Outer Synchronization of Complex Networks with Multiple Coupling Time-varying Delays*

Ping He[1], Chun-Guo Jing[2], Tao Fan[3] and Chang-Zhong Chen[3]

[1]*School of Information Science & Engineering, Northeastern University, Shenyang, Liaoning, 110819, People's Republic of China.*
pinghecn@126.com; pinghecn@qq.com
[2]*School of Computer and Communication Engineering, Northeastern University at Qinhuangdao, Qinhuangdao, Hebei, 066004, People's Republic of China.*
jingchunguo@163.com
[3]*School of Automation and Electronic Information, Sichuan University of Science & Engineering, Zigong, Sichuan 643000, People's Republic of China.*
fantaoid@qq.com; 709332544@qq.com

Abstract

This paper addresses the outer synchronization problem of complex networks with multiple coupling time-varying delays. First of all, some novel synchronization criteria are obtained by employing the Lyapunov stability theory and linear matrix inequality (LMI), we design some synchronization controllers to keep the given complex network synchronizing to the object trajectory. Furthermore, an adaptive outer synchronization scheme is derived to achieve global synchronization, which is simpler than some traditional controllers. Moreover, the presented results here can also be applied to complex networks with single time delay case. Finally, numerical analysis and simulations for two coupled complex networks which are composed of unified chaotic systems are given to demonstrate the effectiveness and feasibility of the proposed complex network control and synchronization schemes.

Keywords: *Complex networks, Synchronization, Multiple time-varying delays, LMI, Adaptive control, Unified chaotic systems.*

1 Introduction

In the last few years, the research problem of complex networks has received a compelling attention from various different fields such as physics sciences, mathematics sciences, economic science and engineering application [1, 2], which extensive exist everywhere in our daily lives including power grids, ecosystems, food webs, and so on. Therefore, discussing the dynamical behaviour of complex networks is quite importance to comprehending the effects of the real-world network.

* Correspondence to: School of Information Science & Engineering, Northeastern University, 11 Alley 3, Wenhua Road, Shenyang, Liaoning, 110819, People's Republic of China.

one hand, the time-delay was always single in many these existing literatures [39, 41]. On the other hand, the multiple coupling time-varying delays were usually considered in inner synchronization [44]. To the best of the author's knowledge, there has been few results on outer synchronization of complex networks with multiple coupling time-varying delays until now, so how to solve the outer synchronization problem for complex networks with multiple coupling time-varying delays still remains largely and challenging.

Based on the foregoing discussions, in this paper, we propose theoretical analysis and numerical simulations for outer synchronization of complex networks with multiple coupling time-varying delays. The most important contribution of this paper is to establish some LMI synchronization conditions and to present a effective adaptive synchronization result for a general complex networks with multiple coupling time-varying delays.

The remaining sections of this paper are organized as follows. In Section 2, we formulate and describe the outer synchronization problem of complex networks with multiple coupling time-varying delays. In Section 3, some linear state feedback controllers are designed and the gain matrix is calculated analytically based on Lyapunov function stability and LMI. Adaptive synchronization scheme is achieved by adaptive control technology. Numerical simulations are given in Section 4. Finally, conclusions are presented in Section 5.

2 Problem description

2.1 Complex networks model

We consider a complex network consisting of N identical dynamical nodes with multiple coupling time-varying delays

$$\dot{x}_i(t) = f(x_i(t)) + \varepsilon \sum_{j=1}^{N} c_{ij}\Gamma x_j(t-\tau_j(t)), i = 1, 2, \cdots, N. \tag{1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t)]^T \in R^n$ is the state variables of the node i, $f(x_i(t)) = [f_1(x_i(t)), f_2(x_i(t)), \cdots, f_n(x_i(t))]^T \in R^n$ is a continuously differentiable nonlinear vector-valued function, $\varepsilon > 0$ is the coupling strength, $C = [c_{ij}] \in R^{n\times n}$ is the coupling configuration weight matrix representing the topological structure of the complex network, whose entries c_{ij} are defined as follows: if there is a connection from node j to node i ($j \neq i$), $c_{ij} \neq 0$; otherwise, $c_{ij} = 0$ ($j \neq i$), Γ is an inner-coupling matrix, $\tau_j(t) \geq 0$ is the coupling time-varying delay.

To achieve outer synchronization between two complex networks with multiple coupling time-varying delays, the complex network (1) is referred as the drive network, and the response network can be defined as follows

$$\dot{y}_i(t) = f(y_i(t)) + \varepsilon \sum_{j=1}^{N} c_{ij}\Gamma y_j(t-\tau_j(t)) + u_i(t), i = 1, 2, \cdots, N. \tag{2}$$

where $y_i(t) = [y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t)]^T \in R^n$ is the state variables of the node i, $u_i(t) \in R^n$ is the linear synchronization controller for node i to be designed later.

Then, the outer synchronization error signal can be defined as follows

$$e_i(t) = y_i(t) - x_i(t). \tag{3}$$

Second, we can design a non-fragile robust synchronization controller as follows

$$u_i(t) = (K + \Delta K)e_i(t), i = 1, 2, \cdots, N. \tag{7}$$

where $\Delta K \in R^{n\times n}$ represents additive gain perturbation as follows:

$$\Delta K = M\Phi(t)N.$$

where M and N are known constant matrices, and the uncertain matrix $\Phi(t)$ satisfying the following condition

$$\Phi^T(t)\Phi(t) \leq I_n.$$

Thirdly, we can design a decoupling synchronization controller as follows

$$u_i(t) = ke_i(t), i = 1, 2, \cdots, N. \tag{8}$$

where k is a constant gain to be designed later.

Finally, we can design a adaptive synchronization controller of the following form:

$$u_i(t) = -k_i(t)e_i(t), \tag{9}$$

$$\dot{k}_i(t) = \gamma_i e_i^T(t)e_i(t), i = 1, 2, \cdots, N. \tag{10}$$

where $\gamma_i > 0$ is the adaptive gain to be designed later.

Then, substitution controllers (6)-(9) into the error dynamics network (4), we can obtain controlled closed-loop error dynamical networks of the following forms

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon\sum_{j=1}^{N} c_{ij}\Gamma e_j(t - \tau_j(t)) + Ke_i(t), \tag{11}$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon\sum_{j=1}^{N} c_{ij}\Gamma e_j(t - \tau_j(t)) + (K + \Delta K)e_i(t), \tag{12}$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon\sum_{j=1}^{N} c_{ij}\Gamma e_j(t - \tau_j(t)) + ke_i(t), \tag{13}$$

and

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon\sum_{j=1}^{N} c_{ij}\Gamma e_j(t - \tau_j(t)) + k_i(t)e_i(t). \tag{14}$$

respectively.

3 Main Results

In this section, we are in the next position to propose our main results for outer synchronization of complex networks with multiple coupling time-varying delays constraint upon LMI and by using the adaptive control schemes.

Substituting (18), (19) and (20) into (17), yield

$$\begin{aligned}
& \dot{V}(e(t)) \mid_{(11)} \qquad (21) \\
\leq\ & \sum_{i=1}^{N}\{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \varepsilon\sum_{j=1}^{N}c_{ij}^2 e_i^T(t)\Gamma\Gamma^T e_i(t) + \frac{\varepsilon N}{1-\mu_i}e_i^T(t)e_i(t)\} \\
=\ & \sum_{i=1}^{N}\{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \frac{\varepsilon N}{1-\mu_i}e_i^T(t)e_i(t)\} + e^T(t)(\varepsilon[c_{ij}^2]\otimes\Gamma\Gamma^T)e(t) \\
=\ & e^T(t)\{I_N\otimes[2LI_n + K + K^T] + diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N})\otimes I_n \\
& +\varepsilon([c_{ij}^2]\otimes\Gamma\Gamma^T)\}e(t).
\end{aligned}$$

where $e(t) = [e_1^T(t), e_2^T(t), \cdots, e_N^T(t)]^T$.

According to (15)

$$\dot{V}(e(t)) < 0. \tag{22}$$

Thus, according to definition 1, the drive network (1) and the response network (2) have achieved complete outer synchronization under the controller (6).

The proof is completed.

Remark 3 *In a large of existing literature, C is supposed to be symmetric [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], which is not practical. C is supposed to be irreducible and reducible which are two parts to discuss [49], respectively. However, in this paper, no assumption on C is needed. The configuration matrix C needs not be symmetric, diffusive, or irreducible. This means that the networks (1) or (2) can either be undirected or directed networks and contain isolated nodes. The complex network structure in this paper is very general and this theorem can be applied to a great many complex networks in the real world.*

Theorem 2 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the nonfragile robust synchronization controller (7) if there exists a controller gain matrix K such that the following LMI condition holds*

$$\begin{aligned}
& I_N\otimes[K^T + K + 2LI_n + MM^T + N^TN] \qquad (23) \\
& +diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N})\otimes I_n + \varepsilon^2([c_{ij}^2]\otimes\Gamma\Gamma^T) < 0.
\end{aligned}$$

Proof: The proof process is omitted since the conclusion is obvious.

Remark 4 *If there is no constraints of (M, N), the non-fragile outer synchronization of complex networks reduces to theorem 1.*

Theorem 3 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the decoupling synchronization controller (8) if there exists a constant k such that the following condition holds*

$$k < -L - \frac{1}{2}\max_{1\leq i\leq N}(\frac{\varepsilon N}{1-\mu_i}) - \lambda_{max}(\frac{1}{2}\varepsilon([c_{ij}^2]\otimes\Gamma\Gamma^T)). \tag{24}$$

Proof: The proof process is omitted since the conclusion is obvious.

where $\Xi = I_{nN} \otimes [2L - 2\rho_i] + diag(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \cdots, \frac{\varepsilon N}{1-\mu_N}) \otimes I_n + \varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)$.

We can choose a sufficiently large ρ_i such that $\Xi < 0$, according to this

$$\dot{V}(e(t)) < 0. \tag{28}$$

Thus, according to Definition 1, the drive network (1) and the response network (2) have achieved adaptive synchronization under the adaptive synchronization controller (9)-(10).

The proof is completed.

Remark 7 *Some stability criteria for the synchronization between drive and response complex networks with multiple coupling time-varying delays are derived, which can also be applied to the complex network with single time delay. Thus, the results presented in this paper improve and generalize the corresponding results of recent works.*

4 Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness and feasibility of the outer synchronization controller obtained in the previous section. Without loss of generality, we take unified chaotic systems as the local node dynamics.

4.1 Unified chaotic systems

Based on the Lorenz system, Lü system and Chen system, the unified chaotic system was proposed by Lü [47]. The unified chaotic system is described by

$$\begin{cases} \dot{x}_1 = (25\alpha + 1)(x_2 - x_1), \\ \dot{x}_2 = (28 - 35\alpha)x_1 - (29\alpha - 1)x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - \frac{8+\alpha}{3}x_3. \end{cases} \tag{29}$$

where $[x_1, x_2, x_3]^T \in R^3$ is the state variables group of unified chaotic system and $\alpha \in [0, 1]$ is a system parameter.

Due to the chaotic system (29) is chaotic for arbitrarily $\alpha \in [0, 1]$ and the system (29) belongs to the generalized Lorenz chaotic system for $0 \leq \alpha < 0.8$, the system (29) belongs to the Lü chaotic system at $\alpha = 0.8$ and the system (29) belongs to the generalized Chen chaotic system for $0.8 < \alpha \leq 1$, so the chaotic system (29) is regarded as unified chaotic systems.

In this subsection, the main theorems are illustrated by the following numerical simulations. For some special parameter values, from (29) , the parameter α is selected by $\alpha = 0.8$ and $\alpha = 0$, respectively, the initial condition $x(0) = [1, 0, -1]^T$ is chosen. The chaotic trajectories of unified chaotic system are illustrated in **Figures** 1, 3, respectively. The time evolution of state variable is shown in **Figures** 2, 4, respectively. It is easy to know that the trajectories of the unified chaotic system are bounded.

4.2 Systems simulation

Clearly, assumption 1 is verified if the $L = 150$, assumption 2 is verified if the $\tau_1(t) = \frac{1}{2} - \frac{1}{2}e^{-t}$ and $\bar{\tau}_1 = \frac{1}{2}$, $\mu_1 = \frac{1}{2}$, $\tau_2(t) = \frac{2}{5} - \frac{2}{5}e^{-t}$ and $\bar{\tau}_2 = \frac{2}{5}$, $\mu_2 = \frac{2}{5}$, $\tau_3(t) = \frac{3}{10} - \frac{3}{10}e^{-t}$ and $\bar{\tau}_3 = \frac{3}{10}$, $\mu_3 = \frac{3}{10}$.