Complexity Aspects of the Classification of Synchronizing Graphs for Kuramoto Coupled Oscillators

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Abstract

In this work, we give a complete presentation of the idea of synchronizing graphs for the Kuramoto model of coupled oscillators. We present the dynamical model and the main relationships between the system dynamics and the underlying interconnection topology. A synchronizing graph is an interconnection that ensures synchronization of all the oscillators for almost every initial condition. We present the main properties that help in the classification of synchronizing graphs and also some considerations about the structure of this family of graphs and the complexity of the classification task.

Keywords: *Network synchronization, coupled oscillators, synchronizing graphs, graph complexity.*

1. Introduction

Synchronization, coordination and emergent behaviors have become important issues for the control community in the last decade. Objects arriving from Biology and Physics, together with classical and modern systems analysis tools promote an interdisciplinary approach to explain known phenomena, to derive new models and to develop new frameworks and techniques. In this context, the Kuramoto model, postulated in the mid seventies [1-3], has shown to be a good model for biological, physical and engineering problems and phenomena, all involving some kind of sychronization or coordination between interacting agents: cardiac pacemaker cells, brain cells' synchronization, fireflies flashing synchronously, the Josephson junction of superconductors, microwave antennas arrays, robot formation, etc. [4-9].

Several control researches have studied different aspects of synchronization problems, which naturally can be tackled from two sides: the dynamic of the agents (see the survey [10]) and the underlying interconnection between agents [11-21]. For the classical Kuramoto model with identical oscillators, the synchronization property relies entirely on the way the agents interact, that is, it can be fully explained studying the algebraic and topological properties of the underlying interconnection graph [22]. In particular, we look for conditions on the graph to ensure the synchronization of the agents for almost every initial condition, *almost global synchronization* (AGS), as it is explained in the next Section. A graph that ensures the AGS property is called *synchronizing*. Characterize this family of graphs is our goal. Following this line of research, we have derived some results and introduced some concepts that show the non triviality of the influence of the underlying graph on the synchronization property [23-29]. The present work surveys those results and introduces new ones.

The article is organized as follows. Section 2 presents the Kuramoto model, the main dynamics considerations and concepts and the definition of synchronizing graphs. Section 3 introduces the main results for local classification of equilibrium points. In Section 4, we present the idea of twin vertices and its relationship with synchronizing graphs. Section 5 contains a result implying the big structural complexity of synchronizability. Finally, some conclusions and research directions are provided.

2. The Kuramoto Model and Synchronizing Graphs

Following the works of Winfree on biological clocks [29], Y. Kuramoto derived a mathematical model for the explanation of several synchronization and coordination properties of biological systems [1,4], like heart cells, fireflies flashing and cricket chorus. Each agent is represented by an oscillator, a dynamical system with a globally attractor limit cycle. At a steady state, the dynamic of the oscillator can be described by the phase of the trajectory on the limit cycle. So, the dynamic of the *i*-th oscillator is:

$$\dot{\theta}_i = \omega_i$$

Interaction between agents is modeled as a perturbation of the angular velocity under the assumption that a limit cycle still exists (*weak coupling*). So, the Kuramoto model takes the following form:

$$\dot{\theta}_i = \omega_i + \sum_{ik} f_{ik}\,(\theta_k - \theta_i) \quad i, k = 1, \dots, n$$

where the periodic functions f_{ik} describe the weak coupling and only depends on the phase differences between agents. The most classical Kuramoto model, the one we use along this article, uses sinusoidal interaction functions. For each agent i, define the set N_i as the set of agents interacting with agent i, the *neighbors* of i. So, the Kuramoto model we use is:

$$\dot{\theta}_i = \omega_i + \sum_{i \in N_i} sin\,(\theta_k - \theta_i)$$

Moreover, we will work with identical oscillators. Using this fact, we may reparameterize time and get equation (1), which will be the one used along the article.

$$\dot{\theta}_i = \textstyle\sum_{i \in N_i} sin\,(\theta_k - \theta_i) \quad , \quad i = 1, \dots, n \tag{1}$$

The natural state space for n oscillators is the n-dimensional torus T^n, since the influence functions are 2π-periodic. Thus, we will consider each phase θ_i in the interval $[0,2\pi)$ for $i=1,\dots,n$. We work with mutual influence between agents

$$i \in N_k \leftrightarrow k \in N_i$$

Synchronization of oscillators is a classical topic in electrical engineering [30]. In the last years, several works in the control community have addressed the problem of analyzing local stability properties of the Kuramoto model while global properties must deal with the existence of multiple equilibria. Following [31], we say a dynamical system has the *almost global stability* property if the origin attracts all the initial conditions, with the possible exception of a zero Lebesgue measure set of the state space. From an engineering point of view, it is a desirable situation, especially when the system has several equilibrium points. For a Kuramoto model, the origin (all the phases equal to zero) represents the *consensus* or *synchronization* of all the agents. *Almost global synchronization* (AGS) means that almost every initial condition leads to synchronization [22]. We denote by θ a vector of phases of T^n. Interaction between agents is modeled by a non-oriented graph *G*, with *n* nodes representing agents and *m* links showing the interconnections (good references for graph theory are [32,33]). *We will only deal with connected graphs, because we are concerning on the synchronization of all the agents*. If we endow *G* with an arbitrary orientation, we can obtain an incidence matrix *B* and a compact description for equation (1) [12]:

$$\dot{\theta} = -B\sin(B^T\theta) \tag{2}$$

We can introduce the function

$$V(\theta) = m - 1_m^T\cos(B^T\theta)$$

and show that (2) is a gradient system (see [12,8]). Here, 1_m^T denotes the *n*-dimensional row vector with all its elements equal to 1. Since the *Laplacian* of *G* is the square matrix $L = BB^T$ [33], it follows that

$$\frac{\partial^2 V}{\partial \theta^2}(\theta) = -B\, diag[\cos(B^T\theta)]B^T$$

can be seen as a *weighted Laplacian*. This fact gives a direct link between dynamical properties of system (2) and algebraic properties of graph *G*. When system (2) has the AGS property, we say the graph *G synchronizes*, is *synchronizing* or is *AGS*. As Kuramoto suggested, each oscillator can be thought as a running particle on the unit circumference, where the respective phase defines its position. So, each oscillator may be represented by a phasor $V_i = e^{j\theta_i}$, *i*=1, …, *n*, where $j = \sqrt{-1}$. For each agent *i*, we introduce the complex numbers

$$\alpha_i(\theta) = \frac{1}{V_i}\sum_{k\in N_i} V_k = \sum_{k\in N_i}\frac{V_k}{V_i} = \sum_{k\in N_i} e^{j(\theta_k-\theta_i)} \tag{3}$$

We must emphasize that the system has a kind of *rotational invariance*, in the sense that if $\bar{\theta} \in T^n$ is an equilibrium point of (2), then, for every real number *c*, the vector $\bar{\theta} + c\,1_m \in T^n$ is also an equilibrium point. In this sense, consensus or synchronization involves a whole closed curve, a circumference, in the torus, and so does every other equilibrium. Stability assertions will refer to this set, but we will not make it explicit every time.

As we have mentioned,

$$\dot{V}(\theta) = -\left\|\dot{\theta}\right\|^2 \tag{4}$$

So, V decreases along the trajectories of system (2). Since $V(0) = 0$ and V is non-negative definite, it is a local Lyapunov function for the consensus (and also generalizes the *order parameter* introduced by Kuramoto [1,12]). Moving towards global properties, we wonder what happens when we start far from the consensus set. Equation (4) says that the potential is not increasing along the trajectories of the system. Since we are working on a compact state space, we may apply LaSalle's result and conclude that every trajectory converges to an equilibrium point of (2) [34]. Moreover, the only attractors of the system are the equilibria. So, in order for the system to have the AGS property, the consensus must be the only attractor of the state space or, equivalently, every non-consensus equilibrium point must be unstable. Our main tool for classifying the equilibria is Jacobian linearization. Another approach, using density functions, was shown to work only for few nodes due to the characteristics of the system [22]. At an equilibrium point $\bar{\theta} \in T^n$, the Jacobian n x n matrix $M(\bar{\theta})$ is symmetric and takes the explicit form:

$$m_{ii} = -\alpha_i(\bar{\theta}) \quad , \quad m_{hi} = cos(\bar{\theta}_h - \bar{\theta}_i) \ \ if \ \ h \in N_i \quad , \quad m_{hi} = 0 \ \ if \ \ h \in {N_i}^c \qquad (5)$$

or, in a more compact notation,

$$M(\bar{\theta}) = \frac{\partial^2 V}{\partial \theta^2}(\theta) = -B \ diag[cos(B^T \bar{\theta})]B^T$$

Observe that $M(\bar{\theta})$ has real eigenvalues and we always have the identity $M(\bar{\theta})1_n = 0$, and $M(\bar{\theta})$ has the null eigenvalue. This is related to the rotational invariance already mentioned. Due to this fact, we only care about the rest of the eigenvalues. If $M(\bar{\theta})$ has a positive eigenvalue, then $\bar{\theta}$ is unstable; if $M(\bar{\theta})$ has n-1 negative eigenvalues, then $\bar{\theta}$ is stable (and we have asymptotical convergence to the equilibrium curve through $\bar{\theta}$). If the null eigenvalue is not simple, then Jacobian linearization may fail to classify the equilibrium point. Without looking directly to the eigenvalues, we can work with the quadratic form induced by $M(\bar{\theta})$. Let $x \in R^n$ and denote by ik the link between nodes i and k, when it exists. Then,

$$x^T M(\bar{\theta}) x = -\textstyle\sum_{ik} (x_i - x_k)^2 \cos(\bar{\theta}_i - \bar{\theta}_k) \qquad (6)$$

where the sum is taken over all the graph's edges.

3. Main Results

In this Section, we will focus on the general properties of equation (2), its equilibria and its relationships with the underlying interconnection graph. We have the following general results.

Lemma 1. $\bar{\theta} \in T^n$ is an equilibrium point of (2) if and only if $\alpha_i(\bar{\theta})$ is real, for i=1,...,n.

***Proof*:** From equation (3), is clear that for every i,

$$\alpha_i(\bar{\theta}) = \sum_{k \in N_i} cos(\bar{\theta}_k - \bar{\theta}_i) + j \sum_{k \in N_i} sin(\bar{\theta}_k - \bar{\theta}_i)$$

So, at an equilibrium point, the imaginary part vanishes and the numbers $\alpha_i(\bar{\theta})$ are all real.

Lemma 2. Let $\bar{\theta} \in T^n$ be an equilibrium point of (2).

i) If $cos(\bar{\theta}_i - \bar{\theta}_j) > 0$ for every connected pair of nodes i, j, then $\bar{\theta}$ is stable.
ii) If for some i, the respective number $\alpha_i(\bar{\theta})$ is non positive, then $\bar{\theta}$ is unstable.
iii) If for a suitable reference, some $\bar{\theta}_i \in (0, \pi/2)$ and the rest of the agents' phases are in $(\pi, 3\pi/2)$, then $\bar{\theta}$ is unstable.
iv) If all the agents' phases are located inside a semi circumference, then $\bar{\theta}$ is a consensus equilibrium point.
v) If $\bar{\theta}$ is a partial synchronized equilibrium point, then $\bar{\theta}$ is unstable.
vi) Consider an agent i with only two neighbors: $N_i = \{h, k\}$. Let $\bar{\theta}$ be a stable equilibrium point and define the angles $\varphi_{ik} = |\bar{\theta}_i - \bar{\theta}_k|$ and $\varphi_{ih} = |\bar{\theta}_i - \bar{\theta}_h|$. Then $\varphi_{ik} = \varphi_{ih}$.

***Proof*:** If $cos(\bar{\theta}_i - \bar{\theta}_j) > 0$ for every connected pair of nodes i, j, the Jacobian matrix $M(\bar{\theta})$ is minus the Laplacian L of the graph with a positive definite weight. Since the Laplacian has n-1 positive eigenvalues, we obtain the stability of $\bar{\theta}$. Let us consider now the case when some $\alpha_i(\bar{\theta})$ is negative. Since this number appears at the diagonal of the symmetric matrix $M(\bar{\theta})$, it implies that this matrix has at least one positive eigenvalue. A different approach we must use when there is a null $\alpha_i(\bar{\theta})$, since in this case, the Jacobian may have a multiple null eigenvalue. We may re-write the function $V(\bar{\theta})$ as follows:

$$V(\bar{\theta}) = m - \frac{1}{2}\sum_{i=1}^{n}\sum_{k\in N_i} cos(\bar{\theta}_k - \bar{\theta}_i) = m - \frac{1}{2}\sum_{1}^{n}\alpha_i(\bar{\theta})$$

Consider the k-th element of the canonical base e_k, a small positive number δ and a perturbation $\tilde{\theta} = \bar{\theta} + \delta e_k$. Then, after some calculations, we obtain

$$V(\tilde{\theta}) = m - \frac{1}{2}\sum_{\substack{i=1,\\ i\neq k}}^{n}\sum_{\substack{k\in N_i\\ h\neq k}} cos(\bar{\theta}_h - \bar{\theta}_i) - \sum_{h\in N_k} cos(\bar{\theta}_k + \delta - \bar{\theta}_h)$$

Using the identity $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, we have that

$$\cos(\bar{\theta}_k + \delta - \bar{\theta}_h) = \cos(\delta)\cos(\bar{\theta}_k - \bar{\theta}_h) - \sin(\delta)\sin(\bar{\theta}_k - \bar{\theta}_h)$$

So,

$$\sum_{h\in N_k} cos(\bar{\theta}_k + \delta - \bar{\theta}_h) = \cos(\delta)\sum_{h\in N_k} cos(\bar{\theta}_k - \bar{\theta}_h) - \sin(\delta)\sum_{h\in N_k}\sin(\bar{\theta}_k - \bar{\theta}_h)$$

Then, it follows that

$$\sum_{h\in N_k} cos(\bar{\theta}_k + \delta - \bar{\theta}_h) = \cos(\delta).Re\left(\alpha_i(\bar{\theta})\right) - \sin(\delta).Im\left(\alpha_i(\bar{\theta})\right) = 0$$

It turns out that $U(\tilde{\theta}) = U(\bar{\theta})$ for all δ. We have proved that arbitrarily close to $\bar{\theta}$, we can find non equilibrium points with the same potential value. This implies that, arbitrarily close to $\bar{\theta}$, there are points with more and less potential value. So, $\bar{\theta}$ must be unstable.

Now, we prove the third assertion. Consider a n-dimensional vector x such that its i-th component is 0 if $\bar{\theta}_i \in (0, \pi/2)$ and 1 otherwise. Then, the quadratic form (6) takes a positive value and $M(\bar{\theta})$ has a positive eigenvalue, implying instability of $\bar{\theta}$.

In order to prove *iv*), suppose, by contradiction that exists unsynchronized agents. Then, $\bar{\theta}_m = min_i\{\bar{\theta}_i\} \leq max_i\{\bar{\theta}_i\} = \bar{\theta}_M$. We claim there should exist and index k achieving the minimum, but unsynchronized with at least one of its neighbors. Indeed, it suffices to consider a length l walk $v_m = v_0, v_1, .., v_l = v_M$ in the graph from vertex v_m to vertex v_M, with respective phases $\bar{\theta}_m = \bar{\theta}_0, \bar{\theta}_1, \ldots, \bar{\theta}_l = \bar{\theta}_M$. Let $K = max\{i \mid \bar{\theta}_0 = \bar{\theta}_1 = \cdots = \bar{\theta}_i\}$. Then, $K < l$ (otherwise, it would be $\bar{\theta}_m = \bar{\theta}_M$). Besides, $\bar{\theta}_K \neq \bar{\theta}_{K+1}$. Since the angles $\bar{\theta}_i$ are all in a semi circumference, $\sin(\bar{\theta}_i - \bar{\theta}_K)$ have the same sign of $(\bar{\theta}_i - \bar{\theta}_K)$ for all i and then

$$\sum_{i\in N_K}\sin(\bar{\theta}_i - \bar{\theta}_K) > 0$$

But this contradicts the equilibrium condition of $\bar{\theta}$.

Consider now a partially synchronized equilibrium point. We may recognize two clusters of agents, located at opposite sides on the circumference. We define a n-dimensional vector x, whose components takes the value 0 on one cluster and 1 on the other. Then, the number $x^T M(\bar{\theta})x$ is positive and $\bar{\theta}$ is unstable.

We prove the final item. The equilibrium condition at agent i implies that $\sin(\varphi_{ik}) = \sin(\varphi_{ih})$. Then, it must be true that either $\varphi_{ik} = \varphi_{ih}$ or $\varphi_{ik} + \varphi_{ih} = \pi$. But in the last case, we would have $\alpha_i(\bar{\theta}) = 0$ and $\bar{\theta}$ should be unstable.

These results have some direct consequences on the classification of certain graph families. For example, the equilibria set for a tree only has partial or fully synchronized points, so a tree is always AGS. A complete graph has only three classes of equilibrium points: consensus, partial consensus and balanced ($\sum_{i=1}^{n} V_i = 0$). For the last case, the respective numbers $\alpha_i(\bar{\theta})$ are equal to -1 for every i, given that a compete graph is synchronizing [22]. In some sense, *sparse* graphs like trees, and the *dense* ones like complete graphs, have the AGS property. Between them, there are synchronizing and non-synchronizing graphs, like the cycles with 5 or mode nodes [26], and the problem of classification is not trivial. The existence of a *cut vertex* allows us to reduce the classification problem to the *blocks* of the graph or, more generically, to *2-connected* graphs [24]. In the next Sections, we will show some results concerning the complexity of the classification task.

4. Twin Vertices

In this Section we present the concept of *twin vertices* and some useful related properties.

Definition 1. Consider two nodes u and v of a given graph G. We say they are twins if they have the same set of neighbors: $N_u = N_v$.

A slightly modification of the previous definition leads us to *adjacent twins*, when u and v are adjacent and they share neighbors. Concerning synchronization, twin vertices act as a *team* in order to get equilibrium in equation (2). Next result imposes a necessary behavior of twins and will be useful for the complexity analysis we will perform in Section 4.

Lemma 3. Consider the system (2) with graph G. Let $\bar{\theta}$ be an equilibrium point and v a vertex of G, with associated phasor V_v. Let T be the set of twins of v and N the set of common neighbors. If the real number

$$\alpha_v(\bar{\theta}) = \frac{1}{V_v} \sum_{w \in N} V_w$$

is nonzero, the twins of v are partially or fully coordinated with it, i.e., the phasors V_h, $h \in N_v$, are parallel to V_v. Moreover, if $\bar{\theta}$ is stable, the agents in T are fully synchronized.

Proof: Let $u \in T$ and consider the real numbers

$$\alpha_u = \frac{1}{V_u} \sum_{w \in N} V_w \qquad \text{and} \qquad \alpha_v = \frac{1}{V_v} \sum_{w \in N} V_w$$

Then, it follows that $\alpha_u . V_u = \alpha_v . V_v$, for all $u \in T$. If there are $u_1, u_2 \in T$ linearly independent, their respective α_{u_1} and α_{u_2} must be zero, and so are the numbers α in T. Then, if there is some $\alpha_u \neq 0$, $u \in T$, all phasors in T are parallel. So, all the nodes in T are partially or fully coordinated (and $\bar{\theta}$ is a partial or a full consensus equilibrium point).

Suppose now that $\bar{\theta}$ is stable and that there are $u_1, u_2 \in T$ such that $V_{u_1} = -V_{u_2}$. Then

$$\alpha_{u_1} = \frac{1}{V_{u_1}} \sum_{w \in N} V_w = -\frac{1}{V_{u_2}} \sum_{w \in N} V_w = -\alpha_{u_2}$$

and we have at least one negative number α. Thus, $\bar{\theta}$ is unstable, by Lemma 3-ii).

A direct consequence of this result is that *complete k-partite* graphs always synchronize [28]. We may define an equivalence relationship in the node set of the graph: two nodes are equivalent if they are twins. So, we can obtain a *quotient* graph by direct identification of equivalent nodes. Relationships between the synchronizability of a given graph and its quotient graph can be stated, but we do not include these properties here. As an example, we just present a result without its proof (it can be found in [25]):

Proposition 1. If the quotient graph is a tree, then the original graph is AGS.

The ***quotient*** graph can be seen as a induced subgraph of the original and, reciprocally, we can cover a given graph by a larger graph, obtained by the addition of twins. These lead us to the following results.

Theorem ***1.*** Any connected graph *G* admits an AGS twin cover.

Proof: Let us suppose that the set of vertices of G is $\{1,\dots,n\}$ and that we have constructed a cover $\bar{G}$ by splitting each vertex i of G in a number a_i of *adjacent* twin vertices. Then, if $\bar{\theta}$ is a linearly stable equilibrium point of $\bar{G}$, by Lemma 3, the twins must be synchronized and, for a given vertex *i*, we have $\sum_{j\in N_i} a_j sin(\bar{\theta}_j-\bar{\theta}_i)$=0, since al the twins of vertex *i* have the same $\bar{\theta}_i$. Then, for any $k\in N_i$:

$$a_k sin(\bar{\theta}_k-\bar{\theta}_i) = -\sum_{j\in N_i, j\neq k} a_j sin(\bar{\theta}_j-\bar{\theta}_i) \quad , \quad \left|sin(\bar{\theta}_k-\bar{\theta}_i)\right| \leq \frac{\sum_{j\in N_i,j\neq k} a_j}{a_k}$$

Then, if the second member of the inequality is small enough to ensure that the sine in the first member is small, we are done. Indeed, small sines, say, smaller than a given number $\sqrt{2}/D$, where *D* is the diameter of *G*, implies phasors in opposite quarter circumference, but by Lemma 2-iii), this implies instability. Then, phasors must be located in a circumference quarter, which by Lemma 2-iv) implies consensus. We can in fact do something weaker, bounding the sines of adjacent vertices in a spanning tree of *G* by $\varepsilon=\sqrt{2}H$, where H is the height of the tree. In order to do this, we will construct a rooted directed spanning tree *T* of *G* and then, for each *i* we will take as *k* to be the father of *i* in this tree. Let S_h be the vertices at distance *h* from vertex 1 (the root). Then, sort each set S_h with an order $<_h$. We consider the following *lexicographical* order on the vertices: given two vertices $u\in S_i$ and $w\in S_j$, we say that $u<w$ if $i>j$ or if $i=j$ and $u<_i w$. The order defined in this way is total. Therefore, we can label the vertices following this order, so that $1=v_1>v_2>\cdots>v_n$. Next, set

$$a_i = \left\lceil \left({}^{\Delta}/_{\varepsilon}\right)^{|V|-i} \right\rceil$$

where Δ is the maximum degree of a vertex in *G* and $|V|$ is the cardinal of set V. Then, the arcs of *T* will be those $u_k v_i$ such that *ki* is an arc of *G* and

$$a_k = max_{j\in N_i} a_j$$

Notice that $v_k>v_i$. We must prove that T is indeed a tree. Is easy to see it is acyclic, because for each $i>1$ any vertex in S_i is adjacent to exactly one vertex in S_{i-1}. Besides, v_1 reaches every other vertex, so T is connected. Let us now find an upper bound for the sine of the difference between adjacent vertices of T. Let v_i and v_k be adjacent vertices of T with $i>k$. Then, for $\varepsilon<\Delta$,

$$\left|sin(\bar{\theta}_k-\bar{\theta}_i)\right| \leq \frac{1}{a_k}\sum_{j\in N_i,j\neq k} a_j \leq \frac{(\Delta-1)\left[\left({}^{\Delta}/_{\varepsilon}\right)^{n-k-1}+1\right]}{\left({}^{\Delta}/_{\varepsilon}\right)^{n-k}} < \varepsilon$$

This result tells that the class of synchronizing graphs could be quite large, but not necessarily complex, since its complement could be small. Nevertheless, we will prove that any 2-connected graph is homeomorphic to a non AGS graph, which implies that the complement to AGS graphs is also quite big and complex. In order to do it, we will prove a stronger result, namely, that doing an enough amount of elementary subdivisions in any non bridge edge of a given graph, it produces a non AGS graph. We conjecture that it is enough to do three subdivisions.

Theorem 2. If *e* is a non-bridge edge of a given graph *G* then there is an integer n_0 such that the graph obtained from *G* by making $n > n_0$ elementary subdivisions of *e* is non AGS.

Proof: The idea is the following: since the cycle C_n is non AGS for $n \geq 5$, we can replace one of the edges, say *uv* by *G-e*, identifying the extremes of *e* with *u* and *v*. If *n* is large enough, the *force* induced by C_n will be sufficiently weak to change a consensus of *G-e* to another stable equilibrium point. Let us denote by $v_1, \ldots, v_m$ the vertices of *G* and let $e = v_1 v_2$. Since *e* is not a bridge, *G'*=*G-e* is connected and the consensus equilibrium point $\bar{\theta}^T = (0, \ldots, 0)$ is stable. Now, connect the vertices v_1 and v_2 of *G'* through a path $P_n = v_1 = w_1, \ldots, w_n = v_2$ to obtain a new graph $\tilde{G}$ with vertices $\tilde{V} = \{w_1, \ldots, w_n, v_3, \ldots, v_m\}$. We want to prove that for *n* large enough, there exist a positive number ε and phases θ_i^ε, for $3 \leq i \leq m$, such that v_1 defined by:

$$\theta_x^\varepsilon = \begin{cases} i\varepsilon & , \quad if\ x = w_i \\ \theta_x^\varepsilon & , \quad if\ x = v_i \end{cases}$$

is a stable ***equilibrium*** point of $\tilde{G}$. In order for θ^ε to be an equilibrium, it must satisfies

$$\sum_{y \in N_x} \sin(\theta_y^\varepsilon - \theta_x^\varepsilon) = 0$$

for all $x \in \tilde{V}$. These equations are trivially satisfied for $x = w_2, \ldots, w_{n-1}$. Thus, we still have the following equations:

$$\begin{cases} \sum_{y \in N_{v1}} \sin(\theta_y^\varepsilon - \theta_{v_1}^\varepsilon) + \sin(\varepsilon) = 0 \\ \sum_{y \in N_{v2}} \sin(\theta_y^\varepsilon - \theta_{v_2}^\varepsilon) - \sin(\varepsilon) = 0 \\ \sum_{y \in N'_x} \sin(\theta_y^\varepsilon - \theta_x^\varepsilon) = 0 \quad , \quad x \in \tilde{V} \backslash \{v_1, v_2\} \end{cases}$$

where *N* and *N'* denotes the neighbors in *G* and *G'* respectively. This system can be thought as an ε-perturbation of the system that defines the equilibrium of *G'*. Moreover, if we add an adequate equation, e.g. θ_{v1}, the system verifies the hypothesis of the Implicit Function Theorem for $\theta = 0_m$ and $\varepsilon = 0$. Thus, it implicitly defines the angles θ_x^ε as a function of ε, for each node of *G*, in a neighborhood $(-\varepsilon_0, \varepsilon_0)$ of 0. We will have that θ^ε is a C^∞-curve in R^n passing through 0_m for $\varepsilon = 0$. Finally, in order to prove the stability of this equilibrium point, we notice that the numbers $cos(i\varepsilon - (i-1)\varepsilon)$ are all positive for small ε. Besides, when $\varepsilon = 0$ all the cosines $cos(\theta_i^\varepsilon - \theta_j^\varepsilon)$ are positive and since ε is small enough, by the continuity of θ_i^ε as a function of ε, the cosines will remain positive. So, by Lemma 2-i), the equilibrium is stable. Therefore, it suffices to take $n_0 > 2\pi/\varepsilon_0$.

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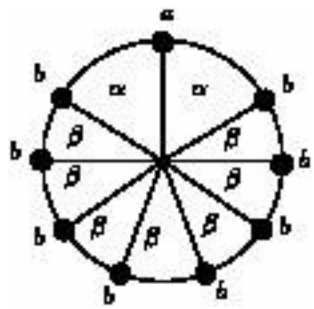
5. A Complexity Analysis of a Graph Family

In this Section we use the previous ideas in order to analyze some complexity issues of the classification of AGS graphs. It is clear that the classification of a given graph requires some knowledge of the structure of the set of equilibrium points of (2). In some cases, this set is very simple or has a well understood behavior (for example, when G is a tree, a cycle or a complete graph [22,29]). But in general, this set is very complicated and its study can be hard. In order to illustrate the complexity of the problem, we present a result concerning a class of graphs whose classification is as hard as the computation of $\sqrt{2}$. Let n, a and b be natural numbers and consider the cycle C_{n+2} with $n + 2$ nodes. We construct a new graph $G_n(a,b)$, adding a twins to agent 1 and b twins to the rest of the agents. We will explore the existence of other equilibrium points besides consensus. We will also impose conditions on n, a and b in order to have or not have the AGS property.

Theorem 3. Given positive integers a, b and n, the graph $G_n(a,b)$ is AGS if and only if

$$n > 3 \quad , \quad a < sin(^{\pi}/_{n})b \quad or \quad n = 3 \quad , \quad \frac{\sqrt{3}}{2}b < a < \sqrt{2}b \qquad (7)$$



Φιγυρε 1: Χανδιδατε εθυιλιβριυμ ποιντ φορ $X_{\nu+2}$. Ατ εαχη ποσιτιον, τηερε αρε α ορ β τωινσ

Proof: from Lemma 2-vi) and Lemma 3, a stable non-consensus equilibrium candidate configuration can be easily derived, as shown in Figure 1.

We retain the references nodes 1 to n+2 from the original graph. We denote by α and β the two ***involved*** angles. Equilibrium conditions for node 1 and nodes i, i=3,…,n+1, are trivially satisfied. Equilibrium conditions for nodes 2 and n+2 results in the following equations:

$$\text{asin}(\alpha) = bsin(\beta) \quad , \quad 2\alpha + n\beta = 2\pi$$

So, since

$$\alpha = \pi - n\frac{\beta}{2} \quad , \quad sin\left(\pi - n\frac{\beta}{2}\right) = sin\left(n\frac{\beta}{2}\right)$$

we have the implicit equation for β:

$$\frac{a}{b}sin\left(n\frac{\beta}{2}\right) = \sin(\beta)$$

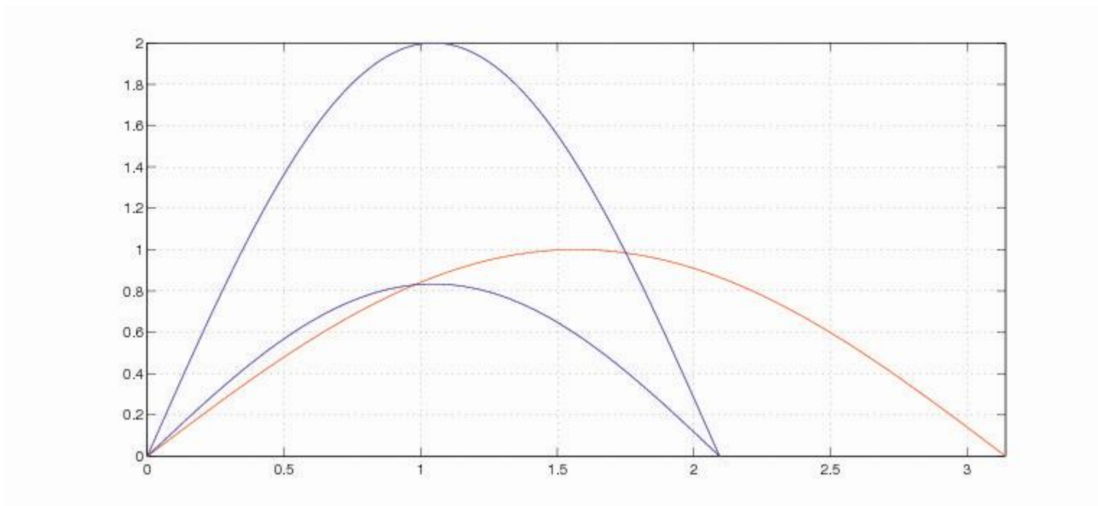
We define the auxiliary function

$$f_n(\beta) = \frac{a}{b}sin\left(n\frac{\beta}{2}\right)$$

Notice we have the trivial solution $\beta = 0$, and the respective $\alpha = \pi$, which correspond to a partial consensus configuration, and it is unstable. So, we rule out this solution.

In Figure 2, we show many distinct possibilities of the relative position of the curves $\sin(\beta)$ and $f_n(\beta)$, for the case $n = 3$. We will only look for solutions $\beta \in [0,\pi]$ and $n > 1$. A β greater than π is only possible for $n = 1$, but the respective angle $\alpha = \pi - \beta/2$ will be greater than $\pi/2$ and will have a negative cosine, given an unstable equilibrium point. Note that the number n affects the frequency of function $f_n(\beta)$. An immediate condition for the existence of a non-consensus equilibrium is $f_n^{'}(0) > 1$:

$$f_n^{'}(0) = \frac{na}{2b} > 1 \leftrightarrow a > \frac{2}{n} b \quad (8)$$



Φιγυρε 2: Σιτυατιον φορ ν=3.

If it is not the case, $G_n(a, b)$ is synchronizing. Now, we assume that condition (8) is fulfilled and we have at least one solution $\beta^* \in [0, \pi]$ (and its respective α^*). We perform a stability analysis of the corresponding equilibrium point. According to Lemma 2, a sufficient condition for stability is that $\cos(\beta^*) > 0$ and $\cos(\alpha^*) > 0$, while a sufficient condition for instability is either $cos(\beta^*) < 0$ or $cos(\alpha^*) < 0$, since the numbers directly defines the sign of all the involved numbers $\alpha_i(\bar{\theta})$. Consider first $\cos(\beta^*)$. Then, $\cos(\beta^*) > 0$ if and only if n>3 or n=3 and $f_n(\pi/2) < 1$. This implies that n>3 or n=3 and $(a/b)\sin(n\pi/4) < 1$. So, we have the following condition for synchronizability:

$$n > 3 \quad or \quad n = 3 \quad , \quad \frac{\sqrt{3}}{2}\, b < a < \sqrt{2} b$$

Consider now $\cos(\alpha^*)$. Then, $\cos(\alpha^*) > 0$ if and only if $\pi - n\beta/2 < \pi/2$, which is equivalent to $\beta > \pi/n$. This implies the inequality $f_n(\pi/n) > \sin(\pi/n)$, since f attains its maximum at π/n (see Figure 2 for the case n=3). So, we obtain the following condition for synchronizability: $a < b \sin(\pi/n)$.

Therefore, putting all things together, and considering that

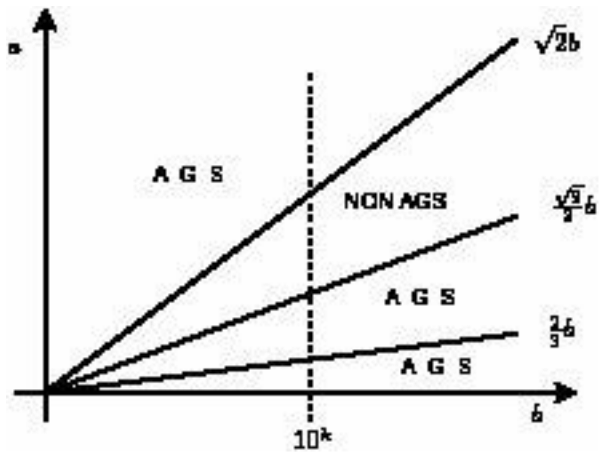
$$sin\left(\frac{\pi}{n}\right) > \frac{2}{n} \quad , \quad sin\left(\frac{\pi}{3}\right) > \frac{\sqrt{3}}{2}$$

we have that the candidate equilibrium is stable if

$$\begin{array}{ll} n > 3 \quad , & a < b \sin(\pi/n) \\ n = 3 \quad , & \frac{\sqrt{3}}{2} b < a < \sqrt{2} b \end{array} \quad (9)$$

Conditions (9) are summarized in Figures 3 and 4.

Φιγυρε 3: Σψνχηρονιζαβιλιτψ χονδιτιον ον α/β.



Φιγυρε 4: (α,β) διαγραμ φορ σψνχηρονιζαβιλιτψ (ν=3).

As an immediate consequence, we have the following result on the complexity of the classification of AGS graphs.

Theorem 4. The classification of the following family of graphs has the same order of complexity than the computation of $\sqrt{2}$:

$$G_{ak} = \{G_3(a, 10^k)\,,\ \ a > 0\ \ ,\qquad k \geq 0\}$$

Proof: from Theorem 3 we know that $G_3(a, 10^k)$ is AGS if

$$\frac{\sqrt{3}}{2} 10^k < a < \sqrt{2} 10^k$$

Suppose we know c_k and d_k, the k-th elements of the decimal expansion of $\sqrt{3}$ and $\sqrt{2}$ respectively. Then, a sufficient condition for synchronizability is $c_k/2 < a < d_k$.

6. Conclusions

In this work we give an introduction to the concept of synchronizing graphs for the Kuramoto model of coupled oscillators, together with a complete presentation of the main properties and characteristics we have studied in the last few years. We include properties that are very useful for the classification of synchronizing graphs. We also present the idea of twin vertices and results that show the complexity of the class of synchronizing graphs. On one hand, the class of synchronizing graphs is a combinatorial one, but on the other hand, its definition is made in terms of the differential equations associated to the Kuramoto model. This makes harder to answer questions like those about the computational complexity of the decision problem of classification. In other contexts, where similar combinatorial definitions have been made, like for planar graphs, an structural problem arises that enable the study of the computational complexity. However, in our case, Theorems 1 and 2 show that every connected graph can be made synchronizing and that every graph with a non bridge edge can be made not synchronizing. This tells us that the class of synchronizing graphs is quite complex from a structural point of view, and not such theorems like the Kuratowski one, can be given. Nevertheless, in this work we present an infinite family of graphs whose classification is computational equivalent to find the n-th digit of the square root of 2. This result seems to tell that there is no combinatorial definition of synchronizing graphs. However, there are many combinatorial problems where the square root of 2 appears, but they are of a enumeration nature. Therefore, if a combinatorial classification theorem exists, it seems to be in terms of amount of structures in the graphs.

As further works, we should find new families whose classification gives new light over the computational complexity of the general classification problem. However, this could be a difficult task, since our present result rest over the well known equilibrium points of the cycles, while we do not have a complete knowledge of the equilibrium of other 2-connected graphs. Indeed, if we identify the twin vertices of the graph $G_n(a,b)$, we obtain a cycle, and that is why the equilibria of $G_n(a,b)$ are quite similar to those of the cycle. But, for other 2-connected synchronizing graphs, named, complete graphs and wheels, the former are invariant over twins operations, while for the latter, we have no idea of its equilibria.

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