Survey of a New Note on Karnaugh Maps

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Abstract

Since 1953, using the Karnaugh maps to minimize the Boolean algebraic functions has been considered as the best method to simplify logic expressions. Due to the multiplicity of Boolean functions at number of variables, and variety of maps, it is better appropriate to present a method to determine the number of groupings of each map. This paper aims at specifying a number for each group containing 2^k cells separately, and finally, with exhibiting a main proposition, the desired method will be formulated.

Keywords: *Karnaugh maps, Boolean algebraic expressions, groupings, n-variables, Newton's binomial expansion*

1. Introduction

Karnaugh maps to simplification of Boolean logic functions have an abundant applications in designing of digital systems and logic circuits. Among all existent methods for minimizing algebraic functions like tabulation method, using the Karnaugh maps has been mentioned as the dominant tool in digital design books. [2] This method was developed at the Harvard computation laboratory for the first time by Maurice Karnaugh for the minimization of Boolean algebraic functions. [1]

The most important issue in minimizing of algebraic expressions is the cells classifying and awareness of their variety and number of their grouping. In this paper, with a precise definition of Karnaugh maps and their matrix declarations, we can reveal the details on the style of these groups' classifications.

2. Definition

Each n-variables Karnaugh map can be defined as a square or rectangular matrix in the following form:

- For even n, the desired matrix would be in square form and its dimensions is $2^{\frac{n}{2}} \times 2^{\frac{n}{2}}$.
- For odd n, the desired matrix would be in rectangular form and its dimensions is $2^{\frac{n-1}{2}} \times 2^{\frac{n+1}{2}}$.

It is obvious that, while n is odd, the number of columns in rectangular matrix is twice than its rows, because of:

$$2^{\frac{n+1}{2}} = 2\left(2^{\frac{n-1}{2}}\right)$$

At an each above mentioned maps, square or rectangular, the groups of adjacent cells are classified. Each of these groups are like as small squares or rectangles, and are like in 2^k form that $k = 0,1,\dots,n$.

The main purpose of this article is to obtain the number of 2^k groups and finally to gain the total number of possible groups in the n-variables Karnaugh map.

3. Method of Obtaining Different Types and the Number of Groupings

In Karnaugh maps with few variables, it is very simple to express the variety and number of groupings, and this work is specified by trial and error, but whenever the number of variables of a Boolean algebraic expression increases, (For example for $n = 5,6$) the number of groups will increase quickly.

Before finding the total number of groupings, it is better to discuss about their variety.

Lemma 1. Each n-variables Karnaugh map has $(n + 1)$ kind of grouping.

Since all groups may be in sizes that are powers of 2, so each n-variables Karnaugh map contains $2^0, 2^1, \dots, 2^{n-1}$ groups, this means the total number of them is $(n + 1)$ groups.

Lemma 2. At each n-variables Karnaugh map, the number of 2^0 groups is equal to $\binom{n}{0}2^n$, the number of 2^1 groups is equal to $\binom{n}{1}2^{n-1}$, … , the number of 2^{n-1} groups is equal to $\binom{n}{n-1}2^1$, and eventually, the number of 2^n group is equal to $\binom{n}{n}2^0$.

Lemma 2 discusses just the number of 2^k groups of Karnaugh maps. It is quite clear that, the number of 2^0 groups (the smallest possible grouping) is equal to the number of each cells of Karnaugh map, it means 2^n. $\left(\binom{n}{0}2^n = 2^n\right)$

On the other hand, the number of 2^n group (the biggest possible grouping) is equal to 1, it means that, the whole of Karnaugh map. $\left(\binom{n}{n}2^0 = 1\right)$

For the desired remaining groups, using of mathematical induction, we can start with 2^1 groups and extend this method to 2^{n-1} groups.

By using the derived results from lemma 1 and 2, we can obtain the total number of possible grouping at each n-variables Karnaugh map, easily.

Example 1. Consider a 4-variable square Karnaugh map. The number of possible groupings for this map that has shown in Figure 1 as following:

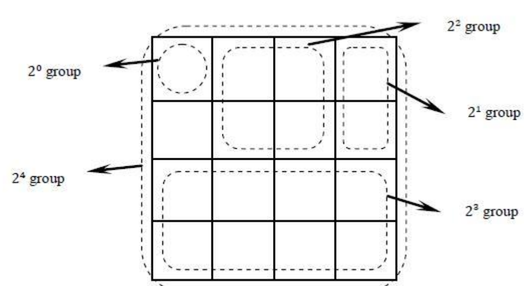

Figure 1. Four Variable Karnaugh Map with its Different Groupings

- Number of 2^0 groups : $\binom{4}{0}2^4 = 16$
- Number of 2^1 groups : $\binom{4}{1}2^{4-1} = 32$
- Number of 2^2 groups : $\binom{4}{2}2^{4-2} = 24$
- Number of 2^3 groups : $\binom{4}{3}2^{4-3} = 8$
- Number of 2^4 group : $\binom{4}{4}2^{4-4} = 1$

$$\begin{aligned} Total\ number\ of\ whole\ possible\ groupings &= 16 + 32 + 24 + 8 + 1 \\ &= 81. \qquad (1) \end{aligned}$$

By considering the above example, we can find out that the total number of possible groups, at each n-variables Karnaugh map has been analyzable by a Newton's binomial expansion. We state this result in the following proposition.

Proposition 1. The total number of possible groups at each n-variables Karnaugh map is equal to 3^n.

Proof. By lemma 2, we can deduce that the total number of existence groups at each n-variables Karnaugh map is as following:

$$\binom{n}{0}2^n + \binom{n}{1}2^{n-1} + \binom{n}{2}2^{n-2} + \cdots + \binom{n}{n-1}2^1 + \binom{n}{n}2^0 \qquad (2)$$

The above summation then can be rewrite as:

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k} = \sum_{k=0}^{n}\binom{n}{k}2^{n-k}.1^k . \qquad (3)$$

On the other hand, the Newton's binomial expansion of degree n , can be written as

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}.b^k. \qquad (4)$$

With comparing of (3) and (4), we can deduce that, $a = 2$ and $b = 1$, so we can write the summarized of (3) as

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}.1^k = (2+1)^n = 3^n. \qquad (5)$$

Thus the desired result is satisfied and the proposition is proven.
With further reviewing of example 1 and proposition 1, we can find out that for 4-variable Karnaugh map, the total number of all groups is $3^4(= 81)$, that confirms (1).

Finally, for comparison, the results of groupings and their variety and number of them, have been presented in Table 1, separately. In this table, variable n has been assigned the numbers 1 up to 8. Also, in the upper border, "n.o. 2^k g.," means the number of 2^k groups.

Table 1. Deduced Results for Groupings in n-variables Karnaugh Maps $(n = 1, 2, \dots, 8)$

n	$\begin{cases} 2^{\frac{n}{2}} \times 2^{\frac{n}{2}} & ; for\ even\ n \\ 2^{\frac{n-1}{2}} \times 2^{\frac{n+1}{2}} & ; for\ odd\ n \end{cases}$	n.o. 2^0g.	n.o. 2^1g.	n.o. 2^2g.	n.o. 2^3g.	n.o. 2^4g.	n.o. 2^5g.	n.o. 2^6g.	n.o. 2^7g.	n.o. 2^8g.	Total 3^n
1	1×2	2	1	-	-	-	-	-	-	-	3
2	2×2	4	4	1	-	-	-	-	-	-	9
3	2×4	8	12	6	1	-	-	-	-	-	27
4	4×4	16	32	24	8	1	-	-	-	-	81
5	4×8	32	80	80	40	10	1	-	-	-	243
6	8×8	64	192	240	160	60	12	1	-	-	729
7	8×16	128	448	672	560	280	84	14	1	-	2187
8	16×16	256	1024	1792	1792	1120	448	112	16	1	6561

4. Conclusion

This paper has presented a method for obtaining the number of n-variables Karnaugh map's grouping based on Newton's binomial expansion of degree n. In future studies, we can use this idea, for minimizing of the Boolean algebraic expressions with minimizing of the total number of groupings.

References

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He received M.Sc. degree in electrical engineering from University of Tabriz, Iran, in 1998. Early in his career, he served as a senior lecturer with the Azad University of Oroumieh, Iran, in 2001. He is currently working toward the Ph.D. degree in electrical engineering at University of Malaya, Kuala Lumpur, Malaysia. His teaching and research interests are signals and systems, digital filter design, digital image processing, and pattern recognition.