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Singular Perturbation Approximation of Balanced Infinite-Dimensional Systems

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Abstract

This paper concerned with a model reduction of infinite dimensional systems by using the singular perturbation approximation. The system considered is that of the exponentially stable linear system with bounded and finite-rank input and output operators such that the balanced realization can be performed on the system. Furthermore, the singular perturbation method is applied to reduce the order of the balanced infinite dimensional systems. A reduced-order model can be obtained by setting to zero of derivative all states corresponding to smaller Hankel singular values. To show the effectiveness of the proposed method, numerical simulations are applied to the heat conduction.

Keywords: *Singular perturbation approximation, infinite-dimensional systems, balanced systems, reciprocal transformation*

1. Introduction

The control design is a study that examines the system settings through feedback, so that the closed loop system behaves as expected with a minimum cost. Control design is one of the central themes in system theory.

There are many problems of dynamics in science and engineering which are modelled in term of partial differential equations. The state space formulation for such model requires infinite dimensionality. Design control for such state space is also of infinite dimension. For the purpose of computation and implementation, this is not practical. Therefore, it is important to find a low order controller for the infinite dimensional systems.

Model reduction is one of the most important methods to obtain low order controller. A number of methods have been proposed in the literature to reduced order of infinite dimensional linear time invariant (FDLTI) systems such as balanced truncation (BT) [1], Hankel norm approximation [2] and singular perturbation approximation (SPA) [3]. All these methods give the stable reduced systems and guarantee the upper bound of the error reduction. Although balanced truncation and SPA methods gives the same of the upper bound of error reduction, but the characteristics of both methods are contrary to each other [11]. It has been shown that the reduced systems by balanced truncation have a smaller error at high frequencies, and tend to be larger at low frequencies. Moreover, the reduced systems through SPA method behave otherwise, i.e., the error goes to zero at low frequencies and tend to enlarge at high frequencies.

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The balanced truncation and Hankel norm approximation techniques have been generalized to infinite dimensional systems [4, 5]. Curtain and Glover [5] generalized the balanced truncation techniques to infinite-dimensional systems and the upper bound of the error reduction have been published in [6]. Meanwhile, by using Lemma 7.3.1 of [8], it can be shown that the reduced systems through balanced truncation method in infinite dimensional systems preserve the behavior of the original system in infinite frequency. This condition is sometimes not desirable in applications. It is therefore necessary to improvise SPA technique such that can be applied to infinite dimensional systems. Many of the properties of the SPA approach FDLTI system can be connected through balanced reciprocal system as shown in [3]. Fatmawati, *et al.*, [12, 13] have generalized reciprocal transformation method to reduce the infinite-dimensional system. This fact motivates generalization SPA method of infinite dimensional systems to obtain reduced order models that have perform well at low frequencies.

2. Preliminary Notes

The infinite dimensional systems dynamics can be presented in the following abstract form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t). \qquad (1)$$

where A is the infinitesimal generator of C_0-semigroup $S(t)$ on Hilbert space X, and operators B and C are finite rank and bounded; $B \in L(C^m, X)$, $C \in L(X, C^k)$. In this paper, we will assume the system (1) is exponentially stable means that the operator A generates the exponentially stable C_0-semigroup $S(t)$ on X. The C_0-semigroup $S(t)$ on X is exponentially stable [8] if there exist positive constants M and α such that $\|S(t)\| \le Me^{-\alpha t}$, for all $t \ge 0$.

We shall denote the state linear system given by (1) as (A, B, C, D) and transfer function given by G, with realization $G(s) = C(sI - A)^{-1}B + D$. We denote the space of bounded linear operator mapping U to Y by $L(U, Y)$. The adjoint of operator A is written as A^* and domain of A is denoted by $\mathrm{D}(A)$. A symmetric operator A is *self-adjoint* if $\mathrm{D}(A^*) = \mathrm{D}(A)$. A self-adjoint operator A on the Hilbert space X with its inner product $\langle \cdot,\cdot \rangle$ is *nonnegative* if $\langle Az, z \rangle \ge 0$ for all $z \in \mathrm{D}(A)$ and *positive* if $\langle Az, z \rangle > 0$ for all nonzero $z \in \mathrm{D}(A)$. We shall use the notation $A \ge 0$ for nonnegativity of the self-adjoint operator A, and $A > 0$ for positivity. The resolvent set of A is the set of all complex numbers λ for which $(\lambda I - A)^{-1}$ is exists as a bounded linear operator on X.

3. Reciprocal Transformation of Balanced Systems

This section review some results of reciprocal transformation of the balanced realization for the infinite dimensional systems. The theory of reciprocal transformation has been developed by Curtain in [9].

The reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the system (A, B, C, D) is defined as

$$\hat{A} = A^{-1}, \quad \hat{B} = A^{-1}B, \quad \hat{C} = -CA^{-1}, \quad \hat{D} = D - CA^{-1}B = G(0), \qquad (2)$$

where G the transfer function of (A,B,C,D).

Let $\hat{G}$ be transfer function of the system $(\hat{A},\hat{B},\hat{C},\hat{D})$, that is, $\hat{G}(s)=\hat{D}+\hat{C}(sI-\hat{A})^{-1}\hat{B}$, then

$$
\begin{aligned}
G(s) &= C[(sI-A)^{-1}+A^{-1}]B+G(0) \\
&= -CA^{-1}\left(\frac{1}{s}I-A^{-1}\right)^{-1}A^{-1}B+G(0) \\
&= \hat{G}(\frac{1}{s}).
\end{aligned}
\tag{3}
$$

In [12, 13], the authors applied the theory of balanced realization for the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$. A system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is the balanced realization of the system (A,B,C,D) if its transfer function is also G and the controllability and observability gramians are both equal to diagonal operator on the state space ℓ_2, the space of square summable sequences [5]. The gramians always satisfy their respective Lyapunov equations.

In this paper, we assume that operator $\tilde{A}$ has bounded inverse such that the reciprocal transformation can be applied to the balanced system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. For simplicity of notation, we write $(\hat{A},\hat{B},\hat{C},\hat{D})$ for the reciprocal system of the balanced system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$. Furthermore, the balanced realization of the reciprocal system is given the following lemma. We will use the infinite matrix representation for the operators with respect to the standard orthonormal basis on ℓ_2.

Lemma 3.1 [12]

If the system $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ is balanced with gramian Σ, then the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$ is also balanced with the same gramian.

Based on Lemma 3.1, we can apply the balanced truncation method to the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$. Let L_B and L_C be the controllability and observability gramians, respectively. Choose a positive integer r such that $\sigma_r > \sigma_{r+1}$ and partition the balanced system $(\hat{A},\hat{B},\hat{C},\hat{D})$ appropriately, with $L_B = L_C = diag(\Sigma_1,\Sigma_2)$, $\Sigma_1 = diag(\sigma_1,...,\sigma_r)$, $\Sigma_2 = diag(\sigma_{r+1},\sigma_{r+2},...)$, as

$$
\hat{A}=\begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}, \hat{B}=\begin{bmatrix}\hat{B}_1\\ \hat{B}_2\end{bmatrix}, \hat{C}=\begin{bmatrix}\hat{C}_1 & \hat{C}_2\end{bmatrix}. \tag{4}
$$

Then the r th order truncated system of $(\hat{A},\hat{B},\hat{C},\hat{D})$ is given by the finite dimensional system $(\hat{A}_{11},\hat{B}_1,\hat{C}_1,\hat{D})$ with its transfer function is $\hat{G}^r$. In this paper, we assume that the Hankel singular values, σ_i, $i=1,2,...,r$ are all distinct, such that $\sigma_1 > \sigma_2 > ... > \sigma_r > \sigma_{r+1} > ... > 0$. From this condition, we have $\Sigma_1 > 0$.

Lemma 3.2 [12]

The truncated system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced with gramian Σ_1 and asymptotically stable.

4. Balanced Singular Perturbation Approximation

In this section we discuss the model reduction method via balanced singular perturbation approximation (BSPA) for infinite dimensional systems. In BSPA method, all balanced states are divided into the slow and fast modes by representing the smaller Hankel singular values as the fast mode, and the rest as the slow mode. Next, a reduced-order model can be obtained by setting the derivative of all states corresponding the fast mode equal to zero.

Suppose $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is the balanced realization of infinite dimensional system (A, B, C, D) on the state space ℓ_2. Using the infinite matrix representation for the operators on ℓ_2, the balanced system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ can be partitioned conformably with with $L_B = L_C = diag(\Sigma_1, \Sigma_2)$, $\Sigma_1 = diag(\sigma_1, ..., \sigma_r)$, $\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, ...)$, $\sigma_r > \sigma_{r+1}$ as

$$\begin{bmatrix} \dot{\tilde{x}}_1(t) \\ \dot{\tilde{x}}_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \tilde{D}u(t). \tag{5}$$

Since the system is balanced, states corresponding to smaller Hankel singular values Σ_2 represent the fast dynamics of the systems. Based on the concept of the SPA method [11], we set to zero the derivative of all states corresponding to Σ_2 to approximate the system (5). The r th-order BSPA of the system $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is given by finite dimensional system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, where

$$\bar{A} = \tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21}$$

$$\bar{B} = \tilde{B}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_2$$

$$\bar{C} = \tilde{C}_1 - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{A}_{21}$$

$$\bar{D} = \tilde{D} - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{B}_2, \tag{6}$$

assuming that $\tilde{A}_{22}$ has bounded inverse.

One of the main results shown in this paper is the following theorem which the properties of reduced order model obtained by using balanced truncation can be extended to BSPA.

Theorem 4.1 The reduced order model by BSPA $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ is balanced with gramian Σ_1 and asymptotically stable.

Proof. Based on Lemma 3.2, the reduced reciprocal system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$ is balanced with gramian Σ_1 that satisfy Lyapunov equations

$$\hat{A}_{11}\Sigma_1 + \Sigma_1 \hat{A}_{11}^{'} + \hat{B}_1 \hat{B}_1^{'} = 0 \tag{7}$$

$$\hat{A}_{11}^{'}\Sigma_1 + \Sigma_1 \hat{A}_{11} + \hat{C}_1^{'} \hat{C}_1 = 0. \tag{8}$$

Multiplying (7) from the left by $\hat{A}_{11}^{-1}$ and from right by $(\hat{A}_{11}^{-1})^{'}$, and then multiplying (8) from the left by $(\hat{A}_{11}^{-1})^{'}$ and from right by $\hat{A}_{11}^{-1}$, yields

$$\Sigma_1 (\hat{A}_{11}^{-1})^{'} + \hat{A}_{11}^{-1}\Sigma_1 + \hat{A}_{11}^{-1}\hat{B}_1\hat{B}_1^{'}(\hat{A}_{11}^{-1})^{'} = 0 \tag{9}$$

$$\Sigma_1 \hat{A}_{11}^{-1} + (\hat{A}_{11}^{-1})^{'}\Sigma_1 + (\hat{A}_{11}^{-1})^{'}\hat{C}_1^{'}\hat{C}_1\hat{A}_{11}^{-1} = 0. \tag{10}$$

By using equations (2), (4), (5) and applying the matrix inversion formula [14], we obtain

(i) $$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} = \tilde{A}^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}^{-1} \tag{11}$$

From block (1,1) of the equation (11), we have $\hat{A}_{11} = (\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})^{-1}$. This condition equivalent to $\hat{A}_{11}^{-1} = \overline{A}$.

(ii) $$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = \tilde{A}^{-1}\tilde{B} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \tag{12}$$

It follows from (12) that

$$\hat{B}_1 = (\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})^{-1}\left(\tilde{B}_1 - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{B}_2\right) = \hat{A}_{11}\overline{B} \text{ or equivalent to } \overline{B} = \hat{A}_{11}^{-1}\hat{B}_1.$$

(iii) $$\hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} = -\tilde{C}\tilde{A}^{-1} = -\begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}^{-1}. \tag{13}$$

From equation (13), we can see that

$$\hat{C}_1 = -(\tilde{C}_1 - \tilde{C}_2\tilde{A}_{22}^{-1}\tilde{A}_{21})(\tilde{A}_{11} - \tilde{A}_{12}\tilde{A}_{22}^{-1}\tilde{A}_{21})^{-1} = -\overline{C}\hat{A}_{11}.$$

Hence, we have $\overline{C} = -\hat{C}_1\hat{A}_{11}^{-1}$.

According to the above result, we obtain the following equations

$$\overline{A} = \hat{A}_{11}^{-1}, \quad \overline{B} = \hat{A}_{11}^{-1}\hat{B}_1, \quad \overline{C} = -\hat{C}_1\hat{A}_{11}^{-1}. \tag{14}$$

Substituting (14) to (9) and (10), we have

$$\bar{A}\Sigma_1 + \Sigma_1 \bar{A}' + \bar{B}\bar{B}' = 0 \tag{15}$$

$$\bar{A}'\Sigma_1 + \Sigma_1 \bar{A} + \bar{C}'\bar{C} = 0 \tag{16}$$

which implies that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is balanced with gramian Σ_1. Since $\hat{A}_{11}$ is asymptotically stable, then $Re(\lambda) < 0$, with λ be any eigenvalue of $\hat{A}_{11}$. Hence, the eigenvalue of $\bar{A}$ is $\frac{1}{\lambda}$ due to $\bar{A} = \hat{A}_{11}^{-1}$. We conclude that $Re(\frac{1}{\lambda}) < 0$, i.e., the reduced order model by BSPA $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is asymptotically stable. □

From (2), (4), (5) and the matrix inversion formula [14], we have

$$\hat{D} = \tilde{D} - \tilde{C}_2 \tilde{A}_{22}^{-1} \tilde{B}_2 = \tilde{D} - (\bar{C}\bar{A}^{-1}\bar{B} - \tilde{C}_2 \tilde{A}_{22}^{-1} \tilde{B}_2) = \bar{D} - \bar{C}\bar{A}^{-1}\bar{B}. \tag{17}$$

The equations (14) can be rewritten as follows

$$\hat{A}_{11} = \bar{A}^{-1}, \quad \hat{B}_1 = \bar{A}^{-1}\bar{B}, \quad \hat{C}_1 = -\bar{C}\bar{A}^{-1}. \tag{18}$$

Let G^r be the transfer function of the r th order model by BSPA $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$, that is $G^r = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}$ and suppose $\hat{G}^r$ is transfer function of the reduced reciprocal system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1, \hat{D})$. Using equations (17) and (18), we can connect the transfer functions G^r of the reduced BSPA system and $\hat{G}^r$ of the reduced reciprocal system.

$$\begin{aligned} G^r(s) &= \bar{C}[(sI - \bar{A})^{-1} + \bar{A}^{-1}]\bar{B} + \bar{D} - \bar{C}\bar{A}^{-1}\bar{B} \\ &= -\bar{C}\bar{A}^{-1}\left(\frac{1}{s}I - \bar{A}^{-1}\right)^{-1}\bar{A}^{-1}\bar{B} + \bar{D} - \bar{C}\bar{A}^{-1}\bar{B} \\ &= \hat{G}^r(\frac{1}{s}). \end{aligned} \tag{19}$$

The following theorem gives the error bound of reduced order model via BSPA and the convergence in the H_∞ norm.

Theorem 4.2 Let G^r be the transfer function of the r th order model by BSPA $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of G. Then we have

$$\left\| G(s) - G^r(s) \right\|_\infty \le 2\sum_{i=r+1}^{\infty} \sigma_i. \tag{20}$$

In particular $\left\| G(s) - G^r(s) \right\|_\infty \to 0$ as $r \to \infty$.

***Proof*.** Using the triangle inequality gives

$$\left\|G(s)-G^r(s)\right\|_\infty \le \left\|G(s)-\hat{G}(\frac{1}{s})\right\|_\infty + \left\|\hat{G}(\frac{1}{s})-\hat{G}^r(\frac{1}{s})\right\|_\infty + \left\|\hat{G}^r(\frac{1}{s})-G^r(s)\right\|_\infty .$$

From (3) dan (19) we may deduce that $G(s)=\hat{G}(\frac{1}{s})$, $\hat{G}^r(\frac{1}{s})=G^r(s)$, such that we obtain

$$\left\|G(s)-G^r(s)\right\|_\infty \le \left\|\hat{G}(\frac{1}{s})-\hat{G}^r(\frac{1}{s})\right\|_\infty . \tag{21}$$

Note that $\hat{G}^r$ is transfer function of the r th order truncations of the balanced and output normal realizations of $\hat{G}$ By mapping $\frac{1}{s}\to s$, we have $\hat{G}(\frac{1}{s})=\hat{G}(s)$ and $\hat{G}^r(\frac{1}{s})=\hat{G}^r(s)$. Applying the theory in [6, Theorem 5.1], the upper bound on the H_∞-errors is given

$$\left\|G(s)-G^r(s)\right\|_\infty \le \left\|\hat{G}(\frac{1}{s})-\hat{G}^r(\frac{1}{s})\right\|_\infty \le 2\sum_{i=r+1}^{\infty}\sigma_i. \tag{22}$$

In [10], it is shown that an exponentially stable infinite dimensional system with finite rank input and output operators has nuclear Hankel operator, i.e., $\sum_{i=1}^{\infty}\sigma_i<\infty$. So (22) would imply that $\left\|G(s)-G^r(s)\right\|_\infty \to 0$ as $r\to\infty$. □

An important property of BSPA method is that it the reduced order model G^r is equal to G at zero frequency, since

$$G^r(0) = -\overline{C}\,\overline{A}^{-1}\overline{B}+\overline{D}=\hat{D}=G(0), \quad \text{by (2) and (17)} \tag{23}$$

with assumption that zero is in the resolvent set of $\tilde{A}$. It is evident that the reduced order model via BSPA method for infinite dimensional systems preserves the steady-state gain of the system.

5. Numerical Computation

In the previous section it was assumed that the operator-valued Lyapunov equations can be derived analytically to construct the balanced realization of infinite dimensional systems. In general, we cannot obtain an exact solution of the operator-valued Lyapunov equations. Therefore, a convergent numerical computation must be used as in [7]. This section devoted the numerical algorithm to computing the model reduction by BSPA procedure according to result described in the last section. The algorithm of the BSPA method of infinite dimensional system can be described as follows.

1. Perform the approximating sequence (A^n,B^n,C^n,D^n) to (A,B,C,D), for sufficiently large n.
2. Solve the Lyapunov equations corresponding to (A^n,B^n,C^n,D^n).

3. Obtain the Lyapunov balanced realization $(\tilde{A}^n, \tilde{B}^n, \tilde{C}^n, \tilde{D}^n)$ of the system (A^n, B^n, C^n, D^n) using a similarity transformation.

4. Partition the balanced realization $(\tilde{A}^n, \tilde{B}^n, \tilde{C}^n, \tilde{D}^n)$ corresponding to $\Sigma = diag(\Sigma_1, \Sigma_2)$.

5. Apply the BSPA method to determine the r th order system $(\overline{A}, \overline{B}, \overline{C}, \overline{D})$ as defined in (6), with r as small as possible.

6. Numerical Example

In this section the construction of a low order model via BSPA for a heat conduction with Dirichlet boundary conditions is demonstrated. This model can be written as

$$\frac{\partial x}{\partial t}(t,\xi) = \frac{\partial^2 x}{\partial \xi^2}(t,\xi) + b(\xi)u(t), \quad x(0,\xi) = x_0(\xi)$$

$$x(t,0) = x(t,1) = 0, \tag{24}$$

$$y(t) = \begin{bmatrix} 2\int_0^{\frac{1}{2}} x(t,\xi)d\xi \\ 2\int_{\frac{1}{2}}^{1} x(t,\xi)d\xi \end{bmatrix},$$

where $x(\xi,t)$ represents the temperature at position $\xi \in (0,1)$ at time t, $x_0(\xi)$ the initial temperature and $b(\xi)u(t)$ represent the addition of heat along the bar according to the shape of $b(\xi)$. The output $y(t)$ is two measures of average of the temperature. Formally, the model can be expressed in abstract form (1), on the state space $X = L_2(0,1)$ and the state function $x(t) = \{x(\xi,t), 0 \le \xi \le 1\}$ with $D = 0$. The operators $A, B,$ and C are defined by

$$Ax(t) = \frac{d^2 x(t)}{d\xi^2}, \quad \mathrm{D}(A) = \left\{ z \in X : Ax \in X, z(0) = z(1) = 0 \right\}, \tag{25}$$

$$Bu = b(\xi)u, \quad \text{for} \quad u \in R, \quad b \in X$$

$$Cx(t) = \begin{bmatrix} 2\int_0^{\frac{1}{2}} x(t,\xi)d\xi \\ 2\int_{\frac{1}{2}}^{1} x(t,\xi)d\xi \end{bmatrix}.$$

Here, the input and the output operators are of finite rank, linear and bounded; $B \in L(R,X), C \in L(X,R^2)$. It can be shown that A is a Riesz-spectral operator with eigenvalues $\lambda_n = -n^2\pi^2, n \ge 1$ and orthonormal eigenvectors $\phi_n(\xi) = \sqrt{2}\sin(n\pi\xi)$ for

$n \geq 1$. They form an orthonormal basis for $L_2(0,1)$. Notice that the exponential stability of the Riesz-spectral operator is equivalent to the condition of $\sup\{Re(\lambda_i)\} < 0$. We see that the semigroup with generator A is exponentially stable, since $\sup\{Re(\lambda_i)\} = -\pi^2 < 0$. The shaping functions b is chosen as follows [8]

$$b(\xi) = \frac{1}{2\varepsilon} 1_{[x_0-\varepsilon, x_0+\varepsilon]}(\xi),$$

where

$$1_{[\alpha,\beta]}(\xi) = \begin{cases} 1 & for \quad \alpha \leq \xi \leq \beta, \\ 0 & \quad elsewhere, \end{cases}$$

with the control point $x_0 = \frac{3}{4}$, and $\varepsilon = \frac{1}{4}$. The algorithm of the model reduction approach discussed in Section 5 will be applied to the systems in (24) using the Galerkin finite element method with a linear spline basis [15]. To formulate a full order system, the spatial interval of the heat equation is divided into 80 elements.

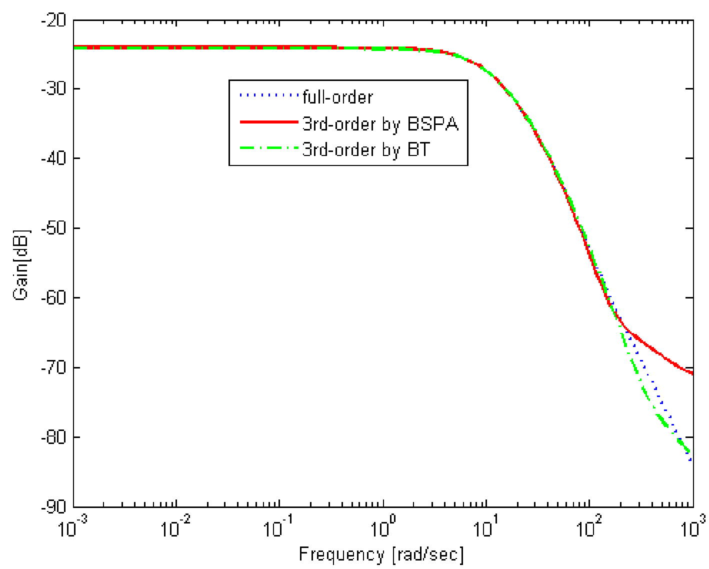

Figure 1. Frequency responses of full-order and reduced-order for output 1

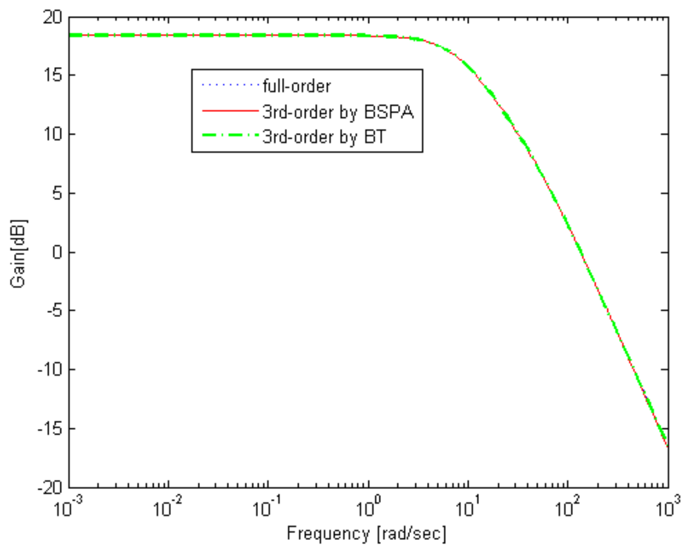

Figure 2. Frequency responses of full-order and reduced-order for output 2

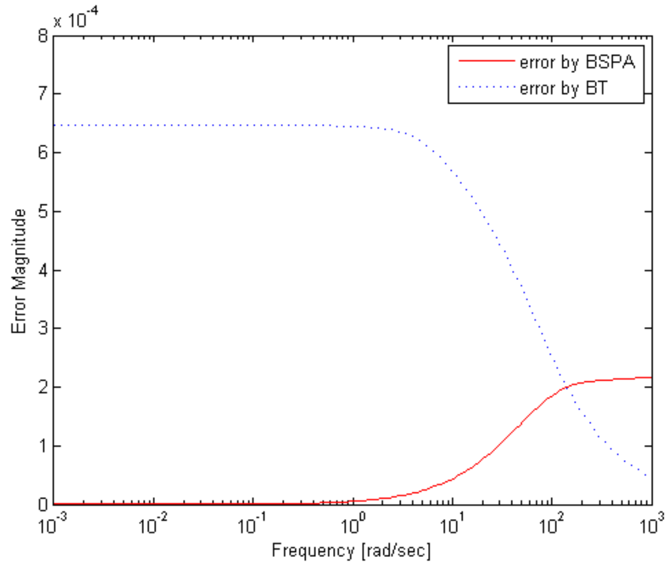

Figure 3. Frequency errors of full-order and reduced-order for output 1

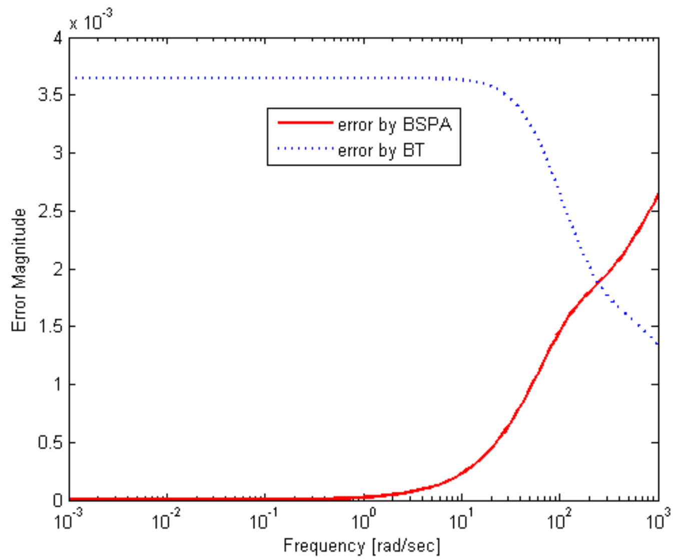

Figure 4. Frequency errors of full-order and reduced-order for output 2

The frequency responses of the full-order systems and of reduced-order systems by the balanced truncation (BT) and the BSPA method (BSPA) are shown in Figure 1 and Figure 2, respectively. The frequency response errors of both model reduction methods are given in Figure 3 and Figure 4, respectively. It is clear that the BSPA method gives a better approximation than the balanced truncation at low frequencies.

7. Conclusions

We have extended the model reduction using singular perturbation approximation method of the balanced infinite dimensional systems. The reduced-order model was obtained by setting the derivative of all states corresponding to the smaller Hankel singular values of the balanced systems equal to zero. The proposed method preserves the steady-state gain of the original system. From the simulation results show that the proposed method provide a good reduction errors than the balanced truncation at low frequencies.

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