

Outer Synchronization of Complex Networks with Multiple Coupling Time-varying Delays*

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Abstract

This paper addresses the outer synchronization problem of complex networks with multiple coupling time-varying delays. First of all, some novel synchronization criteria are obtained by employing the Lyapunov stability theory and linear matrix inequality (LMI), we design some synchronization controllers to keep the given complex network synchronizing to the object trajectory. Furthermore, an adaptive outer synchronization scheme is derived to achieve global synchronization, which is simpler than some traditional controllers. Moreover, the presented results here can also be applied to complex networks with single time delay case. Finally, numerical analysis and simulations for two coupled complex networks which are composed of unified chaotic systems are given to demonstrate the effectiveness and feasibility of the proposed complex network control and synchronization schemes.

Keywords: *Complex networks, Synchronization, Multiple time-varying delays, LMI, Adaptive control, Unified chaotic systems.*

1 Introduction

In the last few years, the research problem of complex networks has received a compelling attention from various different fields such as physics sciences, mathematics sciences, economic science and engineering application [1, 2], which extensive exist everywhere in our daily lives including power grids, ecosystems, food webs, and so on. Therefore, discussing the dynamical behaviour of complex networks is quite importance to comprehending the effects of the real-world network.

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A large amount of nonlinear dynamical behaviors of complex systems, synchronization is a long-standing ever hot research topic [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The large-scale system decomposition method was employed to discuss consensus complex network model [4]. Global asymptotic synchronization condition of neural networks with node delay and hybrid coupling was proposed by the Lyapunov functional method and LMI [6, 7, 8, 9, 10]. The complex dynamical network with neutral delay weights and adaptive coupling weights was also investigated [11, 13], respectively.

Nevertheless, according to above-mentioned relatively literatures, on the one hand, the coupling configuration weight matrix is constantly supposed to be symmetric and irreducible, which are very restrictive in engineering practice. On the other hand, it should be emphasized that a large of previous results on complex network synchronization were analysis the internal synchronization behavior of complex networks, which is called as “inner synchronization”. Instead of the synchronization between two or more complex networks is regarded as outer synchronization, which was firstly considered by Li et al. [16]. The complete outer synchronization (COS) problem of complex networks with identical topological structures was investigated by employing an open-plus-closed-loop controller (OPCL) scheme [16]. Other researchers also have investigated outer synchronization [17, 18, 19, 20, 21, 22, 23, 24, 25], etc. Afterwards, the outer synchronization for discrete-time complex networks was also investigated [17]. The complete synchronization of complex networks with nonidentical and identical topological structures were investigated by using adaptive control scheme [18, 19, 20], respectively.

There exist a large research results focusing on outer synchronization of complex networks [26, 27, 28, 29, 30, 31]. Soon after, the adaptive synchronization controller was proposed to achieve outer synchronization between complex networks with different strengths and nonidentical topological structures [31]. On the other hand, the outer synchronization also furnishes a platform and tool for identifying unknown network parameters or dynamical topology of a complex network [32, 33, 34].

In these early research results on network outer synchronization [26, 27, 28, 29, 30, 31, 32, 33, 34], it is often supposed that the each node in networks is identical, and the corresponding nodes in networks have completely the same dynamics. So, strictly speaking, it is COS between two complex networks. To overcome such a limitation, inverse outer synchronization (IOS) of complex networks was presented by constructing a nonlinear type OPCL [35]. The generalized outer synchronization (GOS) scheme of complex networks was also proposed [36]. A nonlinear synchronization controller and a adaptive updated law were presented to achieve the GOS between two complex networks with coupling time-varying delay [37].

Furthermore, for large-scale complex networks, since the finite information transmission and computer processing speeds among complex network nodes, time-delays are compelling attention and should be considered. Time-delay coupling extremely exists in many real-world systems such as gene regulatory networks and electrical power grids, etc. Therefore, it is quite important to investigate the synchronization of complex networks with coupling time-delay [38].

The problem of function projective synchronization for general complex networks with time delay was investigated [39]. Adaptive synchronization and pinning control of colored networks were considered [40]. Finite-time mixed outer synchronization controllers for complex networks with time-varying coupling delay were presented [41]. Finite-time stochastic outer synchronization between two complex networks with different topologies was considered again [42]. The outer synchronization between two complex networks with discontinuous coupling was investigated [43]. The inner synchronization was generalized by using input passivity and output passivity [44].

However, the investigation of synchronization of complex networks is still insufficient. On the

one hand, the time-delay was always single in many these existing literatures [39, 41]. On the other hand, the multiple coupling time-varying delays were usually considered in inner synchronization [44]. To the best of the author's knowledge, there has been few results on outer synchronization of complex networks with multiple coupling time-varying delays until now, so how to solve the outer synchronization problem for complex networks with multiple coupling time-varying delays still remains largely and challenging.

Based on the foregoing discussions, in this paper, we propose theoretical analysis and numerical simulations for outer synchronization of complex networks with multiple coupling time-varying delays. The most important contribution of this paper is to establish some LMI synchronization conditions and to present a effective adaptive synchronization result for a general complex networks with multiple coupling time-varying delays.

The remaining sections of this paper are organized as follows. In Section 2, we formulate and describe the outer synchronization problem of complex networks with multiple coupling time-varying delays. In Section 3, some linear state feedback controllers are designed and the gain matrix is calculated analytically based on Lyapunov function stability and LMI. Adaptive synchronization scheme is achieved by adaptive control technology. Numerical simulations are given in Section 4. Finally, conclusions are presented in Section 5.

2 Problem description

2.1 Complex networks model

We consider a complex network consisting of N identical dynamical nodes with multiple coupling time-varying delays

$$\dot{x}_i(t) = f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij} \Gamma x_j(t - \tau_j(t)), i = 1, 2, \dots, N. \quad (1)$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \dots, x_{in}(t)]^T \in R^n$ is the state variables of the node i , $f(x_i(t)) = [f_1(x_i(t)), f_2(x_i(t)), \dots, f_n(x_i(t))]^T \in R^n$ is a continuously differentiable nonlinear vector-valued function, $\varepsilon > 0$ is the coupling strength, $C = [c_{ij}] \in R^{n \times n}$ is the coupling configuration weight matrix representing the topological structure of the complex network, whose entries c_{ij} are defined as follows: if there is a connection from node j to node i ($j \neq i$), $c_{ij} \neq 0$; otherwise, $c_{ij} = 0$ ($j \neq i$), Γ is an inner-coupling matrix, $\tau_j(t) \geq 0$ is the coupling time-varying delay.

To achieve outer synchronization between two complex networks with multiple coupling time-varying delays, the complex network (1) is referred as the drive network, and the response network can be defined as follows

$$\dot{y}_i(t) = f(y_i(t)) + \varepsilon \sum_{j=1}^N c_{ij} \Gamma y_j(t - \tau_j(t)) + u_i(t), i = 1, 2, \dots, N. \quad (2)$$

where $y_i(t) = [y_{i1}(t), y_{i2}(t), \dots, y_{in}(t)]^T \in R^n$ is the state variables of the node i , $u_i(t) \in R^n$ is the linear synchronization controller for node i to be designed later.

Then, the outer synchronization error signal can be defined as follows

$$e_i(t) = y_i(t) - x_i(t). \quad (3)$$

Consequently, the error dynamical network between the drive network (1) and the response network (2) can be given by

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij} \Gamma e_j(t - \tau_j(t)) + u_i(t), i = 1, 2, \dots, N. \quad (4)$$

2.2 Mathematical preliminaries

Before stating the main results of this paper, the following lemma and assumptions are necessary to prove our main results.

Assumption 1 *The nonlinear function $f(\cdot)$ satisfies the global Lipschitz condition; that is, there exists a constant $L > 0$ such that*

$$\|f(\alpha) - f(\beta)\| \leq L \|\alpha - \beta\|, \forall \alpha, \beta \in \mathbb{R}^n.$$

Remark 1 *It is easy to verify that some general chaotic systems such as the original Chua's circuit [45], hyperchaotic unified systems [46], unified chaotic systems [47] and modified Chua's circuit [45] can hold.*

Assumption 2 *The coupling time-varying delay $\tau_j(t)$ is a continuously differential function satisfies*

$$0 \leq \dot{\tau}_j(t) \leq \mu_j < 1, 0 \leq \tau_j(t) \leq \bar{\tau}_j.$$

Lemma 1 *(Matrix Cauchy inequality [48]) For any symmetric positive definite matrix $M \in \mathbb{R}^{n \times n}$ and vector $x, y \in \mathbb{R}^n$, yield*

$$\pm 2x^T y \leq x^T M x + y^T M^{-1} y.$$

Then, the complete outer synchronization between two complex networks can be defined as follows:

Definition 1 *The drive network (1) and the response network (2) are said to be achieved complete outer synchronization if there exists a controller $u_i(t)$ for any initial states $x_i(0), y_i(0)$ such that*

$$\lim_{t \rightarrow \infty} \|y_i(t, y_i(0)) - x_i(t, x_i(0))\| = 0. \quad (5)$$

Remark 2 *From definition 1, we can see that outer synchronization means that the corresponding nodes of coupled networks could achieve complete synchronization regardless of synchronization behaviour of inner networks. In this paper, we go beyond any restrictions on the Laplacian matrix. In general, the Laplacian matrix is not assumed to be symmetric or irreducible. Therefore, these two complex networks can be undirected or directed and may also contain clusters and isolated nodes.*

2.3 Designing of controllers

First of all, we can design a nominal synchronization controller as follows

$$u_i(t) = K e_i(t), i = 1, 2, \dots, N. \quad (6)$$

where $K \in \mathbb{R}^{n \times n}$ is the controller gain matrix to be designed later.

Second, we can design a non-fragile robust synchronization controller as follows

$$u_i(t) = (K + \Delta K)e_i(t), i = 1, 2, \dots, N. \quad (7)$$

where $\Delta K \in R^{n \times n}$ represents additive gain perturbation as follows:

$$\Delta K = M\Phi(t)N.$$

where M and N are known constant matrices, and the uncertain matrix $\Phi(t)$ satisfying the following condition

$$\Phi^T(t)\Phi(t) \leq I_n.$$

Thirdly, we can design a decoupling synchronization controller as follows

$$u_i(t) = ke_i(t), i = 1, 2, \dots, N. \quad (8)$$

where k is a constant gain to be designed later.

Finally, we can design an adaptive synchronization controller of the following form:

$$u_i(t) = -k_i(t)e_i(t), \quad (9)$$

$$\dot{k}_i(t) = \gamma_i e_i^T(t)e_i(t), i = 1, 2, \dots, N. \quad (10)$$

where $\gamma_i > 0$ is the adaptive gain to be designed later.

Then, substitution controllers (6)-(9) into the error dynamics network (4), we can obtain controlled closed-loop error dynamical networks of the following forms

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij}\Gamma e_j(t - \tau_j(t)) + Ke_i(t), \quad (11)$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij}\Gamma e_j(t - \tau_j(t)) + (K + \Delta K)e_i(t), \quad (12)$$

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij}\Gamma e_j(t - \tau_j(t)) + ke_i(t), \quad (13)$$

and

$$\dot{e}_i(t) = f(y_i(t)) - f(x_i(t)) + \varepsilon \sum_{j=1}^N c_{ij}\Gamma e_j(t - \tau_j(t)) + k_i(t)e_i(t). \quad (14)$$

respectively.

3 Main Results

In this section, we are in the next position to propose our main results for outer synchronization of complex networks with multiple coupling time-varying delays constraint upon LMI and by using the adaptive control schemes.

3.1 Complete outer synchronization of complex networks

In this subsection, we first introduce a complete outer synchronization control strategy via LMI to present our main results.

Theorem 1 *Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the synchronization controller (6) if there exists a controller gain matrix K such that the following LMI condition holds*

$$I_N \otimes [K^T + K + 2LI_n] + \text{diag}\left(\frac{\varepsilon N}{1 - \mu_1}, \frac{\varepsilon N}{1 - \mu_2}, \dots, \frac{\varepsilon N}{1 - \mu_N}\right) \otimes I_n + \varepsilon^2([c_{ij}^2] \otimes \Gamma\Gamma^T) < 0. \quad (15)$$

Proof: Choose a Lyapunov-Krasovskii functional candidate for the error dynamical network (11) as follows

$$V(e(t)) = \sum_{i=1}^N [e_i^T(t)e_i(t) + \frac{\varepsilon N}{1 - \mu_i} \int_{t-\tau_i(t)}^t e_i^T(s)e_i(s)ds]. \quad (16)$$

Differentiating the Lyapunov-Krasovskii functional (16) along the trajectory of (11), yield

$$\begin{aligned} \dot{V}(e(t)) |_{(11)} & \quad (17) \\ &= \sum_{i=1}^N [2e_i^T(t)\dot{e}_i(t) + \frac{\varepsilon N}{1 - \mu_i} e_i^T(t)e_i(t) - \frac{\varepsilon N(1 - \dot{\tau}_i(t))}{1 - \mu_i} e_i^T(t - \tau_i(t))e_i(t - \tau_i(t))] \\ &= \sum_{i=1}^N [2e_i^T(t)[f(y_i(t)) - f(x_i(t))] + 2\varepsilon \sum_{j=1}^N c_{ij}e_i^T(t)\Gamma e(t - \tau_j(t)) + 2e_i^T(t)Ke_i(t)] \\ & \quad + \frac{\varepsilon N}{1 - \mu_i} e_i^T(t)e_i(t) - \frac{\varepsilon N(1 - \dot{\tau}_i(t))}{1 - \mu_i} e_i^T(t - \tau_i(t))e_i(t - \tau_i(t)). \end{aligned}$$

Applying assumption 1, yield

$$\begin{aligned} e_i(t)^T [f(y_i(t)) - f(x_i(t))] & \leq \|e_i(t)\| \cdot \|f(y_i(t)) - f(x_i(t))\| \\ & \leq \|e_i(t)\| \cdot L \cdot \|e_i(t)\| = Le_i^T(t)e_i(t). \end{aligned} \quad (18)$$

Applying lemma 1, yield

$$\begin{aligned} & 2 \sum_{i=1}^N \sum_{j=1}^N c_{ij}e_i^T(t)\Gamma e(t - \tau_j(t)) \\ & \leq \sum_{i=1}^N \sum_{j=1}^N c_{ij}^2 e_i^T(t)\Gamma\Gamma^T e_i(t) + \sum_{i=1}^N \sum_{j=1}^N e_j^T(t - \tau_j(t))e_j(t - \tau_j(t)) \\ & = \sum_{i=1}^N \sum_{j=1}^N c_{ij}^2 e_i^T(t)\Gamma\Gamma^T e_i(t) + N \sum_{i=1}^N e_i^T(t - \tau_i(t))e_i(t - \tau_i(t)). \end{aligned} \quad (19)$$

Applying assumption 2, yield

$$1 - \frac{1 - \dot{\tau}_i(t)}{1 - \mu_i} < 0. \quad (20)$$

Substituting (18), (19) and (20) into (17), yield

$$\begin{aligned}
 & \dot{V}(e(t)) \Big|_{(11)} \\
 \leq & \sum_{i=1}^N \{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \varepsilon \sum_{j=1}^N c_{ij}^2 e_i^T(t)\Gamma\Gamma^T e_i(t) + \frac{\varepsilon N}{1 - \mu_i} e_i^T(t)e_i(t)\} \\
 = & \sum_{i=1}^N \{e_i^T(t)[2LI_n + K + K^T]e_i(t) + \frac{\varepsilon N}{1 - \mu_i} e_i^T(t)e_i(t)\} + e^T(t)(\varepsilon[c_{ij}^2] \otimes \Gamma\Gamma^T)e(t) \\
 = & e^T(t)\{I_N \otimes [2LI_n + K + K^T] + \text{diag}(\frac{\varepsilon N}{1 - \mu_1}, \frac{\varepsilon N}{1 - \mu_2}, \dots, \frac{\varepsilon N}{1 - \mu_N}) \otimes I_n \\
 & + \varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)\}e(t).
 \end{aligned} \tag{21}$$

where $e(t) = [e_1^T(t), e_2^T(t), \dots, e_N^T(t)]^T$.

According to (15)

$$\dot{V}(e(t)) < 0. \tag{22}$$

Thus, according to definition 1, the drive network (1) and the response network (2) have achieved complete outer synchronization under the controller (6).

The proof is completed.

Remark 3 In a large of existing literature, C is supposed to be symmetric [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], which is not practical. C is supposed to be irreducible and reducible which are two parts to discuss [49], respectively. However, in this paper, no assumption on C is needed. The configuration matrix C needs not be symmetric, diffusive, or irreducible. This means that the networks (1) or (2) can either be undirected or directed networks and contain isolated nodes. The complex network structure in this paper is very general and this theorem can be applied to a great many complex networks in the real world.

Theorem 2 Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the nonfragile robust synchronization controller (7) if there exists a controller gain matrix K such that the following LMI condition holds

$$\begin{aligned}
 & I_N \otimes [K^T + K + 2LI_n + MM^T + N^T N] \\
 & + \text{diag}(\frac{\varepsilon N}{1 - \mu_1}, \frac{\varepsilon N}{1 - \mu_2}, \dots, \frac{\varepsilon N}{1 - \mu_N}) \otimes I_n + \varepsilon^2([c_{ij}^2] \otimes \Gamma\Gamma^T) < 0.
 \end{aligned} \tag{23}$$

Proof: The proof process is omitted since the conclusion is obvious.

Remark 4 If there is no constraints of (M, N) , the non-fragile outer synchronization of complex networks reduces to theorem 1.

Theorem 3 Under assumptions 1-2, the drive network (1) and the response network (2) can achieve complete outer synchronization under the decoupling synchronization controller (8) if there exists a constant k such that the following condition holds

$$k < -L - \frac{1}{2} \max_{1 \leq i \leq N} (\frac{\varepsilon N}{1 - \mu_i}) - \lambda_{\max}(\frac{1}{2}\varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)). \tag{24}$$

Proof: The proof process is omitted since the conclusion is obvious.

Remark 5 Theorems 1-3 provide the outer synchronization condition of complex networks with multiple coupling time-varying delays. Similarly, outer exponential synchronization conditions can also be presented, which will be investigated in our future work.

Remark 6 Unlike from previous results on nonlinear feedback schemes for outer synchronization [26, 27, 28, 29, 30, 31, 32, 33, 34] which are less flexible, we have designed a linear synchronization controller for achieving out synchronization. Our designed synchronization controller is not only robust, but also easy to implement.

3.2 Adaptive outer synchronization of complex networks

In this subsection, we introducing a adaptive control strategy to present our main results:

Theorem 4 Under assumptions 1-2, the drive network (1) and the response network (2) can achieve adaptive outer synchronization under the adaptive synchronization controller (9).

Proof: Define a Lyapunov-Krasovskii functional candidate for the error dynamical network (13) as follows

$$V(e(t)) = \sum_{i=1}^N [e_i^T(t)e_i(t) + \frac{1}{\gamma_i}(k_i(t) - \rho_i)^2 + \frac{\varepsilon N}{1 - \mu_i} \int_{t-\tau_i(t)}^t e_i^T(s)e_i(s)ds]. \quad (25)$$

Differentiating of Lyapunov-Krasovskii functional (25) along the trajectories (13), yield

$$\begin{aligned} & \dot{V}(e(t)) |_{(13)} \quad (26) \\ &= \sum_{i=1}^N [2e_i^T(t)\dot{e}_i(t) + \frac{2}{\gamma_i}(k_i(t) - \rho_i)\dot{k}_i(t) + \frac{\varepsilon N}{1 - \mu_i}e_i^T(t)e_i(t) \\ & \quad - \frac{\varepsilon N(1 - \dot{\tau}_i(t))}{1 - \mu_i}e_i^T(t - \tau_i(t))e(t - \tau_i(t))] \\ &= \sum_{i=1}^N [2e_i^T(t)[f(y_i(t)) - f(x_i(t))] + 2\varepsilon \sum_{j=1}^N c_{ij}e_i^T(t)\Gamma e(t - \tau_j(t))] \\ & \quad - 2k_i e_i^T(t)e_i(t) + \frac{2}{\gamma_i}(k_i(t) - \rho_i)\gamma_i e_i^T(t)e_i(t) \\ & \quad + \frac{\varepsilon N}{1 - \mu_i}e_i^T(t)e_i(t) - \frac{\varepsilon N(1 - \dot{\tau}_i(t))}{1 - \mu_i}e_i^T(t - \tau_i(t))e_i(t - \tau_i(t)). \end{aligned}$$

Then, it follows from (26), (18), (19) and (20) that

$$\begin{aligned} & \dot{V}(e(t)) |_{(13)} \quad (27) \\ &\leq \sum_{i=1}^N \{e_i^T(t)[2L - 2\rho_i]e_i(t) + \varepsilon \sum_{j=1}^N c_{ij}^2 e_i^T(t)\Gamma\Gamma^T e_i(t) + \frac{\varepsilon N}{1 - \mu_i}e_i^T(t)e_i(t)\} \\ &= e^T(t)\{I_{nN} \otimes [2L - 2\rho_i] + \text{diag}(\frac{\varepsilon N}{1 - \mu_1}, \frac{\varepsilon N}{1 - \mu_2}, \dots, \frac{\varepsilon N}{1 - \mu_N}) \otimes I_n \\ & \quad + \varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)\}e(t) \\ &= e(t)^T \Xi e(t). \end{aligned}$$

where $\Xi = I_{nN} \otimes [2L - 2\rho_i] + \text{diag}(\frac{\varepsilon N}{1-\mu_1}, \frac{\varepsilon N}{1-\mu_2}, \dots, \frac{\varepsilon N}{1-\mu_N}) \otimes I_n + \varepsilon([c_{ij}^2] \otimes \Gamma\Gamma^T)$.

We can choose a sufficiently large ρ_i such that $\Xi < 0$, according to this

$$\dot{V}(e(t)) < 0. \quad (28)$$

Thus, according to Definition 1, the drive network (1) and the response network (2) have achieved adaptive synchronization under the adaptive synchronization controller (9)-(10).

The proof is completed.

Remark 7 *Some stability criteria for the synchronization between drive and response complex networks with multiple coupling time-varying delays are derived, which can also be applied to the complex network with single time delay. Thus, the results presented in this paper improve and generalize the corresponding results of recent works.*

4 Numerical simulations

In this section, numerical simulations are presented to verify the effectiveness and feasibility of the outer synchronization controller obtained in the previous section. Without loss of generality, we take unified chaotic systems as the local node dynamics.

4.1 Unified chaotic systems

Based on the Lorenz system, Lü system and Chen system, the unified chaotic system was proposed by Lü [47]. The unified chaotic system is described by

$$\begin{cases} \dot{x}_1 = (25\alpha + 1)(x_2 - x_1), \\ \dot{x}_2 = (28 - 35\alpha)x_1 - (29\alpha - 1)x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - \frac{8+\alpha}{3}x_3. \end{cases} \quad (29)$$

where $[x_1, x_2, x_3]^T \in R^3$ is the state variables group of unified chaotic system and $\alpha \in [0, 1]$ is a system parameter.

Due to the chaotic system (29) is chaotic for arbitrarily $\alpha \in [0, 1]$ and the system (29) belongs to the generalized Lorenz chaotic system for $0 \leq \alpha < 0.8$, the system (29) belongs to the Lü chaotic system at $\alpha = 0.8$ and the system (29) belongs to the generalized Chen chaotic system for $0.8 < \alpha \leq 1$, so the chaotic system (29) is regarded as unified chaotic systems.

In this subsection, the main theorems are illustrated by the following numerical simulations. For some special parameter values, from (29), the parameter α is selected by $\alpha = 0.8$ and $\alpha = 0$, respectively, the initial condition $x(0) = [1, 0, -1]^T$ is chosen. The chaotic trajectories of unified chaotic system are illustrated in **Figures 1, 3**, respectively. The time evolution of state variable is shown in **Figures 2, 4**, respectively. It is easy to know that the trajectories of the unified chaotic system are bounded.

4.2 Systems simulation

Clearly, assumption 1 is verified if the $L = 150$, assumption 2 is verified if the $\tau_1(t) = \frac{1}{2} - \frac{1}{2}e^{-t}$ and $\bar{\tau}_1 = \frac{1}{2}$, $\mu_1 = \frac{1}{2}$, $\tau_2(t) = \frac{2}{5} - \frac{2}{5}e^{-t}$ and $\bar{\tau}_2 = \frac{2}{5}$, $\mu_2 = \frac{2}{5}$, $\tau_3(t) = \frac{3}{10} - \frac{3}{10}e^{-t}$ and $\bar{\tau}_3 = \frac{3}{10}$, $\mu_3 = \frac{3}{10}$.

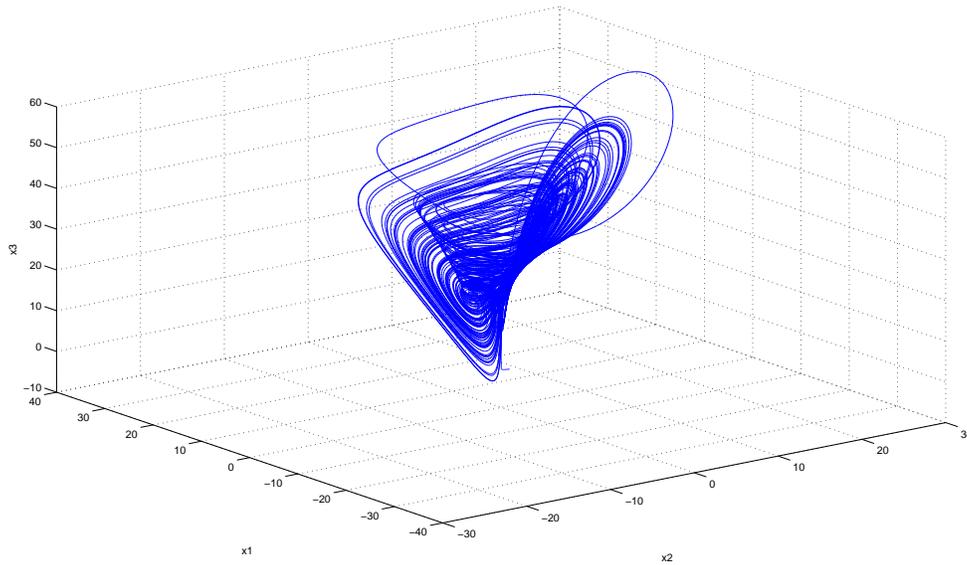


Figure 1. Chaotic attractor of unified chaotic systems (29) with $\alpha = 0.8$.

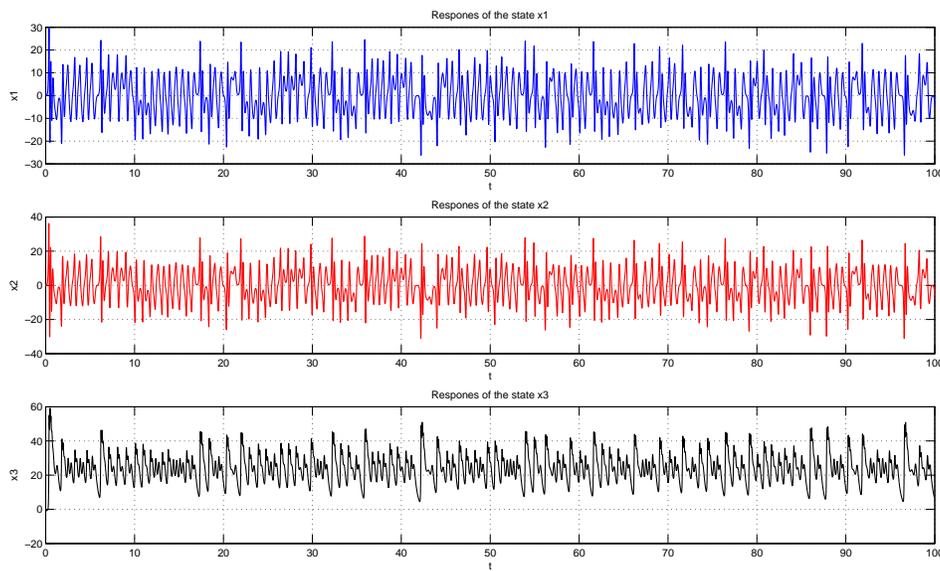


Figure 2. State response of unified chaotic systems (29) with $\alpha = 0.8$.

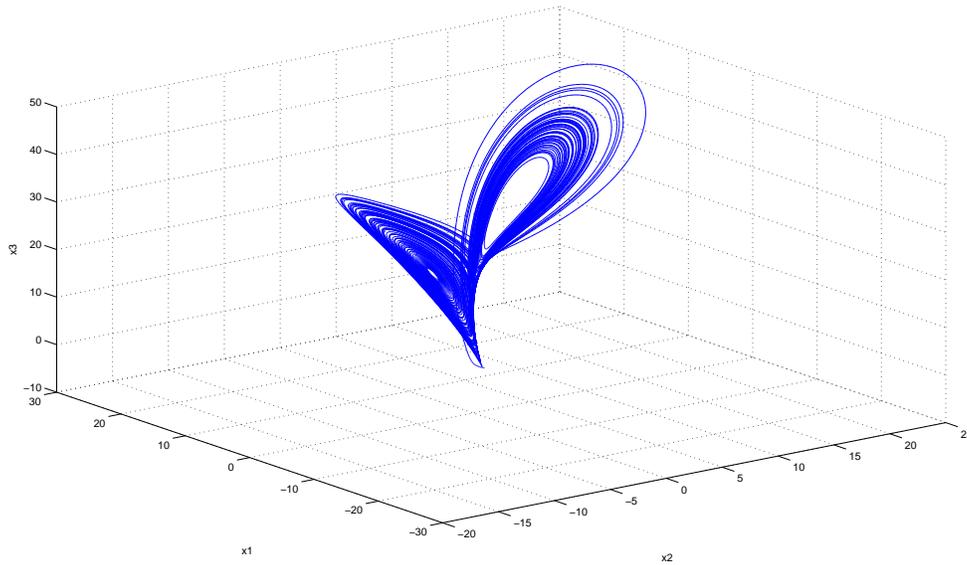


Figure 3. Chaotic attractor of unified chaotic systems (29) with $\alpha = 0$.

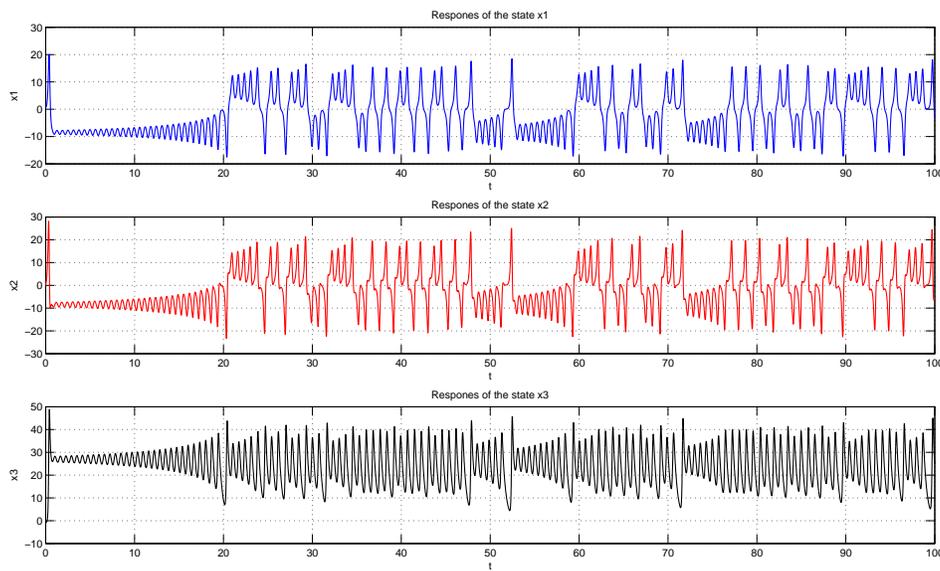


Figure 4. State response of unified chaotic systems (29) with $\alpha = 0$.

Example 1 In this example, a diffusively coupled complex network containing 3 subsystems of (29) is investigated. The coupling configuration weight matrix C is chosen as follows

$$C = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

The inner coupling matrix Γ and the coupling strength ε are chosen as follows

$$\varepsilon = 1,$$

$$\Gamma = \begin{bmatrix} 1 & 4 & -1 \\ -2 & 3 & -4 \\ 2 & -1 & 5 \end{bmatrix}.$$

By using the Matlab LMI control Toolbox to solve the LMI condition in Theorem 1, we can obtain a set of feasible solutions,

$$K = \begin{bmatrix} -118.2570 & 0 & 0 \\ 0 & -118.8285 & 0 \\ 0 & 0 & -118.1885 \end{bmatrix}.$$

Therefore, we take the unified chaotic system ($\alpha = 0.8$) as the local node dynamics, the drive network (1) and the response network (2) can achieve complete outer synchronization under the synchronization controller (6). With initial conditions

$$x_1(0) = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, x_2(0) = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}, x_3(0) = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix},$$

$$y_1(0) = \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix}, y_2(0) = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, y_3(0) = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}.$$

The numerical simulations are proposed in **Figures 5-7**.

From **Figures 5-7**, it is easy to know that, the synchronization errors can fast converge asymptotically to zero, which implies that the drive network (1) and the response network (2) have achieved complete outer synchronization.

Example 2 In this example, a non-diffusively coupled complex network containing 3 subsystems of (29) is investigated. The coupling configuration matrix C is chosen as follows

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

The inner coupling matrix Γ and the coupling strength ε are chosen as follows

$$\varepsilon = 1,$$

$$\Gamma = \begin{bmatrix} 0 & 9 & 8 \\ 7 & 6 & 5 \\ 4 & 3 & 2 \end{bmatrix}.$$

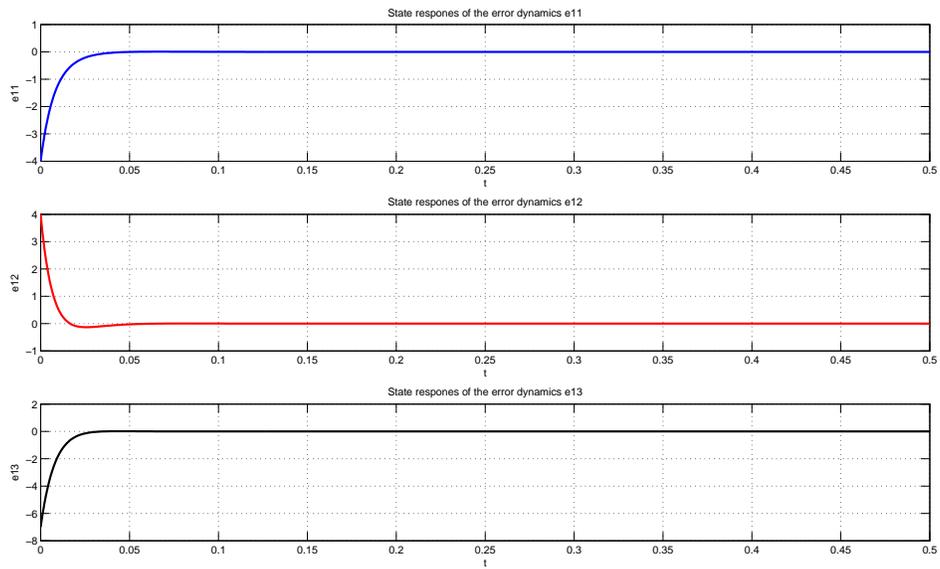


Figure 5. Complete outer synchronization errors $e_{1i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

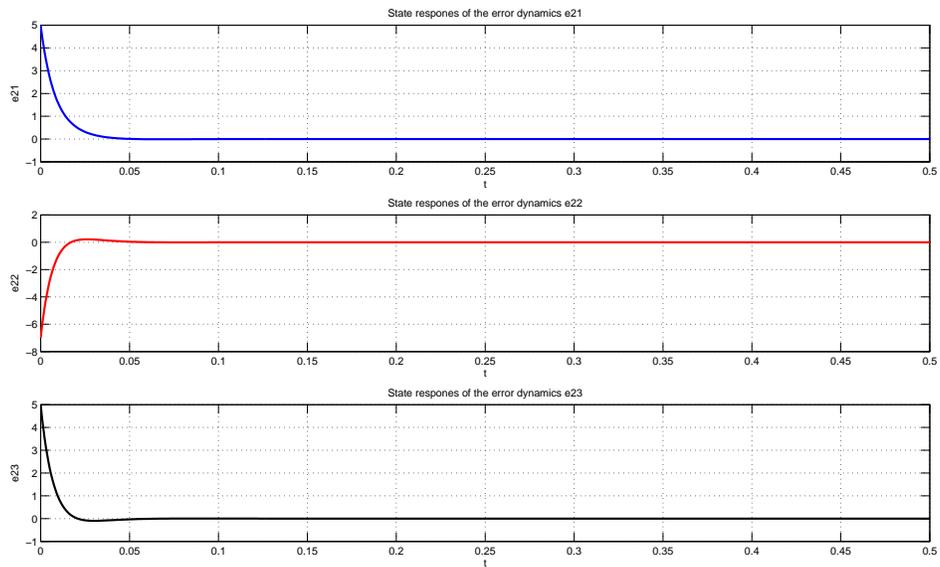


Figure 6. Complete outer synchronization errors $e_{2i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

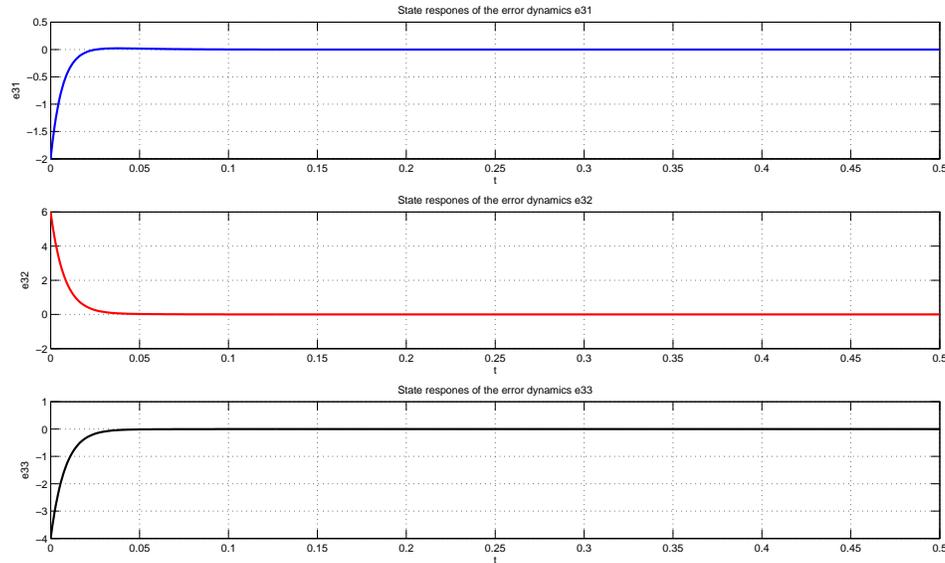


Figure 7. Complete outer synchronization errors $e_{3i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

Therefore, we take the unified chaotic system ($\alpha = 0$) as the local node dynamics, the drive network (1) and the response network (2) can achieve adaptive outer synchronization under the adaptive synchronization controller (9)-(10). With initial conditions

$$x_1(0) = \begin{bmatrix} 8 \\ 6 \\ 7 \end{bmatrix}, x_2(0) = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}, x_3(0) = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}.$$

$$y_1(0) = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, y_2(0) = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}, y_3(0) = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}.$$

The numerical simulations are presented in **Figures 8-10**.

From **Figures 8-10**, it is easy to know that the adaptive outer synchronization errors can fast converge asymptotically to zero, which implies that the drive network (1) and the response network (2) have achieved adaptive outer synchronization.

Remark 8 All the numerical simulation results show that Theorems 1-4 are efficient in ensuring the outer synchronization behavior between coupled complex networks with multiple coupling time-varying delays.

5 Conclusion

In this paper, the problem of outer synchronization of complex networks with multiple coupling time-varying delays is investigated. Based on Lyapunov stability theory, LMI and adaptive control

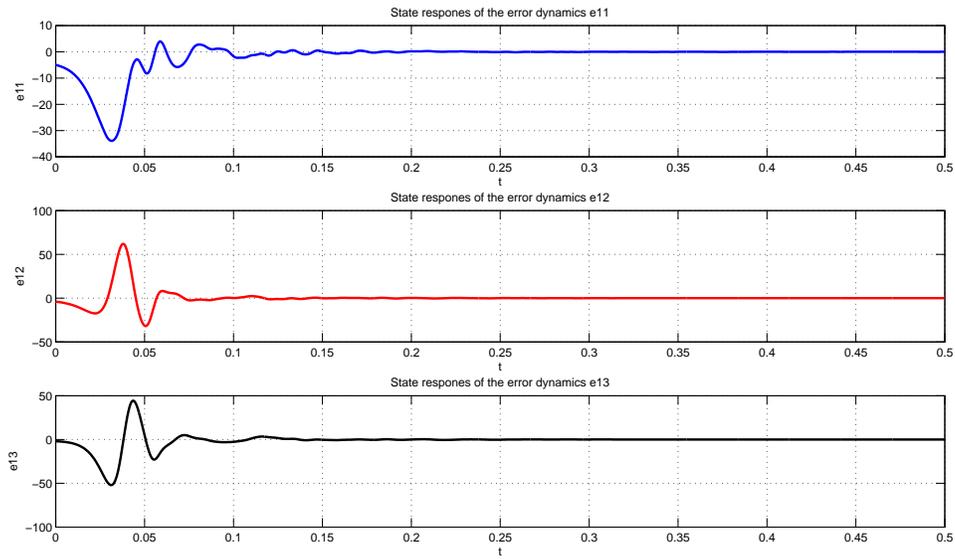


Figure 8. Adaptive outer synchronization errors $e_{1i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

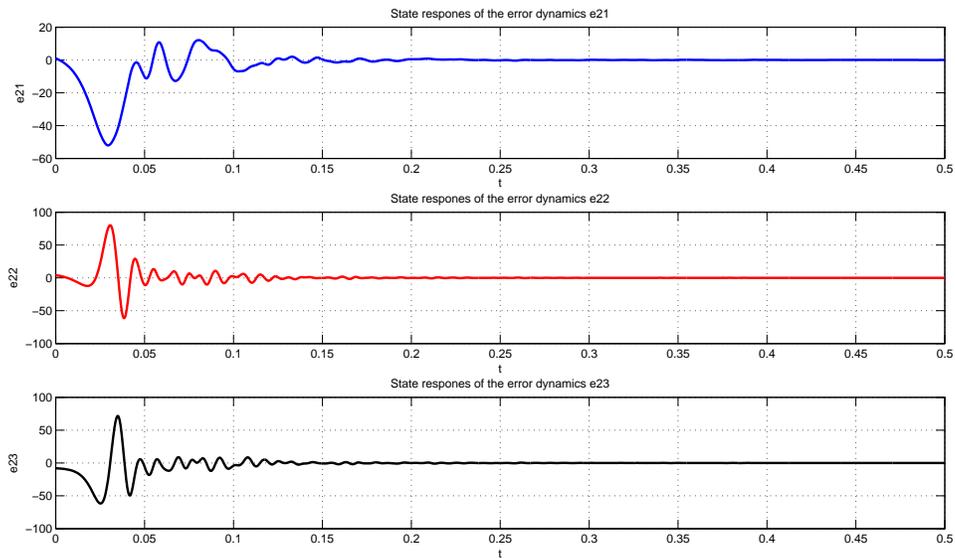


Figure 9. Adaptive outer synchronization errors $e_{2i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

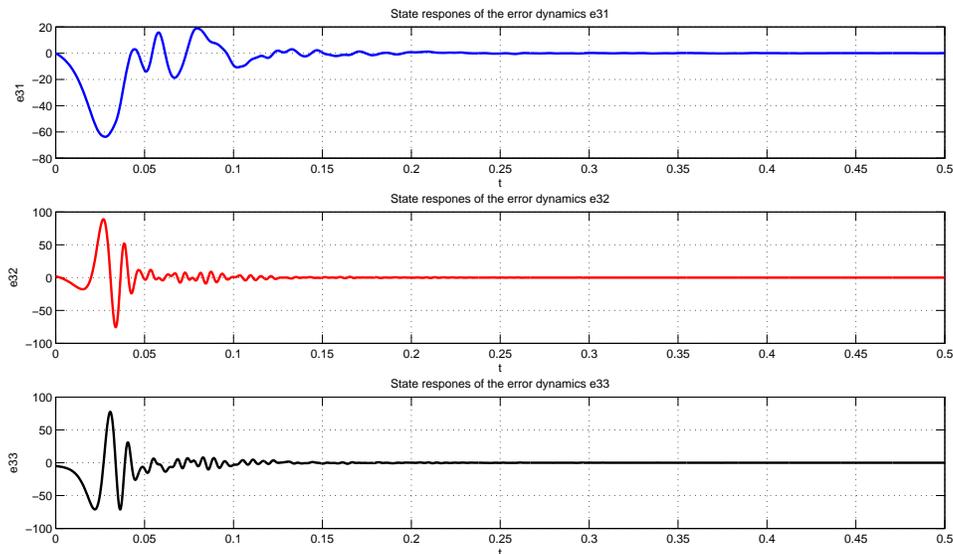


Figure 10. Adaptive outer synchronization errors $e_{3i}(t)$ ($i=1,2,3$) of the drive network (1) and the response network (2).

schemes, some novel synchronization criteria are derived, and some simple synchronization controllers are designed. The results presented in this paper improve and generalize the corresponding results of recent works. Numerical simulations are then presented to show the effectiveness and feasibility of the proposed complex networks control and synchronization schemes.

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References

- [1] Stefano Boccaletti, Vito Latora, Yamir Moreno, Martin Chavez, and D-U Hwang. Complex networks: Structure and dynamics. *Physics Reports*, 424(4):175–308, 2006.
- [2] Xiao Fan Wang and Guanrong Chen. Complex networks: small-world, scale-free and beyond. *IEEE Transactions on Circuits and Systems Magazine*, 3(1):6–20, 2003.

- [3] Alex Arenas, Albert Diaz-Guilera, Jurgen Kurths, Yamir Moreno, and Changsong Zhou. Synchronization in complex networks. *Physics Reports*, 469(3):93–153, 2008.
- [4] Xiao Fan Wang and Guanrong Chen. Synchronization in scale-free dynamical networks: robustness and fragility. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 49(1):54–62, 2002.
- [5] Huijun Gao, James Lam, and Guanrong Chen. New criteria for synchronization stability of general complex dynamical networks with coupling delays. *Physics Letters A*, 360(2):263–273, 2006.
- [6] Jinde Cao, Ping Li, and Weiwei Wang. Global synchronization in arrays of delayed neural networks with constant and delayed coupling. *Physics Letters A*, 353(4):318–325, 2006.
- [7] Qiankun Song and Jinde Cao. Synchronization of nonidentical chaotic neural networks with leakage delay and mixed time-varying delays. *Advances in Difference Equations*, 2011(1):1–17, 2011.
- [8] Jinde Cao, Guanrong Chen, and Ping Li. Global synchronization in an array of delayed neural networks with hybrid coupling. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 38(2):488–498, 2008.
- [9] Yurong Liu, Zidong Wang, Jinling Liang, and Xiaohui Liu. Stability and synchronization of discrete-time markovian jumping neural networks with mixed mode-dependent time delays. *IEEE Transactions on Neural Networks*, 20(7):1102–1116, 2009.
- [10] Jinling Liang, Zidong Wang, Yurong Liu, and Xiaohui Liu. Robust synchronization of an array of coupled stochastic discrete-time delayed neural networks. *IEEE Transactions on Neural Networks*, 19(11):1910–1921, 2008.
- [11] Cheng Hu, Juan Yu, Haijun Jiang, and Zhidong Teng. Pinning synchronization of weighted complex networks with variable delays and adaptive coupling weights. *Nonlinear Dynamics*, 67(2):1373–1385, 2012.
- [12] Yijun Zhang, Shengyuan Xu, Yuming Chu, and Jinjun Lu. Robust global synchronization of complex networks with neutral-type delayed nodes. *Applied Mathematics and Computation*, 216(3):768–778, 2010.
- [13] D. H. Ji, D. W. Lee, J. H. Koo, S. C. Won, S. M. Lee, and Ju H. Park. Synchronization of neutral complex dynamical networks with coupling time-varying delays. *Nonlinear Dynamics*, 65(4):349–358, 2011.
- [14] Xiaodi Li, Changming Ding, and Quanxin Zhu. Synchronization of stochastic perturbed chaotic neural networks with mixed delays. *Journal of the Franklin Institute*, 347(7):1266–1280, 2010.
- [15] Jianquan Lu, Jürgen Kurths, Jinde Cao, Nariman Mahdavi, and Chi Huang. Synchronization control for nonlinear stochastic dynamical networks: pinning impulsive strategy. *IEEE Transactions on Neural Networks and Learning Systems*, 23(2):285–292, 2012.
- [16] Changpin Li, Weigang Sun, and Jürgen Kurths. Synchronization between two coupled complex networks. *Physical Review E*, 76(4):046204, 2007.
- [17] Changpin Li, Congxiang Xu, Weigang Sun, Jian Xu, and Jürgen Kurths. Outer synchronization of coupled discrete-time networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 19(1):013106–013106, 2009.
- [18] Hongwu Tang, Liang Chen, Jun-an Lu, and Chi K Tse. Adaptive synchronization between two

- complex networks with nonidentical topological structures. *Physica A: Statistical Mechanics and its Applications*, 387(22):5623–5630, 2008.
- [19] Guanjun Wang, Jinde Cao, and Jianquan Lu. Outer synchronization between two nonidentical networks with circumstance noise. *Physica A: Statistical Mechanics and its Applications*, 389(7):1480–1488, 2010.
- [20] Song Zheng, Shuguo Wang, Gaogao Dong, and Qinsheng Bi. Adaptive synchronization of two nonlinearly coupled complex dynamical networks with delayed coupling. *Communications in Nonlinear Science and Numerical Simulation*, 17(1):284–291, 2012.
- [21] Xiaoqun Wu, Wei Xing Zheng, and Jin Zhou. Generalized outer synchronization between complex dynamical networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 19(1):013109–013109, 2009.
- [22] Xiangjun Wu and Hongtao Lu. Generalized function projective (lag, anticipated and complete) synchronization between two different complex networks with nonidentical nodes. *Communications in Nonlinear Science and Numerical Simulation*, 17(7):3005–3021, 2012.
- [23] Yongqing Wu, Changpin Li, Yujiang Wu, and Jürgen Kurths. Generalized synchronization between two different complex networks. *Communications in Nonlinear Science and Numerical Simulation*, 17(1):349–355, 2012.
- [24] Yongzheng Sun, Wang Li, and Donghua Zhao. Finite-time stochastic outer synchronization between two complex dynamical networks with different topologies. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2):023152–023152, 2012.
- [25] Mohammad Mostafa Asheghan, Joaquín Míguez, Mohammad T Hamidi-Beheshti, and Mohammad Saleh Tavazoei. Robust outer synchronization between two complex networks with fractional order dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 21(3):033121–033121, 2011.
- [26] Yongzheng Sun and Donghua Zhao. Effects of noise on the outer synchronization of two unidirectionally coupled complex dynamical networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2):023131–023131, 2012.
- [27] Yongqing Wu, Changpin Li, Aili Yang, Lunji Song, and Yujiang Wu. Pinning adaptive anti-synchronization between two general complex dynamical networks with non-delayed and delayed coupling. *Applied Mathematics and Computation*, 218(14):7445–7452, 2012.
- [28] Shu-Guo Wang and Hong-Xing Yao. Impulsive synchronization of two coupled complex networks with time-delayed dynamical nodes. *Chinese Physics B*, 20(9):090513, 2011.
- [29] Wen Sun, Zhong Chen, Yibing Lü, and Shihua Chen. An intriguing hybrid synchronization phenomenon of two coupled complex networks. *Applied Mathematics and Computation*, 216(8):2301–2309, 2010.
- [30] Song Zheng. Exponential synchronization of two nonlinearly non-delayed and delayed coupled complex dynamical networks. *Physica Scripta*, 85(1):015003, 2012.
- [31] Zhuchun Li and Xiaoping Xue. Outer synchronization of coupled networks using arbitrary coupling strength. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 20(2):023106–023106, 2010.
- [32] Hui Liu, Jun-An Lu, Jinhu Lü, and David J Hill. Structure identification of uncertain general complex dynamical networks with time delay. *Automatica*, 45(8):1799–1807, 2009.
- [33] Zhao-Yan Wu and Xin-Chu Fu. Topology identification of general dynamical network with

- distributed time delays. *Chinese Physics Letters*, 26(7):070201, 2009.
- [34] Junchan Zhao, Qin Li, Jun-An Lu, and Zhong-Ping Jiang. Topology identification of complex dynamical networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 20(2):023119–023119, 2010.
- [35] Ranjib Banerjee, Ioan Grosu, and Syamal K Dana. Antisynchronization of two complex dynamical networks. In *Complex Sciences*, pages 1072–1082. Springer, 2009.
- [36] Yongzheng Sun, Wang Li, and Jiong Ruan. Generalized outer synchronization between complex dynamical networks with time delay and noise perturbation. *Communications in Nonlinear Science and Numerical Simulation*, 18(4):989–998, 2013.
- [37] Mei Sun, Changyan Zeng, and Lixin Tian. Linear generalized synchronization between two complex networks. *Communications in Nonlinear Science and Numerical Simulation*, 15(8):2162–2167, 2010.
- [38] Wanli Guo, Francis Austin, and Shihua Chen. Global synchronization of nonlinearly coupled complex networks with non-delayed and delayed coupling. *Communications in Nonlinear Science and Numerical Simulation*, 15(6):1631–1639, 2010.
- [39] Hongyue Du, Peng Shi, and Ning Lü. Function projective synchronization in complex dynamical networks with time delay via hybrid feedback control. *Nonlinear Analysis: Real World Applications*, 14(1):1182–1190, 2013.
- [40] Zhaoyan Wu, Xin-Jian Xu, Guanrong Chen, and Xinchu Fu. Adaptive synchronization and pinning control of colored networks. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):043137–043137, 2012.
- [41] Ping He, Shu-Hua Ma, and Tao Fan. Finite-time mixed outer synchronization of complex networks with coupling time-varying delay. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):043151–043151, 2012.
- [42] Yongzheng Sun, Wang Li, and Donghua Zhao. Finite-time stochastic outer synchronization between two complex dynamical networks with different topologies. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(2):023152–023152, 2012.
- [43] Yongzheng Sun, Wang Li, and Donghua Zhao. Outer synchronization between two complex dynamical networks with discontinuous coupling. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(4):043125–043125, 2012.
- [44] Jin-Liang Wang, Zhi-Chun Yang, and Huai-Ning Wu. Passivity analysis of complex dynamical networks with multiple time-varying delays. *Journal of Engineering Mathematics*, 74(1):175–188, 2012.
- [45] Chua Leono, Komuro Motomasa, and Matsumoto Takashi. The double scroll family. *IEEE Transactions on Circuits and Systems*, 33(11):1072–1118, 1986.
- [46] Xing-Yuan Wang and Guo-Bin Zhao. Hyperchaos generated from the unified chaotic system and its control. *International Journal of Modern Physics B*, 24(23):4619–4637, 2010.
- [47] Jinhu Lü, Guanrong Chen, Daizhan Cheng, and Sergej Celikovsky. Bridge the gap between the lorenz system and the chen system. *International Journal of Bifurcation and Chaos*, 12(12):2917–2926, 2002.
- [48] Mai Viet Thuan and Nguyen Thi Thanh Huyen. Exponential stabilization of linear systems with interval nondifferentiable time-varying delays in state and control. *Differential Equations and Control Processes*, 2011(N4):178–193, 2011.

- [49] Jianquan Lu and Daniel WC Ho. Globally exponential synchronization and synchronizability for general dynamical networks. *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, 40(2):350–361, 2010.

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