New Condition of Stabilisation for Continuous Takagi-Sugeno Fuzzy System based on Fuzzy Lyapunov Function

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Abstract

This paper deals with the stabilisation of continuous Takagi-Sugeno fuzzy models. Using non-quadratic Lyapunov function, new sufficient stabilization criteria are established in terms of Linear Matrix Inequality. Finally, numeric simulations are given to validate the developed approach.

Keywords*: Takagi-Sugeno fuzzy system, Linear Matrix Inequalities LMIs, Fuzzy Lyapunov Function, Parallel Distributed Controller PDC.*

1. Introduction

Fuzzy control systems have experienced a big growth of industrial applications in the recent decades, because of their reliability and effectiveness. Many researches are investigated on the Takagi-Sugeno models [4]–[5] which can combine the flexible fuzzy logic theory and rigorous mathematical theory into a unified framework. Thus, it becomes a powerful tool in approximating a complex nonlinear system.

Two classes of Lyapunov functions are used to analysis these systems: quadratic Lyapunov functions and non-quadratic Lyapunov ones which are less conservative than first class. Many researches are investigated with non-quadratic Lyapunov functions [3]–[11].

In this paper, a new stabilisation conditions for Takagi Sugeno sufficient fuzzy models based on the use of fuzzy Lyapunov function are presented. This criterion is expressed in terms of Linear Matrix Inequalities (LMIs) which can be efficiently solved by using various convex optimization algorithms [12]. The presented method is less conservative than existing results.

The organization of the paper is as follows. In section 2, we present the system description and problem formulation and we give some preliminaries which are needed to derive results. Section 3 will be concerned to stability analysis for T-S fuzzy systems. Illustrative example is given in section 4 for a comparison of previous results to demonstrate the advantage of proposed method. Section 5 concerns the proposed approach to stabilize a T-S fuzzy system with Parallel Distributed Controller (PDC). An approach to determine inputs constraints is presented in section 6. Finally section 7 makes conclusion.

Notation: Throughout this paper, a real symmetric matrix $S > 0$ denotes S being a positive definite matrix. The superscript ‘‘T’’ is used for the transpose of a matrix.

2. System Description and Preliminaries

Consider a T-S fuzzy continuous model for a nonlinear system as follows:

$$\begin{aligned}&IF\ z_1(t)\ is\ M_{i1}\ and \ldots and\ z_p(t)\ is\ M_{ip}\\ &THEN\quad \dot{x}(t)=A_i x(t)+B_i u(t)\quad i=1,\ldots,r\end{aligned} \tag{1}$$

where $M_{ij}\left(i=1,2,\ldots,r,\ j=1,2,\ldots,p\right)$ is the fuzzy set and r is the number of model rules; $x(t)\in\Re^n$ is the state vector, $u(t)\in\Re^m$ is the input vector, $A_i\in\Re^{n\times n}$, $B_i\in\Re^{n\times m}$, and $z_1(t),\ldots,z_p(t)$ are known premise variables.

The final outputs of the fuzzy systems are:

$$\dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)\left\{A_i x(t)+B_i u(t)\right\} \tag{2}$$

where

$$z(t)=\left[z_1(t)z_2(t)\ldots z_p(t)\right]$$

$$h_i\left(z(t)\right)=w_i\left(z(t)\right)\Big/\sum_{i=1}^{r}w_i\left(z(t)\right),\ w_i\left(z(t)\right)=\prod_{j=1}^{p}M_{ij}\left(z_j(t)\right)\ \text{ for all } \underline{t}.$$

The term $M_{i1}\left(z_j(t)\right)$ is the grade of membership of $z_j(t)$ in M_{i1}

Since $\begin{cases}\sum_{i=1}^{r}w_i\left(z(t)\right)\succ 0\\ w_i\left(z(t)\right)\geq 0, \qquad i=1,2,\ldots,r\end{cases}$

We have $\begin{cases}\sum_{i=1}^{r}h_i\left(z(t)\right)=1\\ h_i\left(z(t)\right)\geq 0, \qquad i=1,2,\ldots,r\end{cases}$ for all t.

The time derivative of premise membership functions is given by:

$$\dot{h}_i\left(z(t)\right)=\frac{\partial h_i}{\partial z(t)}\cdot\frac{\partial z(t)}{\partial x(t)}\cdot\frac{dx(t)}{dt}=\sum_{l=1}^{s}\upsilon_{il}\xi_{il}\times\frac{dx(t)}{dt} \tag{3}$$

We have the following property:

$$\sum_{k=1}^{r}\dot{h}_k\left(z(t)\right)=0 \tag{4}$$

The PDC fuzzy controller is represented by

$$u(t)=-\frac{\sum_{i=1}^{r}w_i\left(z(t)\right)F_i x(t)}{\sum_{i=1}^{r}w_i\left(z(t)\right)}=-\sum_{i=1}^{r}h_i\left(z(t)\right)F_i x(t) \tag{5}$$

The fuzzy controller design is to determine the local feedback gains F_i in the consequent parts.

The open-loop system is given by the equation (6)

$$\dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)A_i x(t) \tag{6}$$

By substituting (5) into (2), the closed-loop fuzzy system can be represented as:

$$\dot{x}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left\{A_i-B_iF_j\right\}x(t) \tag{7}$$

Assumption 1

The time derivative of the premises membership function is upper bounded such that $\left|\dot{h}_k\right|\le\phi_k$, for $k=1,\ldots,r$, where, $\phi_k, k=1,\ldots,r$ are given positive constants.

Lemma 1 [2]

Assume that $\left|\dot{h}_k\right|\le\phi_k$, $k=1,\ldots,r$. The Takagi-Sugeno fuzzy system (6) is stable if the following LMIs are satisfied:

$$P_i=P_i^T\succ 0,\ \ i=1,\ldots,r-1 \tag{8}$$

$$P_i-P_r\ge 0,\ \ i=1,\ldots,r \tag{9}$$

$$\bar{P}_\phi+\frac{1}{2}\left(A_i^TP_j+P_jA_i+A_j^TP_i+P_iA_j\right)\prec 0,\ \ i\le j, \tag{10}$$

where $i,j\in\{1,\ldots,r\}$, $\bar{P}_\phi=\sum_{k=1}^{r-1}\phi_k\left(P_k-P_r\right)$ and ϕ_k are scalars.

Lemma 2 [3]

Assume that $\left|\dot{h}_k\right|\le\phi_k$, $k=1,\ldots,r$. The Takagi-Sugeno fuzzy system (6) is stable if the following LMIs are satisfied:

$$P_i=P_i^T\succ 0,\ \ i=1,\ldots,r-1 \tag{11}$$

$$P_i+X\ge 0,\ \ i=1,\ldots,r \tag{12}$$

$$\tilde{P}_\phi+\frac{1}{2}\left(A_i^TP_j+P_jA_i+A_j^TP_i+P_iA_j\right)\prec 0,\ \ i\le j, \tag{13}$$

where $i,j\in\{1,\ldots,r\}$, $\tilde{P}_\phi=\sum_{k=1}^{r}\phi_k\left(P_k+X\right)$ and ϕ_k are scalars, and $X=X^T$.

3. Basic Stability Conditions

Consider the open-loop system (14).

$$\dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)A_i x(t) \tag{14}$$

The aim of the next section is to find conditions for the stability of the unforced T-S fuzzy system by using the Lyapunov theory.

Theorem 1

Under assumption 1 and for $0 \le \varepsilon \le 1$, the Takagi Sugeno fuzzy system (14) is stable if there exist positive definite symmetric matrices $P_k, k = 1, 2, \ldots, r$, matrix $R = R^T$ such that the following LMIs hold.

$$P_k + R \succ 0, \quad k \in \{1, \ldots, r\} \tag{15}$$

$$P_j + \mu R \succ 0, \quad j \in \{1, \ldots, r\} \tag{16}$$

$$\begin{aligned} P_\phi + \frac{1}{2}\Big\{ & A_i^T\left(P_j + \mu R\right) + \left(P_j + \mu R\right) A_i \\ & + A_j^T\left(P_i + \mu R\right) + \left(P_i + \mu R\right) A_j \Big\} \prec 0, \ \ i \le j \end{aligned} \tag{17}$$

where $i, j = 1, 2, \ldots, r$ and $P_\phi = \sum_{k=1}^{r} \phi_k \left(P_k + R\right)$ and $\mu = 1 - \varepsilon$

Proof

Let consider the Lyapunov function in the following form:

$$V\left(x(t)\right) = \sum_{k=1}^{r} h_k\left(z(t)\right) \cdot V_k\left(x(t)\right) \tag{18}$$

with

$$V_k\left(x(t)\right) = x^T(t)\left(P_k + \mu R\right)x(t), \qquad k = 1, 2, \ldots, r$$

where $P_k = P_k^T, R = R^T$, $0 \le \varepsilon \le 1$, $\mu = 1 - \varepsilon$ and $\left(P_k + \mu R\right) \ge 0$, $k = 1, 2, \ldots, r$.

This candidate Lyapunov function satisfies

i) $V\left(x(t)\right)$ is C^1,

ii) $V(0) = 0$ and $V\left(x(t)\right) \ge 0$ for $x(t) \ne 0$,

iii) $\left\| x(t) \right\| \to \infty \Rightarrow V\left(x(t)\right) \to \infty.$

The time derivative of $V\left(x(t)\right)$ with respect to t along the trajectory of the system (14) is given by:

$$\dot{V}\left(x(t)\right) = \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) V_k\left(x(t)\right) + \sum_{k=1}^{r} h_k\left(z(t)\right) \dot{V}_k\left(x(t)\right) \tag{19}$$

The equation (19) can be rewritten as,

$$\dot{V}\left(x(t)\right) = \Upsilon_1(x, z) + \Upsilon_2(x, z) \tag{20}$$

where

$$\Upsilon_1(x, z) = x^T(t)\left(\sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) \cdot \left(P_k + \mu R\right)\right) x(t) \tag{21}$$

$$\begin{aligned} \Upsilon_2(x, z) = \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} & h_i\left(z(t)\right) h_j\left(z(t)\right) x^T(t) \\ & \times \Big\{ A_i^T\left(P_j + \mu R\right) + \left(P_j + \mu R\right) A_i \\ & + A_j^T\left(P_i + \mu R\right) + \left(P_i + \mu R\right) A_j \Big\} x(t) \end{aligned} \tag{22}$$

Then, based on assumption 1, an upper bound of $\Upsilon_1(x,z)$ obtained as:

$\Upsilon_1(x,z) \le \sum_{k=1}^{r} \phi_k \cdot x(t)^T (P_k + \mu R) x(t)$ (23)

Based on(4), it follows that $\sum_{k=1}^{r} \dot{h}_k(z(t)) \varepsilon R = \bar{R} = 0$ where R is any symmetric matrix of proper dimension.

Adding $\bar{R}$ to (21), then

$$\Upsilon_1(x,z) \le \sum_{k=1}^{r} \phi_k \cdot x(t)^T (P_k + R) x(t) \tag{24}$$

Then,

$$\begin{aligned}
\dot{V}(x(t)) \le & \sum_{k=1}^{r} \phi_k x^T(t)(P_k + R)x(t) \\
& + \frac{1}{2}\sum_{k=1}^{r}\sum_{i=1}^{r} h_k h_i x^T(t) \times \{A_i^T(P_k + \mu R) + (P_k + \mu R)A_i \\
& + A_k^T(P_i + \mu R) + (P_i + \mu R)A_k\} x(t) \\
& = x(t)^T \left(\sum_{k=1}^{r} \phi_k \cdot (P_k + R) \right) x(t) \\
& + x^T(t) \frac{1}{2} \sum_{k=1}^{r}\sum_{i=1}^{r} h_k h_i \times \{A_i^T(P_k + \mu R) + (P_k + \mu R)A_i \\
& + A_k^T(P_i + \mu R) + (P_i + \mu R)A_k\} x(t)
\end{aligned}$$

If (17) holds, then $\dot{V}(x(t)) \prec 0$ and (14) is stable. □

4. Stabilization with PDC Controller

Consider the closed-loop system (7) which can be rewritten as

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t)) h_i(z(t)) G_{ii} x(t) + 2\sum_{i=1}^{r}\sum_{i \prec j} h_i(z(t)) h_j(z(t)) \left\{ \frac{G_{ij} + G_{ji}}{2} \right\} x(t), \tag{25}$$

where

$G_{ij} = A_i - B_i F_j$ and $G_{ii} = A_i - B_i F_i$.

In this section we define a fuzzy Lyapunov function and then consider stability conditions. A sufficient stability condition, for ensuring stability is given follows.

Theorem 2

Under assumption 1, and assumption 2 and for given $0 \le \varepsilon \le 1$, the Takagi-Sugeno system (25) is stable if there exist positive definite symmetric matrices $P_k, k = 1,2,\ldots,r$, and R, matrices $F_1,\ldots,F_r$ such that the following LMIs hols.

$$P_k + R \succ 0, \quad k \in \{1,\ldots,r\} \tag{26}$$

$$P_j + \mu R \ge 0, \quad j = 1,2,\ldots,r \tag{27}$$

$$P_\phi + \left\{ G_{ii}^T \left(P_k + \mu R \right) + \left(P_k + \mu R \right) G_{ii} \right\} \prec 0, \\ i,k \in \{1,\ldots,r\} \tag{28}$$

$$\left\{ \frac{G_{ij} + G_{ji}}{2} \right\}^T \left(P_k + \mu R \right) + \left(P_k + \mu R \right) \left\{ \frac{G_{ij} + G_{ji}}{2} \right\} \prec 0, \tag{29}$$

for $i, j, k = 1, 2, \ldots, r$ such that $i \prec j$

where

$$G_{ij} = A_i - B_i F_j,\ G_{ii} = A_i - B_i F_i$$

and $$P_\phi = \sum_{k=1}^{r} \phi_k \left(P_k + R \right)$$

Proof

Let consider the Lyapunov function in the following form:

$$V\left(x(t)\right) = \sum_{k=1}^{r} h_k\left(z(t)\right) \cdot V_k\left(x(t)\right) \tag{30}$$

with

$$V_k\left(x(t)\right) = x^T(t)\left(P_k + \mu R\right) x(t), \qquad k = 1, 2, \ldots, r$$

where $P_k = P_k^T, R = R^T$, $0 \le \varepsilon \le 1, \mu = 1 - \varepsilon,$ and $\left(P_k + \mu R\right) \ge 0, \quad k = 1, 2, \ldots, r$.

The time derivative of $V\left(x(t)\right)$ with respect to t along the trajectory of the system (25) is given by:

$$\dot V\left(x(t)\right) = \sum_{k=1}^{r} \dot h_k\left(z(t)\right) V_k\left(x(t)\right) + \sum_{k=1}^{r} h_k\left(z(t)\right) \dot V_k\left(x(t)\right) \tag{31}$$

The equation (31) can be rewritten as,

$$\begin{aligned} \dot V\left(x(t)\right) = {} & x^T(t) \left(\sum_{k=1}^{r} \dot h_k\left(z(t)\right)\left(P_k + \mu R\right) \right) x(t) \\ & + \dot x^T(t) \left(\sum_{k=1}^{r} h_k\left(z(t)\right)\left(P_k + \mu R\right) \right) x(t) \\ & + x^T(t) \left(\sum_{k=1}^{r} h_k\left(z(t)\right)\left(P_k + \mu R\right) \right) \dot x(t) \end{aligned} \tag{32}$$

By substituting (25) into (32), we obtain,

$$\dot V\left(x(t)\right) = \Upsilon_1(x,z) + \Upsilon_2(x,z) + \Upsilon_3(x,z) \tag{33}$$

where

$$\Upsilon_1(x,z) = x^T(t) \left(\sum_{k=1}^{r} \dot h_k\left(z(t)\right) \cdot \left(P_k + \mu R\right) \right) x(t) \tag{34}$$

$$\begin{aligned} \Upsilon_2(x,z) = {} & x^T(t) \sum_{k=1}^{r} \sum_{i=1}^{r} h_k\left(z(t)\right) h_i^2\left(z(t)\right) \\ & \times \left\{ G_{ii}^T \left(P_k + \mu R\right) + \left(P_k + \mu R\right) G_{ii} \right\} x(t) \end{aligned} \tag{35}$$

$$\Upsilon_3(x,z) = x(t)^T \sum_{k=1}^{r}\sum_{i=1}^{r}\sum_{i \prec j} h_k(z(t)) h_i(z(t)) h_j(z(t)) \times \left\{ \left(\frac{G_{ij}+G_{ji}}{2} \right)^T (P_k + \mu R) + (P_k + \mu R) \left(\frac{G_{ij}+G_{ji}}{2} \right) \right\} \Bigg] x(t) \tag{36}$$

Then, based on assumption 1, an upper bound of $\Upsilon_1(x,z)$ obtained as:

$$\Upsilon_1(x,z) \le \sum_{k=1}^{r} \phi_k \cdot x(t)^T (P_k + \mu R) x(t) \tag{37}$$

Based on(4), it follows that $\sum_{k=1}^{r} \dot{h}_k(z(t)) \varepsilon R = \bar{R} = 0$ where R is any symmetric matrix of proper dimension.
Adding $\bar{R}$ to(37), then

$$\Upsilon_1(x,z) \le \sum_{k=1}^{r} \phi_k \cdot x(t)^T (P_k + R) x(t) \tag{38}$$

Then,

$$\dot{V}(x(t)) \le \sum_{k=1}^{r} \phi_k x^T(t)(P_k + R) x(t) + \Upsilon_2(x,z) + \Upsilon_3(x,z)$$

If (28) and (29) holds, the time derivative of the fuzzy Lyapunov function is negative. Consequently, we have

$$\begin{aligned}\dot{V}(x(t)) \le x^T(t) \Bigg(& \sum_{k=1}^{r}\sum_{i=1}^{r} h_k(z(t)) h_i^2(z(t)) \times \left\{ \left(G_{ii}^T (P_k + \mu R) + (P_k + \mu R) G_{ii} \right) \right\} \\ & + \sum_{k=1}^{r} \phi_k ((P_k + R)) + \sum_{k=1}^{r}\sum_{i=1}^{r}\sum_{i \prec j} h_k(z(t)) h_i(z(t)) h_j(z(t)) \\ & \times \left(\left\{ \frac{G_{ij}+G_{ji}}{2} \right\}^T (P_k + \mu R) + (P_k + \mu R) \left\{ \frac{G_{ij}+G_{ji}}{2} \right\} \right) \Bigg) x(t) \\ & \prec 0\end{aligned}$$

and the closed loop fuzzy system (7) is stable. ■

5. Inputs Constraints

The conditions of Theorem 1 were derived by including an assumption on the time derivative of the premise membership function

$$\left| \dot{h}_k(z(t)) \right|^2 \le \left| \xi_{k\max} \right|^2 \dot{x}(t)^T \dot{x}(t) \le \phi_k^2 \tag{39}$$

so we need to select ϕ_k to satisfy the constraint.

Theorem 3

Assume that $x(0)$ and $z(0)$ are known. The assumption (39) holds if there exist positive definite matrices $P_1, P_2, \ldots, P_r$ and matrices $F_1, F_2, \ldots, F_r$ satisfying

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & (P_i + \mu R)^{-1} \end{bmatrix} \ge 0 \ for\ i = 1, \ldots, r \tag{40}$$

$$\begin{bmatrix} \phi_k\left(P_k+R\right) & \xi_{k\max}\left[A_i-B_iF_j\right]^T \\ \xi_{k\max}\left[A_i-B_iF_j\right] & \phi_k I \end{bmatrix} \geq 0, \quad (41)$$
$$\forall i,k \in \{1,2,\ldots,r\}$$

Proof

The assumption (39) can be rewritten as

$$\left(\frac{\partial h_\rho\left(z(t)\right)}{\partial x(t)}\dot{x}(t)\right)^T\left(\frac{\partial h_\rho\left(z(t)\right)}{\partial x(t)}\dot{x}(t)\right) \leq \phi_\rho^2 \quad (42)$$

Substituting (7) in(39), we obtain

$$\begin{aligned} &\left[\left(\sum_{l=1}^{s}v_{\rho l}\left(z(t)\right)\xi_{\rho l}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)^T\right. \\ &\qquad\left.\times\left(\sum_{l=1}^{s}v_{\rho l}\left(z(t)\right)\xi_{\rho l}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)\right] \\ &\leq \\ &\left[\left(\xi_{\rho\max}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)^T\right. \\ &\left.\times\left(\xi_{\rho\max}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)\right] \quad (43) \\ &\leq \phi_\rho^2 \end{aligned}$$

Dividing by ϕ_k^2, we obtain

$$\begin{aligned} &\frac{1}{\phi_k^2}\left[\left(\xi_{k\max}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)^T\right. \\ &\left.\times\left(\xi_{k\max}\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]x(t)\right\}\right)\right] \\ &\leq 1 \end{aligned}$$

which can be written as

$$\begin{aligned} &\left(\frac{\xi_{k\max}^2}{\phi_k^2}\right)x^T(t)\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]\right\}^T \\ &\qquad\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]\right\}x(t) \quad (44) \\ &\qquad\leq 1 \end{aligned}$$

We assume that for the fuzzy Lyapunov function the inequality (45) holds [2]:

$$V\left(x(t)\right) \leq V\left(x(0)\right) \leq 1, \ t \geq 0 \quad (45)$$

i.e.,

$$\sum_{k=1}^{r} h_k\left(z(t)\right)x^T(t)\left(P_k+\mu R\right)x(t)\le\sum_{k=1}^{r} h_k\left(z(0)\right)x^T(0)\left(P_k+\mu R\right)x(0)\le 1 \quad (46)$$

Then we have

$$1-\sum_{k=1}^{r} h_k\left(z(0)\right)x^T(0)\left(P_k+\mu R\right)x(0)\ge 0 \quad (47)$$

and

$$1-x^T(0)\left(\sum_{k=1}^{r} h_k\left(z(0)\right)\left(P_k+\mu R\right)\right)x(0)\ge 0 \quad (48)$$

which is expressed via LMIs using the *Schur* complement as follows:

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & \left(\sum_{k=1}^{r} h_k\left(z(0)\right)\left(P_k+\mu R\right)\right)^{-1}\end{bmatrix}\ge 0 \quad (49)$$

So it follows:

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & \left(P_k+\mu R\right)^{-1}\end{bmatrix}\ge 0 \text{ for } k=1,\ldots,r$$

which leads the LMI condition (40).

On the other hand, by considering (44) and(46), we deduce that (39) holds if

$$\left(\frac{\xi_{k\max}^2}{\phi_k^2}\right)\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]^T\right\}$$
$$\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]\right\} \quad (50)$$
$$-\sum_{k=1}^{r} h_k\left(z(t)\right)\left(P_k+\mu R\right)\le 0$$

which is equivalent to

$$\begin{bmatrix} \phi_k\sum_{k=1}^{r} h_k\left(z(t)\right)\left(P_k+\mu R\right) & \left(\xi_{\rho\max}\mathrm{M}^T\right) \\ \left(\xi_{\rho\max}\mathrm{M}\right) & \phi_k I\end{bmatrix}\ge 0 \quad (51)$$

where $\mathrm{M}=\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right)h_j\left(z(t)\right)\left[A_i-B_iF_j\right]$

This leads to the LMI condition(41):

$$\begin{bmatrix} \phi_k\left(P_k+\mu R\right) & \Sigma_{ijk}^T \\ \Sigma_{ijk} & \phi_k I\end{bmatrix}\ge 0,\ \forall i,k\in\{1,2,\ldots,r\} \quad (52)$$

where $\Sigma_{ijk}=\xi_{k\max}\left(A_i-B_iF_j\right)$. ■

7. Numerical Examples

In order to show the improvements of proposed approaches over some existing results, in this section, we present two numerical examples. The first one concerns the feasible area for a T-S fuzzy system. Indeed, we compare the fuzzy Lyapunov approach (Theorem 1) with the Lemma 1 in [2], and Lemma 2 in [3]. However, the second example presents the state variables evolution for a two rules T-S fuzzy system.

Example 1. Consider the following T-S fuzzy system:

$$\dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)A_i x(t) \tag{53}$$

with: $r=4$

$$A_1=\begin{bmatrix}-5 & -4\\ -1 & a\end{bmatrix},\ A_2=\begin{bmatrix}-4 & -4\\ \frac{1}{5}(3b-2) & \frac{1}{5}(3b-4)\end{bmatrix},\ A_3=\begin{bmatrix}-3 & -4\\ \frac{1}{5}(2b-3) & \frac{1}{5}(2a-6)\end{bmatrix},\ A_4=\begin{bmatrix}-2 & -4\\ b & -2\end{bmatrix}$$

where $a\in[-10,0]$, $b\in[0,500]$ and the premise functions are given by:

$$h_1=\alpha_1(x_1)\alpha_2(x_2),\qquad h_2=\alpha_1(x_1)\beta_2(x_2)$$
$$h_3=\beta_1(x_1)\alpha_2(x_2),\qquad h_4=\beta_1(x_1)\beta_2(x_2)$$

such that,

$$\alpha_i(x_i)=\begin{cases}\left(1-\sin(x_i)\right)/2, & \text{for } |x_i|\le\frac{\pi}{2},\\ 0, & \text{for } x_i\succ\frac{\pi}{2},\\ 1, & \text{for } x_i\prec\frac{-\pi}{2},\end{cases}\quad \text{, and}\quad \beta_i(x_i)=1-\alpha_i(x_i),$$

The stability of this system is checked using Lemma 1 in [2], Lemma 2 in [3] and proposed Theorem 1, respectively, for several values of pair (a,b), $a\in[-10,0]$, $b\in[0,500]$ and $\left|\dot{h}_k\left(z(t)\right)\right|\prec 0.85$.

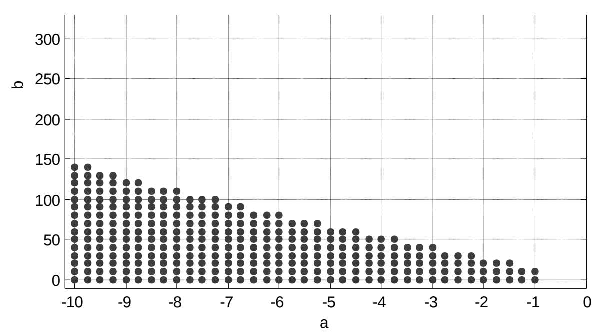

Figure 1. Feasible area of Lemma 1 [2]

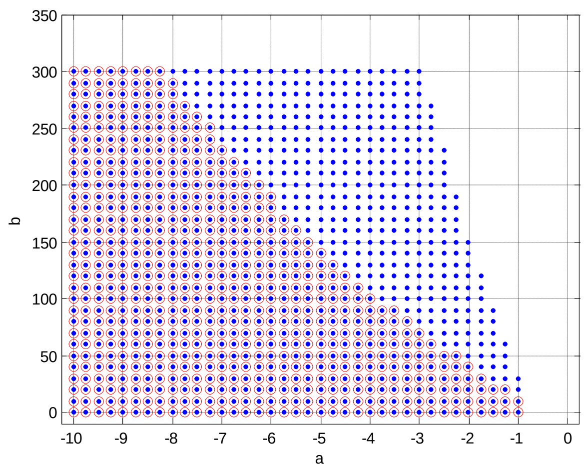

Figure 2. Feasible area of Lemma 4 (□) [3] and Theorem 1 (•)

It is easy to see that if $\varepsilon = 1$ and $R = -P_r$ then Lemma 1 is recovered, and if $\varepsilon = 1$ so is Lemma 2. Therefore, Lemma 1 and 2 are particular cases of Theorem 1. Fig.1 shows the stability margin when we use a Fuzzy Lyapunov Function.

Fig. 2 (□) presents a less conservative condition given by the Lemma 2 [3]. This result is obtained by introducing of a single variable R.

Fig.2 (•) shows the proposed condition stability presented in Theorem 1 for $\varepsilon = 0.1$. Note that the proposed condition involved all previous ones with a larger stability region. With the introduction of a positive constant $\varepsilon \le 1$, a new less conservative condition is obtained.

Example 2. Consider the following T-S fuzzy system:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t) \tag{54}$$

with: $r = 2$

the premise functions are given by:

$$h_1\left(x_1(t)\right) = \frac{1+\sin x_1(t)}{2};\; h_2\left(x_1(t)\right) = \frac{1-\sin x_1(t)}{2};\; A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix};\quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix};$$

It is assumed that $\left|x_1(t)\right| \le \frac{\pi}{2}$. For $\xi_{11} = 0, \xi_{12} = 0.5,\ \xi_{21} = -0.5,$ and $\xi_{22} = 0,$ we obtain

$$P_1 = \begin{bmatrix} 37.7864 & 26.8058 \\ 26.8058 & 36.2722 \end{bmatrix};\; P_2 = \begin{bmatrix} 98.5559 & 28.7577 \\ 28.7577 & 22.9286 \end{bmatrix};\quad R = \begin{bmatrix} -1.2760 & -2.2632 \\ -2.2632 & -0.6389 \end{bmatrix}$$

Figure 3 shows the evolution of the state variables. As can be seen, the conservatism reduction leads to very interesting results regarding fast convergence of this Takagi-Sugeno fuzzy system.

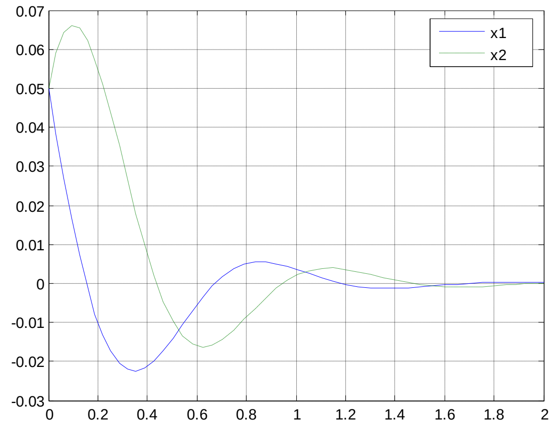

Figure 3. State Variables

8. Conclusion

This paper provided a new condition for the stability and stabilisation of Takagi-Sugeno fuzzy systems in terms of a combination of the LMI approach and the use of non-quadratic Lyapunov function as Fuzzy Lyapunov function.

The stability condition proposed in this note is less conservative than some of those in the literature, which has been illustrated via examples.

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