Finite-Time Stabilization of Polynomial Systems

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Abstract

In this paper finite-time stabilization of nonlinear polynomial systems, is considered. First we provide a formula for a finite time stabilizing state feedback control law, for a second order polynomial system. Then we extend the result to the case of polynomial systems with multiple inputs. Some extensions concerning asymptotic stabilization are also given.

Keywords: *polynomial systems, finite time stability, non affine systems*

1. Introduction

When studying stability of systems with Lyapunov theory one can conclude whether the system is stable or unstable, which is not always completely satisfactory from a practical point of view. Indeed, an equilibrium point may be stable in the Lyapunov sense, but the stability domain could be so small as to render the system practically unstable [20]. However in some cases, the state of a system may be unstable in the sense of Lyapunov but its performance can be considered acceptable since its trajectory doesn't exceed some prescribed bounds; an example considering Van der Pol oscillator is given in [12]. Moreover, many systems are known to operate in finite time interval for which it is necessary to guarantee some boundedness of the state, it follows that definition of stability in the sense of Lyapunov is not appropriate to cover such systems. Many examples fall into this category including the problem, in chemical process, of keeping the temperature or pressure within specified bounds [16]. Problems of this type have given rise to alternative notions of stability which are called finite-time stability (FTS), [3, 4, 5, 9, 16] or a more general one practical stability (PS), [1, 2, 12, 14, 19, 21]. It means that once the finite-time stabilizing control law is determined; we guarantee that the state norm of the system doesn't exceed some bounds during an infinite or finite time interval. Finite Time Stability has several formulations in the literature. In 1955, Chetaev [17] emphasized the need for an alternative notion of stability. He introduced the concept of (S_I, S_A, T) -stability so that if the initial state belongs to the set S_I , the instantaneous state belongs to the set S_A during the time interval *T*. This concept is named finite time stability [16] or short time stability [18]. Then the notion of practical stability was introduced in 1961 by LaSalle and Lefschetz [12] for systems operating over an infinite time interval with prescribed bounds. Weiss and Infante [16] were the first to provide sufficient conditions of FTS where the sets of initial and admissible states (S_I, S_A) are expressed in term of time invariant hyper balls. Note that PS or FTS concepts are defined with respect to certain prescribed sets of the state space rather than a single point like equilibrium as in

Lyapunov analysis. Weiss [15] later provided necessary conditions for practical stability. In [2], Michel studied finite time stability and practical stability of discontinuous systems. Gruyitch [13, 14] introduced the practical stability with the settling time. An extension of this notion based on vector norms was presented in [22]. In [19], the authors

Introduced the practical stability in terms of two measures and they present a systematic study of the theory of practical stability. It is worth noting that in many recent works in the literature [6, 9, 22] practical stability is defined with respect to finite time interval like in finite time stability.

Nowadays, polynomial nonlinear systems appear in many practical fields, such as in magnetic problems [7] and oscillation problems of the electromagnetic oscillator [11]. Therefore, the investigation of such systems is practically relevant and the aim of this paper is to propose a constructive method to finite time stabilize polynomial systems. We are motivated by the fact that it may not exists a method in the literature concerning this objective.

The paper is organised as follows: section 2 introduces some preliminaries. Definition of finite-time stability and stabilization are introduced together with the systems under study which are polynomial systems. Then the problem we deal with, is precisely stated. Section 3 presents the main results which allow the design of state feedback control laws that stabilize polynomial systems in the context of finite time stability framework. An example illustrates the proposed approach. Section 4 gives a particularization of main result to the case of asymptotic stabilization.

2. Finite Time Stabilization

2.1. Preliminaries

Let us consider the following system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

By referring to [6, 9] and [22] we can give the following definition of Finite Time Stability.

Definition 1.

Given c_1, c_2, T positive scalars and $V(x)$ a positive definite function satisfying $V(0) = 0$. System (1) is said finite-time stable with respect to (c_1, c_2, T, V) (we can write (c_1, c_2, T, V)-FTS) if

$$V(x_0) \le c_1 \Rightarrow V(x) < c_2 \qquad \forall t \in [0, T]$$

when $V(x) = x^T x$ or $V(x) = x^T R\, x$, $R = R^T > 0$, the previous definition is the same as the ones in ([21], [3]). For the sequel, we introduce the following definition of finite-time stabilizability.

Consider now a controlled system described by

$$\dot{x} = f(x, u) \tag{2}$$

where u is the control. Definition 1 is extended in the following way.

Definition 2.

System (2) is said finite-time stabilizable with respect to (c_1, c_2, T, V) if there exists a state feedback controller $u(x) = [u_1(x)..u_m(x)]^T$ such that the closed-loop system $\dot{x} = f(x, u(x))$ is (c_1, c_2, T, V)- FTS.

For solving the finite-time stabilization problem, the lemma which follows is central.

Lemma 1.

Let $V(x)$ a positive definite function satisfying $V(0) = 0$. If there exists a positive scalar α such that

$$\frac{dV}{dt}(x) \leq \alpha V(x) \tag{3}$$

System (1) is (c_1, c_2, T, V)-FTS if the following inequality is satisfied

$$\alpha T < \ln\left[\frac{c_2}{c_1}\right] \tag{4}$$

Remark 1. it is worth noting that the finite time stability notion introduced in this paper is different from that stated in [7] and the references therein, where it means fast convergence.

2.2. Problem Formulation

Consider the following nonlinear non affine system:s

$$\dot{x} = f_0(x) + f_1(x)u + f_2(x)u^2 \tag{5}$$

where $x = [x_1 \ldots x_n]^T \in \mathbb{R}^n$, u is the control, and $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $\forall\ i = 0, 1, 2$, are smooth functions on $\mathbb{R}^n$ and $f_0(0) = (0)$.

We are interested by the design of a continuous feedback controller which ensures the finite-time stability of the closed-loop system in the sense of the definition 2.

Remark 2. Continuity of the control is a natural preference because it guarantees the existence of solutions to the differential equations.

The problem investigated in this section is summarized as follows.

Problem 1.

Given c_1, c_2, T positive scalars and $V(x)$ a positive definite function satisfying $V(0) = 0$, design a state feedback controller $u(x)$ which finite-time stabilizes system (4) with respect to (c_1, c_2, T, V).

We denote by

$$a(x)=\frac{dV}{dx}f_2(x) \tag{6a}$$

$$b(x)=\frac{dV}{dx}f_1(x) \tag{6b}$$

$$c(x)=\frac{dV}{dx}f_0(x) \tag{6c}$$

3. Main Results

3.1. Single Input Case

For a single input system, the main result is stated in the following theorem.

Theorem 1. Let c_1, c_2 and T , $T>0$, $c_2>c_1$, be given. If there exists a positive scalar α and a positive definite function $V(x)$, satisfying $\alpha T < \ln\left[\frac{c_2}{c_1}\right]$ and for all $x \neq 0$, $a(x) \neq 0$ and $a(x)b(x)^2+c(x) \leq \alpha V(x)$ then system (5) is (c_1, c_2, T, V) -Finite Time Stabilizable and the control defined by

$$u(x)=\frac{-b(x)+b(x)\sqrt{1+4a(x)^2}}{2a(x)}, \ \forall x \neq 0 \tag{7}$$

with $u(0)=0$, solves problem 1.

Proof. The proof has two steps. First we prove that under the control (7), we satisfy condition (3) that guarantees with condition (4) finite time stabilization of the system as shown in the section above. Second, we prove the continuity of the control over ¡ ^{n}.
we have:

$$\dot{V}(x)=\frac{dV}{dx}\dot{x} \tag{8}$$

If $x=0$, then we have $\dot{V}=0$. It readily follows that $\dot{V}(x) \leq \alpha V(x)$. Now if $x \neq 0$ we consider

$$\dot{V}(x)=\left(c(x)+b(x)u+a(x)u^2\right) \tag{9}$$

Using (7) it is easy to check that:

$$\dot{V}(x)=a(x)b(x)^2+c(x) \tag{10}$$

If $a(x)b(x)^2+c(x) \leq \alpha V(x)$ holds then we have: $\dot{V}(x) \leq \alpha V(x)$

Now we shall prove the continuity of the control over $\mathbb{R}^n$.

As $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} \dfrac{-b(x)+b(x)\sqrt{1+4a(x)^2}}{2a(x)}$

$$= \lim_{x\to\infty} a(x)b(x) = 0$$

$u(x)$ is continuous at the origin. The continuity of the control over $\mathbb{R}^n \setminus \{0\}$ is immediate. This completes the proof. □

We can note that when $V(x) = x^T x$, theorem 1 is reduced to the corollary given below.

Corollary 1. Let c_1, c_2 and T , $T>0$, $c_2 > c_1$, be given. If there exists a positive scalar α, satisfying for all $x \neq 0$, $a(x) \neq 0$, $a(x)b(x)^2 + c(x) \le \alpha\, x^T x$ and $\alpha T < \ln\left[\dfrac{c_2}{c_1}\right]$, then system (5) is (c_1, c_2, T)-finite time stabilizable and the control (7) solves problem 1.

Example 1

Let us consider the system

$$\begin{cases} \dot{x}_1 = 0,5x_1 + 0,25x_1^3 + 0,5\, x_1\, x_2^2 + 0,25\,u - 0,25x_1\, u^2 \\ \dot{x}_2 = 0,5x_2 + x_1^3 + 0,5\, x_2^3 + x_1x_2^2 + 0,5u - 0,5x_2\, u^2 \end{cases} \tag{11}$$

with

$$V(x) = x^T R\, x \text{ and } R = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$\frac{dV}{dx} = \begin{bmatrix} 4x_1 & 2x_2 \end{bmatrix}$$

we have

$$f_0(x) = \begin{bmatrix} 0,5x_1 + 0,25x_1^3 + 0,5\, x_1\, x_2^2 \\ 0,5x_2 + x_1^3 + 0,5\, x_2^3 + x_1x_2^2 \end{bmatrix},\ f_1(x) = \begin{bmatrix} 0,25 \\ 0,5 \end{bmatrix},\ f_2(x) = \begin{bmatrix} -0,25\,x_1 \\ -0,5\,x_2 \end{bmatrix}$$

and

$a(x) = -\left(x_1^2 + x_2^2\right)$, $b(x) = x_1 + x_2$,

$$c(x)=2x_1^{\,2}+x_1^{\,4}+2x_1^{\,2}x_2^{\,2}+2x_1^{\,3}x_2$$
$$+x_2^{\,2}+x_2^{\,4}+2x_2^{\,3}x_1,$$

we have

$$\begin{cases} a(x)\neq 0,\ \forall x\neq 0 \\ a(x)b(x)^2+c(x)=2x_1^{\,2}+x_2^{\,2}\leq \alpha V(x), \\ \qquad\qquad\qquad\qquad\qquad \forall \alpha \geq 1 \end{cases} \tag{12}$$

By using theorem 1, (12) guarantees that system (11) is finite time stabilizable with respect to $(0,2;0,9;1,5;V)$ under the continuous feedback control (7). Figure 1 shows $V(x)=x^T R\,x$ in open-loop for 15 different values of initial state satisfying $x_0^T R\,x_0 \leq 0,2$. It is clear that the system is not $(0,2;0,9;1,5;V)$-Finite Time Stable. Figure 2 shows that $V(x)<0,9$ for closed-loop system during the time interval defined by $T=1,5$ when considering 15 different initial values verifying $x_0^T R\,x_0 \leq 0,2$. Figure 3 shows the control behaviour.

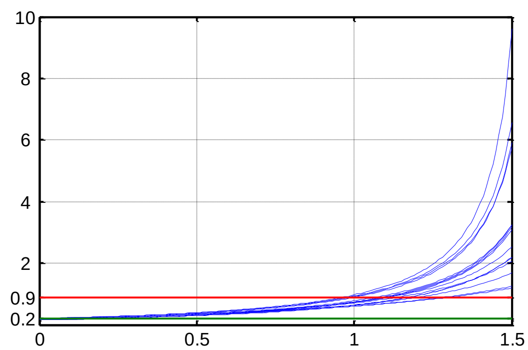

Figure 1. Evolution of $V(x)$ (open loop)

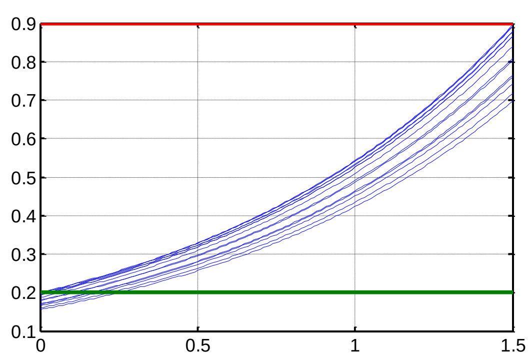

Figure 2. Evolution of $V(x)$ (closed loop)

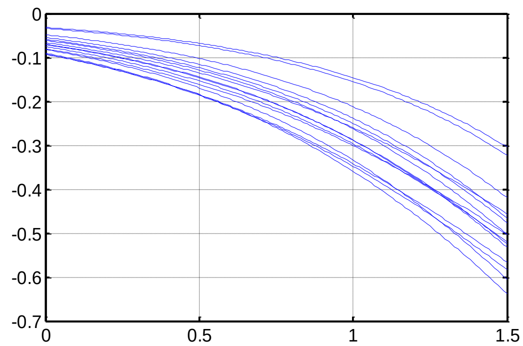

Figure 3. Control

3.2. Multi Inputs Case

We will consider now a more general case of second order polynomial system with multiple inputs. We will restrict our study to the following system

$$\dot{x} = f_0(x) + \sum_{j=1}^{m} f_{1,j}(x) u_j + \sum_{j=1}^{m} f_{2,j}(x) u_j^2 \tag{13}$$

where $x = [x_1 \ldots x_n]^T \in \mathbb{R}^n$, $u = [u_1 \ldots u_m]^T \in \mathbb{R}^m$ and $f_{i,j}$ are smooth functions on $\mathbb{R}^n$ and $f_0(0) = 0$.

We denote for $j = 1, \ldots, m$

$$a_j(x) = \frac{dV}{dx} f_{2,j}(x) \tag{14a}$$

$$b_j(x) = \frac{dV}{dx} f_{1,j}(x) \tag{14b}$$

$$c(x) = \frac{dV}{dx} f_0(x) \tag{14c}$$

The following theorem is a generalization of theorem 1.

Theorem 2. Let c_1, c_2 and T , $T > 0$, $c_2 > c_1$, be given. If there exists a positive scalar α, satisfying for all $x \neq 0$, and for all $j \in \{1, \ldots, m\}$:

$a_j(x) \neq 0$,

$\alpha T < \ln\left[\frac{c_2}{c_1}\right]$

and

$$\sum_{j=1}^{m} a_j(x) b_j(x)^2 + c(x) \leq \alpha V(x)$$

then the control defined by

$$u_j(x) = \frac{-b_j(x) + b_j(x)\sqrt{1+4a_j(x)^2}}{2a_j(x)}, \forall x \neq 0 \quad (15)$$

with $u_j(0)=0$, (c_1, c_2, T)-finite time stabilizes system (13).

Proof. We have $\dot{V}(x) = \dot{x}^T \frac{dV}{dx}$.

If $x=0$, then we have $\dot{V}(0)=0$.

Now if $x \neq 0$ it follows that

$$\dot{V}(x) = c(x) + \sum_{j=1}^{m} b_j(x) u_j + \sum_{j=1}^{m} a_j(x) u_j^2 \quad (16)$$

Associating with (15) it is easy to check that

$$\dot{V}(x) = \sum_{j=1}^{m} a_j(x) b_j(x)^2 + c(x) \quad (17)$$

If $\sum_{j=1}^{m} a_j(x) b_j(x)^2 + c(x) \leq \alpha V(x)$ holds then we have: $\dot{V}(x) \leq \alpha V(x)$

The continuity of the control is proved as for theorem 1. □

We can note that when $V(x) = x^T x$, theorem 2 is reduced to the corollary given bellow.

Corollary 2. Let c_1, c_2 and T , $T>0$, $c_2 > c_1$, be given. If there exists a positive scalar α , satisfying for all $x \neq 0$, and for all $j \in \{1, \ldots, m\}$ that

$a_j(x) \neq 0$,

$\alpha T < \ln\left[\frac{c_2}{c_1}\right]$

and

$$\sum_{j=1}^{m} a_j(x) b_j(x)^2 + c(x) \leq \frac{\alpha}{2} x^T x$$

then the control defined by (15) (c_1, c_2, T)-Finite Time Stabilizes system (13).

4. Asymptotic Stabilization

The next result is concerned with asymptotic stability. We first recall the following definition.

Definition 3. [7] (Control Lyapunov Function) A definite positive, proper and smooth function $V(x)$ is said to be a control Lyapunov function (CLF) with respect to system (1) if it satisfies for all $x \neq 0$:

$$\inf_{u \in \mathbb{R}^n} \left\{ \frac{\partial V}{\partial x} f(x,u) \right\} < 0 \qquad (18)$$

Proposition 1. [7] If there exists a CLF $V(x)$ for system (1) then it is globally asymptotically stabilizable by a state feedback control law $u(x)$ with $u(0)=0$, that is continuous over $\mathbb{R}^n \setminus \{0\}$.

The main result of this section can be now stated.

Corollary 3. The system (5) is globally asymptotically stabilizable under the continuous feedback control law (7) with $u(0)=0$, if there exists a CLF V for system (5) such that for all $x \neq 0$, $a_j(x) \neq 0$ and $a(x)b(x)^2 + c(x) < 0$, The feedback is continuous over $\mathbb{R}^n$.

Proof. we have

$$\dot{V}(x) = c(x) + b(x)u + a(x)u^2 \qquad (19)$$

Note that if there exists a positive proper and smooth function $V(x)$ such that $a(x) < 0$ and $b(x)^2 - 4a(x)c(x) < 0$ for all $x \neq 0$, then the system is stabilizable with any control in particular $u(x) = 0$. Now replacing in (19), the control by its expression in (6), we can conclude that $\dot{V}(x) = a(x)b(x)^2 + c(x) < 0$, and then the control law (7) stabilizes asymptotically the system. Continuity is shown as above and the proof is complete.

It is easy to extend the previous result to deal with asymptotic stabilization of polynomial systems with multiple inputs given by (13).

Corollary 4. The system (13) is globally asymptotically stabilizable under the continuous feedback control law (15) with $u_j(0)=0$, if there exists a CLF $V(x)$ for system (13), such that $\sum_{j=1}^{m} a_j(x) b_j(x)^2 + c(x) < 0$ for all $j \in \{1, \ldots, m\}$ and for all. Note that in [7] the control law is not continuous at the origin which makes the system weakly stabilizable.

Example 2.

Let us consider the system

$$\begin{cases} \dot{x}_1 = x_1^5 + 0,25\, x_1\, u - \left(x_1 + x_1^3\right) u^2 \\ \dot{x}_2 = x_2^5 + 0,5 x_2\, u - x_2\, u^2 \end{cases} \quad (20)$$

with $V(x) = x^T R\, x$ and $R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

so

$$\frac{dV}{dx} = \begin{bmatrix} 4x_1 & 2x_2 \end{bmatrix}$$

we have

$$f_0(x) = \begin{bmatrix} x_1^5 \\ x_2^5 \end{bmatrix},\ f_1(x) = \begin{bmatrix} 0,25\, x_1 \\ 0,5\, x_2 \end{bmatrix},\ f_2(x) = \begin{bmatrix} -\left(x_1 + x_1^3\right) \\ -x_2 \end{bmatrix}$$

and

$$a(x) = -\left(4x_1^2 + 4x_1^4 + 2x_2^2\right),\ b(x) = x_1^2 + x_2^2,\ c(x) = 4x_1^6 + 2x_2^6$$

we have the following inequality which guarantees that $V(x)$ is a CLF for system (20):

$$a(x) < 0, \quad \forall\, x \neq 0$$

Conditions of theorem 3 are satisfied indeed we have

$$\begin{cases} a(x) \neq 0,\ \forall\, x \neq 0 \\ a(x) b(x)^2 + c(x) \leq 0 \end{cases} \quad (21)$$

Hence system (20) is asymptotically stabilizable under the control (7). The system is unstable in open-loop. Figure 4 shows the evolution of $V(x)$, Figure 5 shows the control behavior in closed-loop. The two components of the state are illustrated in Figure 6.

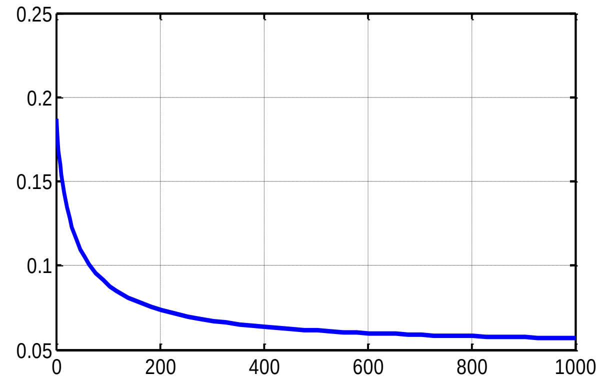

Figure 4. Evolution of $V(x)$ (closed loop)

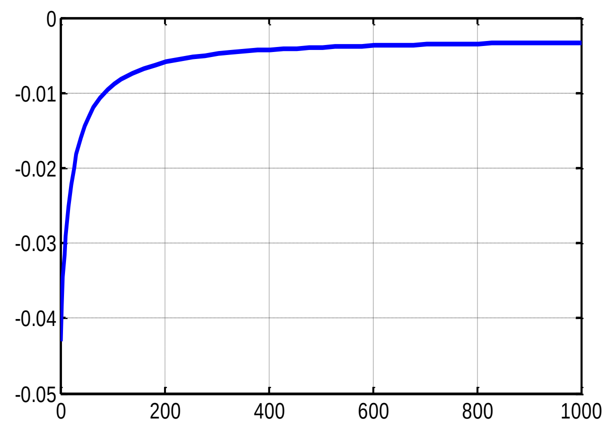

Figure 5. Control

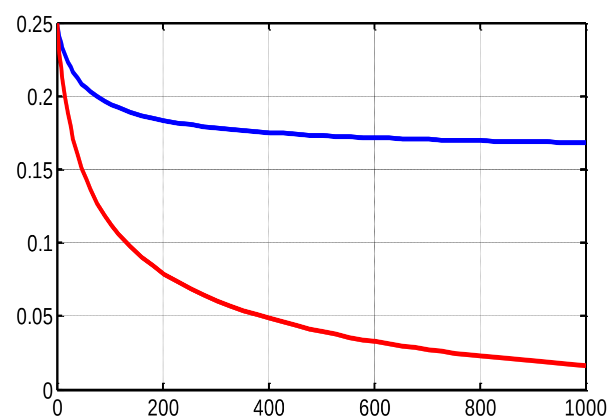

Figure 6. Two Components of the State

5. Conclusion

The problem of finite-time stabilization of nonlinear non affine system is solved by giving an explicit feedback control formula. Our control law ensures for a given nonlinear second order polynomial system with single input that its trajectory remains within fixed bounds during a finite-time interval. The result is extended to the case of systems with multi inputs. A theorem guaranteeing asymptotic stabilization of this class of systems is then proposed.

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