $Q_{total} = Q_{wheel} \cdot (1+\theta)$

The maximum deflection and moment are: $y_{\max} = w(0) = \frac{\beta \cdot Q_{total}}{2 \cdot k}$, $M_{\max} = M(0) = \frac{Q_{total}}{4 \cdot \beta}$

where: k in $lb/inch^2$ is the rail support modulus derived by the relation, p=k·w=k·y and as easily can be found k=ρ/ℓ, and it can be found easily that:

$$\beta = \sqrt[4]{\frac{k}{4 \cdot E \cdot J}} = \sqrt[4]{\frac{\rho}{4 \cdot E \cdot J \cdot \ell}} = \frac{1}{L}$$

where: L is the "elastic length" given previously by the method cited in the German literature. The maximum Reaction/ Action $R_{\max}$ on each support point of the rail (sleeper)] is:

$$R_{\max} = p_{\max} \cdot \ell = k \cdot w_{\max} \cdot \ell = k \cdot y_{\max} \cdot \ell = k \cdot \frac{\beta \cdot Q_{total}}{2 \cdot k} \cdot \ell = \sqrt[4]{\frac{\rho}{4EJ\ell}} \cdot \frac{Q_{total} \cdot \ell}{2} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\rho \cdot \ell^3}{EJ}} \cdot Q_{total} = \bar{A}_{stat} \cdot Q_{total} \quad (10a)$$

The mathematical operations lead to:

$$R_{\max} = \bar{A}_{stat} \cdot \left(1 + \frac{D_{33} \cdot V}{D_{wheel} \cdot 100}\right) \cdot Q_{wheel} \quad (10b)$$

The Giannakos (2004) Method: A research program was conducted by OSE – Hellenic Railways Organization since 1987 till 2002. The research group consisted of: (a) the National Technical University of Athens, (b) the Polytechnic School of the Aristotle University of Thessaloniki, (c) the Hellenic Ministry of Environment, Land Planning and Public Works/Central Public Works Laboratory, (d) the Hellenic Institute of Geological and Mineral Research (IGME), (e) the cooperation of the "Track" Research Department of the French Railways, (f) the University of Graz. The research program was performed under the coordination of the Greek Railways with co-ordinator the present author. The results of the research program have been published and the interested reader should read [1, 2, 5, 7]. This method was developed, proposed and appeared in literature after the aforementioned extensive research program. The following simplified equation is proposed for the calculation of actions on the track panel [1, 2, 5]:

$$R_{\max} = (Q_{wheel} + Q_{\alpha}) \cdot \bar{A}_{dynam} + 3 \cdot \sqrt{\sigma(\Delta Q_{NSM})^2 + \sigma(\Delta Q_{SM})^2} \quad (11a)$$

where Q_{wheel} is the static wheel load, Q_{α} is the load due to cant deficiency (superelevation deficiency), $\bar{A}_{dynam}$ is the dynamic coefficient of sleeper's reaction, λ is the coefficient of dynamic load (3 for a probability of appearence 99.7%), $\sigma(\Delta Q_{NSM})$ is the standard deviation of the dynamic load due to Non-Suspended Masses m_{NSM} of each axle, $\sigma(\Delta Q_{SM})$ is the standard deviation of the dynamic load due to suspended masses m_{SM}:

$$\bar{A}_{dynam} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot h_{TRACK}}{E \cdot J}},$$

and h_{TRACK} is given by the equation (13a) below. The equation (11a) is transformed in:

$$R_{max} = \frac{1}{2\sqrt[8]{2}} \cdot \sqrt[16]{\left(\frac{\rho_{total} \cdot \ell^3}{E \cdot J}\right)^3} \cdot \left(Q_{wheel} + Q_{\alpha}\right) +$$

$$+(3 \cdot \sqrt{\underbrace{\left(k_{\alpha} \cdot V \cdot \sqrt[8]{2^6 \cdot \left(m_{NSM-vehicle} + m_{TRACK}\right)^4 \cdot E \cdot J \cdot \left(\frac{\rho_{total}}{\ell}\right)^3}\right)^2}_{\sigma(\Delta Q_{NSM})} + \underbrace{\left(\frac{V-40}{1000} \cdot N_L \cdot Q_{wheel}\right)^2}_{\sigma(\Delta Q_{SM})}}\ ) \quad (11b)$$

The result of equation (11b) in kN, for a probability of occurrence 99.7 %, where Q_{wheel} = the static wheel load in kN, Q_{α} = the load due to superelevation deficiency in kN, $m_{NSM-vehicle}$ the mass in tons (1t = 2204.62 pounds) of the Non Suspended Masses of the vehicle, m_{TRACK} the track mass participating in their motion in tons also (for the calculation of the m_{TRACK} see [18, 19, 20]), ρ_{total} in kN/mm, ℓ the distance between the sleepers in mm, V in km/h, N_L ranging between 0.7 and 1.5 dependent on the track leveling defaults and k_{α} coefficient of the condition of the rail running table, ranging from $38942.43 \cdot 10^{-7}$ for ground rail running table to $155769.73 \cdot 10^{-7}$ for non-ground rail running table, for tracks of good condition and maybe up to $324520.28 \cdot 10^{-7}$ for secondary lines with rail running table in a very bad condition ([5], [7], [21]), E the modulus of elasticity [kN/mm^2], J the moment of inertia of the rail [mm^4], $\sigma(\Delta Q_{NSM})$ = the standard deviation of the dynamic load due to Non Suspended Masses and $\sigma(\Delta Q_{SM})$ = the standard deviation of the dynamic load due to Suspended Masses, the pad stiffness is calculated through a trial-and-error procedure that ensures equilibrium among the numerous springs-components of the track system. In [22] this method is adopted, by the International Federation of Concrete, for pre-cast concrete railway track systems design.

5. Verification of the four Calculation Methods for the Actions on Sleepers with Real Data from Track Observations

In Greece until 1999, twin-block concrete sleepers of French technology were exclusively used, namely Vagneux U2, U3 with RN fastenings, and U31 with Nabla fastenings. Since then, monoblock sleepers of pre-stressed concrete B70 type with W-14 fastenings have been laid. The available international bibliography did not give any satisfactory justification for the appearance of cracks on over 60% of type U2/U3 twin-block sleepers laid on the Greek network. The extended cracking, over the failure threshold (R3 region) of the twin-block concrete sleepers, observed in the Greek railway network at a large percentage (more than 60 % of the laid on track U2/U3 sleepers equipped with RN fastenings), was not justified by the methods cited in the international literature at that time (French, German, American). The cracking was observed on the twin-block concrete sleepers of French technology, namely Vagneux U2, U3 with RN fastenings, for tracks designed for V_{max} = 200 km/h and temporary operational speed $V_{operational}$ = 140 km/h. The calculations performed by the three methods did not provide results over the failure threshold (140–170 kN) and they were predicting no cracking at all. After the research, the Giannakos (2004) method was developed whose results successfully predicted the extended cracking of the U2/U3 sleepers [1, 2], calculating actions over the cracking threshold and in almost all cases over the failure threshold. This method was derived from theoretical analyses and/or measurements from laboratory tests performed in Greece, Austria, France, and Belgium and observations from real on-track experience. Moreover, International Federation of Concrete (**fib**) has adopted this method for precast concrete railway track systems [22]. The conditions of the Greek network between the 1980s and the beginning of 1990s, consisted of very compacted, stiff support with polluted ballast bed ($\rho_{ballast}$ = 380 kN/mm) and substructure classified according to the fluctuation of coefficient ρsubstructure for the seating of the track from (a) $\rho_{substructure}$ = 40 kN/mm for pebbly substructure to the most adverse conditions of either (b) $\rho_{substructure}$ = 100 kN/mm, which corresponds to frozen ballast bed and substructure or approximately the rigidity of Newly Constructed Lines 1 (NBS1) of the DB – German Railways (107 kN/mm) [23], or (c) $\rho_{substructure}$ = 250 kN/mm for stiff (rigid) subgrade at the bottom of a tunnel or on a concrete bridge with small height of ballast-bed. The calculations according to the three aforementioned methods were programmed in a computer code and parametric investigations were performed varying the stiffness of the substructure as described in [5] and [1]. The results are depicted in Figure 3, with ρ_{total}=100 kN/mm the most characteristic value of the subgrade.

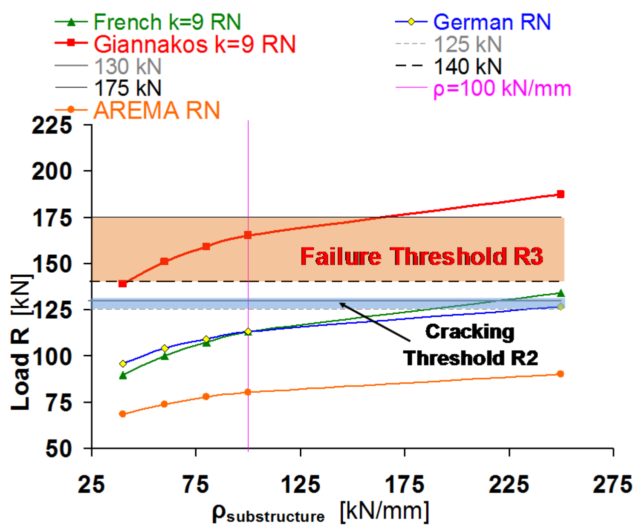

Figure 3. Calculation of actions on U2/U3 twin-block sleepers with the four methods

The forces on the sleeper are calculated according to the French, the German, the AREMA and Giannakos (2004) method. On the same figure the limits of the three regions of "strength" of the U2/U3 sleeper are plotted as described in its technical specifications. The characteristic maximum value for $\rho_{substructure}$ is 100 kN/mm, depicted in the Figure 3, by a vertical line. It is noted that the loads on the sleeper estimated by the AREMA, the French, and the German methods are below the R2 Region/ Cracking Threshold limit (125-130 kN). This means that no cracking of the sleepers is predicted with these three methods, in contrast with the situation observed on track in the Greek Railway Network. On the other hand, Giannakos (2004) method estimates load levels on the sleepers that lie within the R3 Region/ Failure Threshold and is successful in predicting the extended cracking that was observed (over 60% of the number of sleepers laid on track [1]).

6. The Specific Case of an Isolated Defect

The aforementioned methods in paragraph 4, give equations to calculate the actions on track depending on the parametrical analysis of the conditions on the railway track. In this paper we try to relate the depth (sagittal) of an isolated defect to the dynamic component of the load. We focus herein on the term: $A \cdot e^{-\zeta\omega_n t} \cdot \sin\left(\omega_n t\sqrt{1-\zeta^2} - \varphi\right)$, from Equation (6) which represents the transient part of motion. We investigate this term for ζ=0. The theoretical analysis for the additional –to the static and semi-static component– dynamic component of the load due to the Non Suspended Masses and the Suspended Masses of the vehicle, leads to the examination of the influence of the Non Suspended Masses only, since the frequency of oscillation of the Suspended Masses is much smaller than the frequency of the Non Suspended Masses. If m_{NSM} represents the Non Suspended Mass, m_{SM} the Suspended Mass and m_{TRACK} the Track Mass participating in the motion of the Non Suspended Masses of the vehicle, the differential equation is (with no damping ζ=0):

$$m_{NSM} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \Rightarrow \left(m_{NSM} + m_{TRACK}\right) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \qquad (12)$$

where: g the acceleration of gravity and h_{TRACK}, the total dynamic stiffness coefficient of track:

$$h_{TRACK} = 2\sqrt{2} \cdot \sqrt[4]{\frac{EJ\rho_{total}^3}{\ell^3}} \quad , \quad m_{TRACK} = 2\sqrt{2} \cdot m_0 \cdot \sqrt[4]{\frac{EJ\ell}{\rho_{total}}} \qquad (13)$$

where the track mass m_{TRACK} that participates in the motion of the Non Suspended (Unsprung) Masses of the Vehicles, ρ_{total} the total static stiffness coefficient of the track, ℓ the distance among the sleepers, E, J the modulus of elasticity and the moment of inertia of the rail, m_0 the unitary mass of track (per unit of length of the track).

For a comparison of the theoretical track mass to measurement results refer to [18] and [19]. The particular solution of the differential Equation (12) corresponds to the static action of the weight of the wheel: $z = \frac{m_{TRACK} \cdot g}{h_{TRACK}}$. We assume that the rolling wheel runs over an isolated sinusoidal defect of length λ of the form:

$$n = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi x}{\lambda}\right) = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi Vt}{\lambda}\right)$$

where *n* is the ordinate of the defect. Consequently, the ordinate of the center of inertia of the wheel is *n*+*z*. Defining τ_1 as the time needed for the wheel to pass over the defect at a speed V: $\tau_1 = \frac{\lambda}{V}$, then:

$$m_{NSM} \cdot \frac{d^2}{dt^2}(z+n) + m_{TRACK} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = 0 \Rightarrow$$

$$\Rightarrow \left(m_{NSM} + m_{TRACK}\right) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} = -m_{NSM} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1}$$

Since:

$$\frac{dn}{dt} = \frac{a}{2} \cdot \frac{2\pi V}{\lambda} \cdot \sin\frac{2\pi Vt}{\lambda} = \frac{a}{2} \cdot \frac{2\pi\lambda}{\lambda \cdot \tau_1} \cdot \sin\frac{2\pi Vt}{\lambda} \Rightarrow \frac{dn}{dt} = \frac{a}{2} \cdot \frac{2\pi}{\tau_1} \cdot \sin\frac{2\pi Vt}{\lambda} \Rightarrow$$

$$\frac{d^2 n}{dt^2} = -\frac{a}{2} \cdot \left(\frac{2\pi}{\tau_1}\right)^2 \cdot \cos\frac{2\pi Vt}{\lambda} = -\frac{a}{2} \cdot \left(\frac{2\pi}{\tau_1}\right)^2 \cdot \cos\frac{2\pi\lambda t}{\lambda \cdot \tau_1} \Rightarrow \frac{d^2 n}{dt^2} = -\frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1}$$

Where: $x = V \cdot t,\quad \omega_1 = 2 \cdot \frac{\pi \cdot V}{\ell} = \frac{2\pi}{T},\quad \omega_n^2 = \frac{h_{TRACK}}{m_{NSM}}$

and ω_1 the cyclic frequency of the external force and ω_n the natural frequency.

The additional dynamic component of the load due to the motion of the wheel is:

$$-m_{NSM} \cdot \left(z'' + n''\right) = h_{TRACK} \cdot z + m_{TRACK} \cdot z'' \tag{14}$$

To solve equation (12) we divide by (m_{NSM}+m_{TRACK}):

$$\frac{d^2 z}{dt^2} + \frac{h_{TRACK}}{\left(m_{NSM} + m_{TRACK}\right)} \cdot z = -\frac{m_{NSM}}{\left(m_{NSM} + m_{TRACK}\right)} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1} \tag{15}$$

differential equation of motion for an undamped forced harmonic motion ([24], [25]):

$$m \cdot \ddot{z} + kz = p_0 \cos\left(\omega_1 t\right) \Rightarrow \ddot{z} + \frac{k}{m} z = \frac{p_0}{m} \cos\left(\omega_1 t\right) = \omega_n^2 \cdot \frac{p_0}{k} \cos\left(\omega_1 t\right)$$

where: $\omega_n^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega_n^2}$

The complete solution is (see Annex 1) $z(t) = \frac{p_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega_1}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right]$ (16)

when: $k=h_{TRACK}$, $m=m_{NSM}+m_{TRACK}$, and: $\omega_n^2 = \frac{h_{TRACK}}{m_{NSM} + m_{TRACK}},\quad p_0 = -\frac{2 \cdot \alpha \cdot \pi^2 \cdot m_{NSM}}{\tau_1^2}$

The general solution of equation (15) is:

$$z(t) = \underbrace{-\frac{2 \cdot \alpha \cdot \pi^2 \cdot m_{NSM}}{\tau_1^2}}_{p_0} \cdot \frac{1}{h_{TRACK}} \cdot \frac{1}{1 - \left(\frac{\omega_1}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right] = \frac{\alpha}{2} \cdot \frac{m_{NSM}}{\left(m_{NSM} + m_{TRACK}\right)} \cdot \frac{1}{1 - \left(\frac{\omega_n}{\omega_1}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right]$$

and:

$$z(t) = -\frac{1}{2} \cdot \underbrace{\frac{4 \cdot \pi^2}{\tau_1^2}}_{\omega_1^2} \cdot \frac{\alpha \cdot m_{NSM}}{\underbrace{\omega_n^2 \cdot \left(m_{NSM} + m_{TRACK}\right)}_{h_{TRACK}}} \cdot \frac{1}{1 - \left(\frac{\omega_1}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right] = \frac{\alpha}{2} \cdot \frac{m_{NSM}}{\left(m_{NSM} + m_{TRACK}\right)} \cdot \frac{1}{1 - \left(\frac{\omega_n}{\omega_1}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right] \tag{17}$$

where, $T_n=2\pi/\omega_n$ the period of the free oscillation of the wheel circulating on the rail and $T_1=2\pi/\omega_1$ the necessary time for the wheel to run over a defect of wavelength λ: $T_1=\lambda/V$. Consequently, $T_n/T_1=\omega_1/\omega_n$.

From equation (17): $$\frac{\left(m_{NSM} + m_{TRACK}\right)}{m_{NSM}} \cdot z(t) = \alpha \cdot \frac{1}{2} \cdot \frac{1}{1 - \left(\frac{\omega_n}{\omega_1}\right)^2} \cdot \left[\underbrace{\cos\left(\omega_1 t\right)}_{steady-state} - \underbrace{\cos\left(\omega_n t\right)}_{transient-part}\right] \tag{18}$$

We can investigate equation (18) after a sensitivity analysis by variating parameters: for given values of $T_n/T_1=\omega_1/\omega_n$ and for given value of V (for example equal to 1) the time period T_1 is proportional to μ=0.1, 0.2, … 1.0 of defect λ. Equation (18) is transformed:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot z(t)\cdot\frac{1}{\alpha}\right]=\frac{1}{2}\cdot\frac{1}{1-(n)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(n\cdot\omega_1 t)}_{transient-part}\right]=\frac{1}{2}\cdot\frac{1}{1-(n)^2}\cdot\left[\underbrace{\cos(2\pi\cdot\mu)}_{steady-state}-\underbrace{\cos(n\cdot 2\pi\cdot\mu)}_{transient-part}\right] \quad (19)$$

where $n=\omega_n/\omega_1$, $\omega_1=\lambda/V$ and we examine values of $\mu\cdot\lambda=0,\ 0.1\lambda,\ 0.2\lambda,....,\ 0.8\lambda,\ 0.9\lambda,\ \lambda$, for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ. In Figure 4 the equation (19) is depicted.

7. A Defect of Long Wavelegth in High Speed

The first term in the bracket of equation (19) is depicted on the vertical axis while on the horizontal axis the percentages of the wavelength $\mu\cdot\lambda$ are shown. We observe that $z(x)$ has its maximum value for T_1/T_n=0,666667=2/3, equal to 1,465:

$$z(t)=\left[\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\right]\cdot\alpha\cdot 1{,}465 \quad (20)$$

for $x=0{,}91\lambda$. The relation T_1/T_n represents the cases for short and long wavelength of the defects. For T_1/T_n=2-2,5 the wavelength is long and for T_1/T_n << the wavelength is short ([6], p.49). The second derivative of z(x) from equation (17), that is the vertical acceleration that gives the dynamic overloading due to the defect, is calculated:

$$z'(t)=\frac{\alpha}{2}\cdot\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{-\omega_1\cdot\sin(\omega_1 t)}_{steady-state}+\underbrace{\omega_n\cdot\sin(\omega_n t)}_{transient-part}\right] \quad (21a)$$

$$z''(t)=-\frac{\alpha}{2}\cdot\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\omega_1^2\cdot\cos(\omega_1 t)}_{steady-state}-\underbrace{\omega_n^2\cdot\cos(\omega_n t)}_{transient-part}\right] \quad (21b)$$

for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ, and $T_n=0{,}0307$ *sec* as calculated above. The additional subsidence of the deflection z at the beginning of the defect is negative in the first part of the defect. Following the wheel's motion, z turns to positive sign and reaches its maximum and possibly afterwards z becomes again negative. After the passage of the wheel over the defect, one oscillation occurs which approaches to the natural cyclic frequency ω_n (this oscillation is damped due to non-existence of a new defect since we considered one isolated defect) in reality, even if in the present analysis the damping was omitted for simplicity. The maximum value of z is given in Table 1 below, as it is –graphically– measured in Figure 4.

It is observed that the maximum value is shifted towards the end of the defect as the ratio T_1/T_n decreases, that is when the defect's wavelength becomes short. The maximum is obtained for $T_1/T_n = 0{,}666667 = 2/3$. For each combination of "vehicle + track section" the critical value of the speed V, for which the *2/3* are achieved is a function of the wavelength λ.

Since: $T_1 = \frac{\lambda}{V} \Rightarrow V = \frac{\lambda}{T_1} = \frac{3}{2} \cdot \frac{\lambda}{T_n}$, we can calculate the critical speed $V_{critical}$ for any combination of track layers and their corresponding stiffness. As a case study we use the ballasted track depicted in Figure 1, for high speed, equipped with rail UIC60 (ρ_{rail} = 75.000 kN/mm), monoblock sleepers of prestressed concrete B70 type ($\rho_{sleeper}$ = 13.500 kN/mm), W14 fastenings combined with pad Zw700 Saargummi (ρ_{pad} fluctuating from 50,72 to 48,52 kN/mm), ballast fouled after 2 years in circulation ($\rho_{ballast}$ = 380 kN/mm) and excellent subgrade/ substructure for high speed lines: $\rho_{subgrade}$ = 114 kN/mm [e.g. the New Infrastructure (NBS) of the German Railways]. The calculation of the static stiffness coefficient of the subgrade ρsubgrade for a high speed line of this type as it is derived from practice is given in [26] and [27]. For this cross section of ballasted track, h_{TRACK} is equal to 85,396 kN/mm = 8539,6 t/m and m_{TRACK} is equal to 0,426 t (for the calculations see [18, 19]). If we consider an average m_{NSM}=1,0 t, then:

$$m_{NSM} + m_{TRACK} = \frac{1,0+0,426}{9,81} = 0,145 \quad tons-mass$$

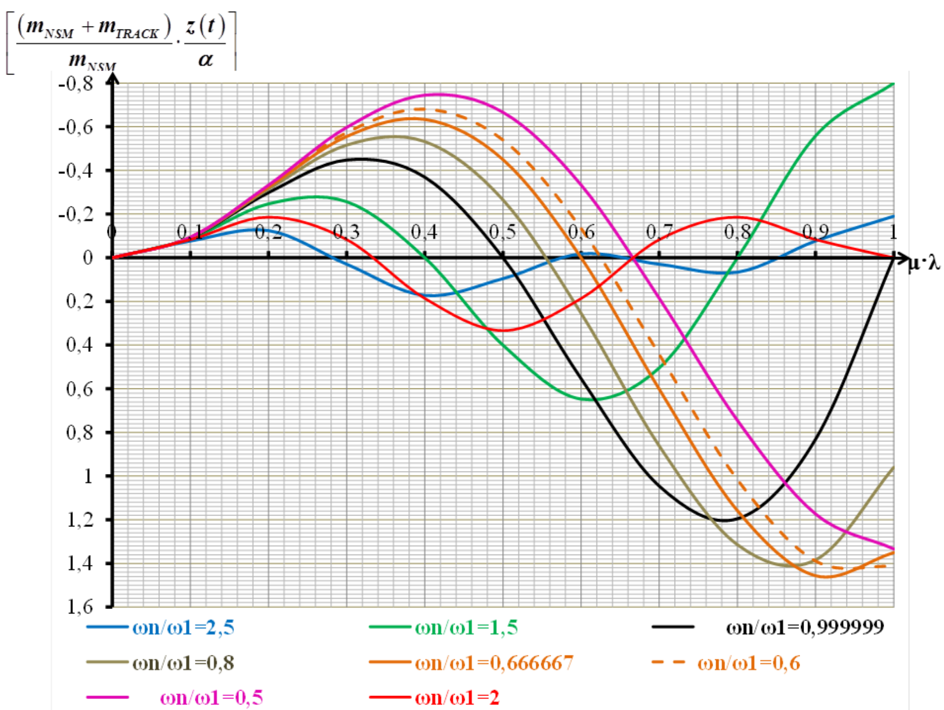

Figure 4. Mapping of Equation (19). On the Horizontal Axis the percentage of the wavelength λ of the defect is depicted. On the Vertical Axis the first term of equation (19), inside the brackets, is depicted

Table 1. Maximum Values of ζ=[(m_{NSM}+m_{TRACK})/m_{NSM}]·[z_{max}/α]

T_1/T_n	2,5	2	1,5	1	0,8	0,66667	0,6	0,5
ζ	0,18	0,335	0,65	1,205	1,415	1,47	1,43	1,34
where: ζ=[(m_{NSM}+m_{TRACK})/m_{NSM}]·[z_{max}/α]								

Where *g = 9,81 m/sec²*, the acceleration of gravity. The period T_n is given by:

$$T_n = 2\pi\sqrt{\frac{1,0+0,426}{9,81\cdot 8539,6}} = 0,026 \ \sec \Rightarrow V_{critical} = \frac{3}{2}\cdot\frac{\lambda}{T_n} = \frac{3}{2}\cdot\frac{\lambda}{0,026} = 57,69\cdot\lambda$$

where an average h_{TRACK}=8539,6 t/m is used and $V_{critical}$ is given in [m/sec], λ in [m]. For a wavelength of 1,0 m, $V_{critical}$ = 57,69 m/sec = 207,7 km/h. If we consider a defect with a wavelength that produces a forced oscillation with:
$\frac{T_1}{T_n} = \frac{\omega_n}{\omega_1} = 2,5$, we calculate (in Figure 4 is 0,19, for x=0,41·λ): $z_{\max} = \left[\frac{m_{NSM}\cdot\alpha\cdot 0,19}{(m_{NSM}+m_{TRACK})}\right] = 0,133\cdot\alpha$

with the values calculated above: T_n = 0,026 sec, T_1 = 0,065 sec, the wavelength ℓ equals:

$$\lambda = V\cdot T_1 = 2,5\cdot V\cdot T_n = 0,065\cdot 57,69 = 3,75m$$

This value represents a defect of long wavelength. The static deflection due to a wheel load of 11,25 t or 112,5 kN is equal to:

$$z_{static} = \frac{Q_{wheel}}{2\sqrt{2}}\cdot\sqrt[4]{\frac{\ell^3}{EJ\rho_{total}^3}} = \frac{112.500N}{2\sqrt{2}}\cdot\sqrt[4]{\frac{600^3 mm^3}{210.000\frac{N}{mm^2}\cdot 3,06\cdot 10^7 mm^4\cdot 85.396^3\frac{N^3}{mm^3}}} =$$

$$= \frac{112.500N}{2\sqrt{2}}\cdot 1,524228617\cdot 10^{-5}\frac{mm}{N} = 0,606mm$$

Consequently, for α=1 mm, that is for every mm of vertical defect, the dynamic increment of the static deflection is equal to (0,133/0,606)=21,9% of the static deflection (for every mm of the depth of the defect).

If we examine the second derivative (vertical acceleration) as a percentage of g, the acceleration of gravity, then [from equation (21)]:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{z''(t)}{\alpha}\right] = -\frac{1}{2}\cdot\frac{\omega_1^2}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\cos\left(\frac{2\pi V}{\lambda}\cdot\frac{\mu\cdot\lambda}{V}\right)}_{steady-state} - \underbrace{\frac{\omega_n^2}{\omega_1^2}\cdot\cos\left(n\cdot\frac{2\pi V}{\lambda}\frac{\mu\cdot\lambda}{V}\right)}_{transient-part}\right] =$$

$$= -\left(\frac{2\pi}{n\cdot T_n}\right)^2\cdot\frac{1}{g}\cdot\left[\frac{1}{2\cdot\left[1-(n)^2\right]}\cdot\left[\underbrace{\cos(2\pi\mu)}_{steady-state} - \underbrace{(n)^2\cdot\cos(2n\pi\mu)}_{transient-part}\right]\right] \quad [\%g] \tag{22}$$

Equation (22) is plotted in Figure 5.

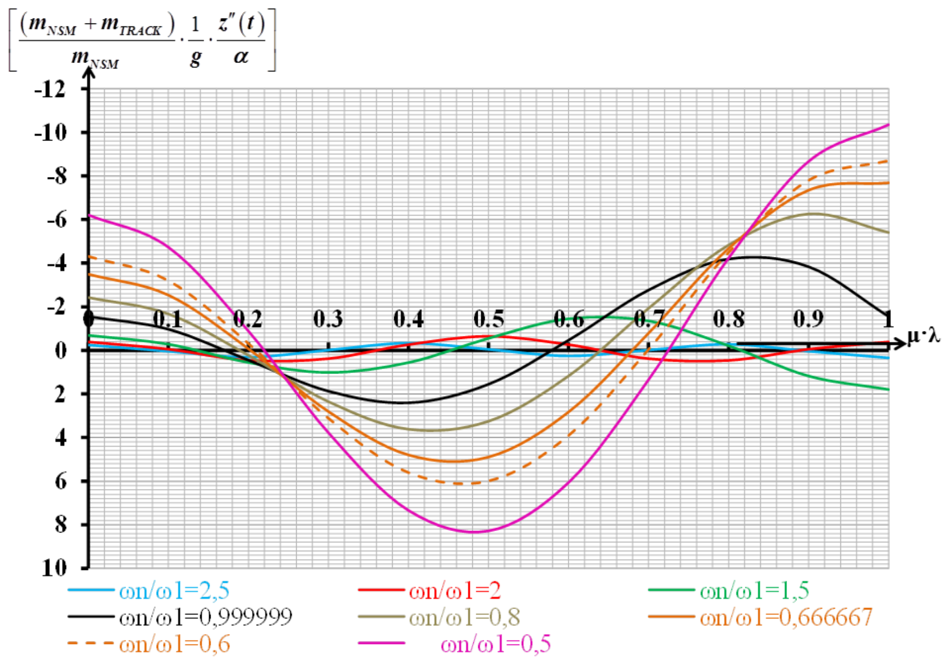

Figure 5. Mapping of the equation (22), for the vertical acceleration due to a defect of long wavelength. In the Horizontal Axis the percentage of the wavelength λ of the defect is depicted. In the Vertical Axis the first term of equation (22), in the brackets, is depicted

The first term in the bracket of Equation (22) is depicted on the vertical axis while on the horizontal axis the percentages of the wavelength μ·λ are shown. For the case calculated above in Figure 5, at the point $x=0,41\cdot\lambda$ the term in bracket has a value of -0.332:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{1}{g}\cdot\frac{z''(t)}{\alpha}\right]=-0.332041\Rightarrow z''(t)=-0.332041\cdot\frac{1,0}{1,0+0,426}\cdot g\cdot\alpha=0,233\cdot g\cdot\alpha$$

Equation (12) (its second part corresponds to the static action of the wheel load) has as particular solution: $z=\frac{m_{NSM}\cdot g}{h_{TRACK}}$

Abandoning the second part leads to the classic solution where *z* is the supplementary subsidence owed to the dynamic increase of the Load. The dynamic increase of the Load is equal to:

$$Q_{dynamic}=h_{TRACK}\cdot z+m_{TRACK}\cdot z''=85.396\cdot 0,133-426kg\cdot 0,233\cdot 9,81\frac{m}{\sec^2}=1,04t \qquad (23)$$

where, from the analysis above: h_{TRACK} = 8539,6 t/m = 85.396 N/mm, m_{TRACK} = 0,426 t = 426 kg. Consequently, for arc height (i.e. sagitta) α=1 mm of a defect of wavelength *λ*, that is for every mm of vertical defect, the dynamic increase of the load is equal to (1,04/ 11,25)=9,24% of the static load of the wheel (for every mm of the depth of the defect). Apparently the increase of the static stiffness coefficient and of the inferred dynamic stiffness coefficient of track leads to lower values of $Q_{dynamic}$ since the h_{TRACK} is in the denominator in the equation for calculation of *z*, consequently the first term of the equation (23) for the $Q_{dynamic}$ will be reduced. The same happens for the track mass participating in the motion of the Non Suspended Masses of the wheel ([18], [19], [7]). Thus finally the $Q_{dynamic}$ will be reduced when the ρ_{total} and the h_{TRACK} are increased.

8. Conclusions

For a defect of wavelength λ and sagitta of 1 mm (depth of the defect), the dynamic increase of the acting load –compared to the static wheel load– is equal to 9,24%. Furthermore from Figure 4 and Figure 5, it is verified that when the speed increases, the period T_1 decreases and the supplementary sagitta (depth of the defect) increases. Supplementary, since it is added to the static deflection and it is owed to the dynamic component of the load. The increase of the dynamic component of the load increases faster since it is dependent on the square of the speed $(\omega_1)^2$. When the dynamic stiffness coefficient h_{TRACK} increases, T_n decreases, T_1/T_n increases, the supplementary sagitta decreases (for the same V), and the dynamic component of the action decreases also. Thus the softer the pad the higher percentage of the load is transmitted through the sleeper under the load. Finally in total the reaction per sleeper in the case of softer pads is smaller due to a distribution of the load along the track in more sleepers, as it can be derived from literature ([1], [2], [5]).

ANNEX 1

For the free oscillation (without external force) the equation is:

$$m \cdot \ddot{z} + k \cdot z = 0 \Rightarrow \ddot{z} + \frac{k}{m} \cdot z = 0 \Rightarrow \ddot{z} + \omega_n^2 \cdot z = 0 \quad (1.1)$$

The general solution is [4]:

$$z(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) \Rightarrow z(t) = z(0) \cdot \cos(\omega_n t) + \frac{\dot{z}(0)}{\omega_n} \cdot \sin(\omega_n t) \quad (1.2)$$

Where:

$$A = z(0), \quad B = \frac{\dot{z}(0)}{\omega_n} \quad (1.3)$$

If we pass to the undamped harmonic oscillation of the form:

$$m \cdot \ddot{z} + k \cdot z = p_0 \cdot \cos(\omega t) \Rightarrow \ddot{z} + \omega_n^2 \cdot z = \frac{p_0}{m} \cdot \cos(\omega t) \Rightarrow \ddot{z} + \omega_n^2 \cdot z = \omega_n^2 \cdot \frac{p_0}{k} \cdot \cos(\omega t) \quad (1.4)$$

where:

$$\omega_n^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega_n^2} \quad (1.5)$$

The particular solution of the linear second order differential equation (1.4) is of the form:

$$z_p(t) = C \cdot \cos(\omega t) \Rightarrow \dot{z}_p(t) = -\omega \cdot C \cdot \sin(\omega t) \Rightarrow \ddot{z}_p(t) = -\omega^2 \cdot C \cdot \cos(\omega t) \quad (1.6)$$

Substituting equation (1.6) to equation (1.4) we derive:

$$-\omega^2 \cdot C \cdot \cos(\omega t) + \omega_n^2 \cdot C \cdot \cos(\omega t) = \frac{p_0}{m} \cdot \cos(\omega t) \Rightarrow -\omega^2 \cdot C + \omega_n^2 \cdot C = \omega_n^2 \cdot \frac{p_0}{k} \Rightarrow$$
$$\Rightarrow C(\omega_n^2 - \omega^2) = \omega_n^2 \cdot \frac{p_0}{k} \Rightarrow C = \frac{\omega_n^2}{(\omega_n^2 - \omega^2)} \cdot \frac{p_0}{k} \Rightarrow C = \frac{p_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \quad (1.7)$$

<u>The general solution for the equation (1.4)</u> is the addition of the solution (1.2) and of the solution of the equation (1.6) combined with equation (1.7):

$$z(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) + \frac{p_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cdot \cos(\omega t) \quad (1.8)$$

$$\dot{z}(t) = -\omega_n \cdot A \cdot \sin(\omega_n t) + \omega_n \cdot B \cdot \cos(\omega_n t) - \frac{p_0}{k} \cdot \frac{\omega}{1-\left(\frac{\omega}{\omega_n}\right)^2} \cdot \sin(\omega t) \quad (1.9)$$

Calculating the values of equation (1.8) and (1.9) at t=0:

$$z(0) = A \cdot \cos(0) + B \cdot \sin(0) + \frac{p_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \cdot \cos(0) \Rightarrow A = z(0) - \frac{p_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \quad (1.10)$$

$$\dot{z}(0) = -\omega_n \cdot A \cdot \sin(0) + \omega_n \cdot B \cdot \cos(0) - \frac{p_0}{k} \cdot \frac{\omega}{1-\left(\frac{\omega}{\omega_n}\right)^2} \cdot \sin(0) \Rightarrow B = \frac{\dot{z}(0)}{\omega_n} \quad (1.11)$$

$$z(t) = \underbrace{\left[z(0) - \frac{p_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \right] \cdot \cos(\omega_n t) + \frac{\dot{z}(0)}{\omega_n} \cdot \sin(\omega_n t)}_{transient-part} + + \frac{p_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \cdot \cos(\omega t) \quad (1.12)$$

and for initial conditions z(0)=ż(0)=0: $z(t) = \frac{p_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos(\omega t)}_{steady-state} - \underbrace{\cos(\omega_n t)}_{transient-part} \right]$ (1.13)

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