LMI-based tracking control for Takagi-Sugeno fuzzy model

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Abstract

This paper deals with the problem of tracking control for Takagi-Sugeno fuzzy model. An LMI (Linear Matrix Inequality) formulation is suggested to make possible the convergence of the state vector of the continuous-time system to a desired one using a new approach, called MultiQuadratic Fuzzy Lyapunov (MQFL). A fourth order unstable nonlinear system is studied to illustrate the efficiency of this formulation and to compare the corresponding simulation results with the obtained stability conditions based on the quadratic Lyapunov function.

Keywords — *Takagi-Sugeno (TS) fuzzy model, stability, Linear Matrix Inequalities (LMI), quadratic Lyapunov function*

1. Introduction

Takagi-Sugeno (TS) fuzzy models have been of great importance in the academic research and the industrial applications [1]-[3]. The main idea is based on the use of sector nonlinearity concept, also known as, the polytopic transformation method, which decomposes a complex nonlinear system into a set of linear subsystems using fuzzy IF-THEN rules. These rules locally represent a linear input-output relation. Then a global nonlinear model is constructed by assembling all the linear subsystems with associated fuzzy membership functions. The stability of these models is, most of the time, studied using the quadratic Lyapunov approach [4]-[10]. The obtained conditions are given in terms of Linear Matrix Inequalities (LMI) and can be efficiently solved by convex programming techniques [11], [12].

The stability conditions based on the use of the quadratic Lyapunov function are conservative as a single common symmetric positive definite matrix verifying all Lyapunov inequalities is required. It is also rejected by certain systems such as the saturated systems, the piecewise linear systems, etc. Some works show the contribution of the polyquadratic and the piecewise quadratic Lyapunov functions, … [13], [14].

Our new approach, called MultiQuadratic Fuzzy Lyapunov functions (MQFL), permits to locally find a symmetric positive definite matrix to each model of the basis. Therefore, the obtained stability conditions for the global fuzzy model are necessary and not sufficient.

In this paper, some basic notions of the TS fuzzy model in the continuous-time case are presented in the first section. Based on the quadratic Lyapunov function, the second section introduces the LMI formulation which allows the convergence of the state vector of the closed-loop TS fuzzy model to the desired vector. In the third section, the same work is redone using the MQFL function. Finally, the case of a fourth order unstable continuous nonlinear system is considered to illustrate the proposed approach.

Notations

- The symbol $(*)$ denotes the transpose elements in the symmetric positions, for example, $X+(*)<0$ stands for $X+X^T<0$ and $\begin{pmatrix} A & B \\ (*) & C \end{pmatrix}<0$ stands for $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}<0$
- $\sum_{i<j} X_i X_j = \sum_{i=1}^{r} \sum_{\substack{j=1 \\ j>i}}^{r} X_i X_j$
- $\sum_{i<j}^{3} a_{ij} = a_{12}+a_{13}+a_{23}$
- BMI : Bilinear Matrix Inequalities
- LMI : Linear Matrix Inequalities

2. Takagi-Sugeno fuzzy model representation

Consider a continuous nonlinear system described by the following form:

$$\begin{cases} \dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right) u(t) \\ y(t) = h\left(x(t)\right) \end{cases} \tag{1}$$

where $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are the nonlinear functions with appropriate dimensions, $x=\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ and $u \in \mathbb{R}^m$ are respectively the state and the input vectors.

Based on the sector nonlinearity concept, the system (1) can be represented with the TS fuzzy model (2). Its *i-th* rule is given by [1]:

<Plant rule i>

If ($z_1(t)$ is F_{1i}) and … and ($z_p(t)$ is F_{pi}) Then $\begin{cases} \dot{x}(t) = A_i\, x(t) + B_i\, u(t) \\ y_i(t) = C_i\, x(t) \end{cases}$ (2)

where r is the number of IF-THEN rules. $(z_1,...,z_p)$ are the premise variables, in general, they are the state variables x_i of the system. F_{ji} $(j = 1, 2, ..., p)$ are the fuzzy sets. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ and $C_i \in \mathbb{R}^{1\times n}$.

The defuzzification process of the TS fuzzy model (2) can be represented as:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(t)\left(A_i\, x(t) + B_i\, u(t)\right) \\ y(t) = \sum_{i=1}^{r} \mu_i(t)\, C_i\, x(t) \end{cases} \tag{3}$$

In the literature, the commonly used validities μ_i are calculated as:

$$\mu_i(t) = w_i(t) / \sum_{i=1}^{r} w_i(t) \tag{4}$$

where the weights w_i given to every fuzzy rule (2) are:

$$w_i(t) = \prod_{j=1}^{p} F_{ji}\left(z_j(t)\right) \tag{5}$$

$\forall \ t$, $F_{ji}(z_j)$ is the grade of membership of z_j in F_{ji}.

Note that:

$$\begin{cases} 0 \le \mu_i(t) \le 1 \ \text{ for } \ i = 1, 2, \ldots, r \\ \sum_{i=1}^{r} \mu_i(t) = 1 \end{cases} \tag{6}$$

In this paper, our objective is to achieve $x(t) - x_d(t) \to 0$ as $t \to +\infty$, where x_d is the desired state vector given by:

$$x_d(t) = \left[x_{1d}(t) \quad x_{2d}(t) \quad \ldots \quad x_{nd}(t)\right]^T \tag{7}$$

We have choosen to apply the geometric method [10] calculating the distance d_i between the state variables x_i of the TS fuzzy model (3) and the desired state variables x_{id} as:

$$d_i(t) = \left| x_i(t) - x_{id}(t) \right| \text{ for } i = 1, 2, \ldots, r \quad (8)$$

The normalized distance d_i^{norm} is given by:

$$d_i^{norm}(t) = d_i(t) / \sum_{j=1}^{r} d_j(t) \quad (9)$$

The validities μ_i are given by:

$$\mu_i(t) = \gamma_i(t) / \sum_{j=1}^{r} \gamma_j(t) \quad (10)$$

with:

$$\gamma_i(t) = \left(1 - d_i^{norm}(t)\right) \prod_{\substack{j=1 \\ j \neq i}}^{r} \left(1 - \exp\left(-\left(\frac{d_j^{norm}(t)}{\sigma} \right)^2 \right) \right) \quad (11)$$

σ represents a variable regulating parameter between 0 and 0.99.

All μ_i defined in (10) verify the same convexity conditions as (6).

Remark 1: If $n \neq r$ then the distance d_i can be calculated between the partial output y_i of the TS fuzzy model (2) and the desired partial output y_{id} as $d_i(t) = \left| y_i(t) - y_{id}(t) \right|$ where $y_d(t) = \sum_{i=1}^{r} \mu_i(t) y_{id}(t)$.

3. Stability conditions based on the quadratic Lyapunov function

Based on the quadratic Lyapunov function, stability analysis of a TS fuzzy model is to seek a common positive definite matrix P verifying $V > 0$ and $\dot{V} < 0$ for all the models of the basis. Its basic results can be found in [4]-[9], [15]-[17].

To achieve $x(t) - x_d(t) \to 0$ as $t \to +\infty$, an elementary fuzzy control law u_i is applied:

<Controller rule i>

If ($z_1(t)$ is F_{1i}) and … and ($z_p(t)$ is F_{pi}) Then $u_i(t) = -K_i\, x(t) + N_i\, x_d(t)$ for $i = 1, 2, ..., r$ (12)

where K_i and N_i are the feedback and the tracking gains, respectively.

The global fuzzy control law u is done by:

$$u(t) = \sum_{i=1}^{r} \mu_i(t)\left(-K_i\, x(t) + N_i\, x_d(t)\right) \tag{13}$$

Then, substituting (13) in (3) leads to:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(t)\, \mu_j(t) \left(\begin{bmatrix} G_{ij} & H_{ij} \end{bmatrix} \begin{bmatrix} x(t) \\ x_d(t) \end{bmatrix} \right) \\ y(t) = \sum_{i=1}^{r} \mu_i(t)\, C_i\, x(t) \end{cases} \tag{14}$$

with $G_{ij} = A_i - B_i K_j$ and $H_{ij} = B_i N_j$.

Note that e is the difference the state vector x of the continuous TS fuzzy model (14) and the desired state vector x_d , supposed constant, as:

$$e(t) = x(t) - x_d(t) \tag{15}$$

The time derivative $\dot{e}(t) = \frac{d\, e(t)}{dt}$ is given by:

$$\dot{e}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(t)\, \mu_j(t) \left(\begin{bmatrix} G_{ij} & Z_{ij} \end{bmatrix} \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} \right) \tag{16}$$

with $Z_{ij} = G_{ij} + H_{ij}$.

The expression of $\dot{e}$ contains the dominant terms (for $i = 1, 2, ..., r$) and the coupled terms (for $1 \le i < j \le r$). It can be rewritten as:

$$\dot{e}(t) = \sum_{i=1}^{r} \left(\mu_i(t)\right)^2 \left(\begin{bmatrix} G_{ii} & Z_{ii} \end{bmatrix} \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} \right) + 2 \sum_{i<j} \mu_i(t)\mu_j(t) \left(\left[\left(\frac{G_{ij} + G_{ji}}{2} \right) \quad \left(\frac{Z_{ij} + Z_{ji}}{2} \right) \right] \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} \right) \tag{17}$$

Theorems 1 and 2 are developed in terms of the BMI and LMI, respectively, to achieve $x(t) - x_d(t) \to 0$ as $t \to +\infty$.

Theorem 1 – The closed-loop continuous system (17) is quadratically globally stable if there exists a common symmetric positive definite matrix P satisfying the following BMI formulation:

$$\begin{pmatrix} PG_{ii}+(*) & PZ_{ii} & 0 \\ (*) & -P & P \\ (*) & (*) & -P \end{pmatrix} < 0 \quad \text{for} \quad i=1,2,\ldots,r \tag{18a}$$

$$\begin{pmatrix} P\left(\dfrac{G_{ij}+G_{ji}}{2}\right)+(*) & P\left(\dfrac{Z_{ij}+Z_{ji}}{2}\right) & 0 \\ (*) & -P & P \\ (*) & (*) & -P \end{pmatrix} \le 0 \quad \text{for} \quad 1 \le i < j \le r \tag{18b}$$

Proof : Based on the quadratic approach of Lyapunov verifying:

$$V\left(e(t)\right) = e^T(t) P\, e(t) > 0 \tag{19a}$$

$$\dot{V}\left(e(t)\right) = \dot{e}^T(t)\, P\, e(t) + e^T(t)\, P\, \dot{e}(t) < 0 \tag{19b}$$

the time derivative $\dot{V}(e)$ is given by:

$$\dot{V}\left(e(t)\right) = \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix}^T \left(\sum_{i=1}^{r} \left(\mu_i(t)\right)^2 \Lambda_{ii} + 2 \sum_{i<j} \mu_i(t)\mu_j(t) \left(\frac{\Lambda_{ij}+\Lambda_{ji}}{2}\right)\right) \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} < 0 \tag{20}$$

with:

$$\Lambda_{ii} \equiv \begin{pmatrix} PG_{ii}+(*) & PZ_{ii} \\ (*) & 0 \end{pmatrix} \quad \text{for} \quad i=1,2,\ldots,r \tag{21a}$$

$$\frac{\Lambda_{ij}+\Lambda_{ji}}{2} \equiv \begin{pmatrix} P\left(\dfrac{G_{ij}+G_{ji}}{2}\right)+(*) & P\left(\dfrac{Z_{ij}+Z_{ji}}{2}\right) \\ (*) & 0 \end{pmatrix} \quad \text{for} \quad 1 \le i < j \le r \tag{21b}$$

As all $0 \le \mu_i \le 1$, for $i=1,2,\ldots,r$, and $\begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} \neq 0_{\mathbb{R}^2}$, (20) is equivalent to:

$$\Lambda_{ii} < 0 \quad \text{for} \quad i=1,2,\ldots,r \tag{22a}$$

$$\frac{\Lambda_{ij}+\Lambda_{ji}}{2} \le 0 \quad \text{for} \quad 1 \le i < j \le r \tag{22b}$$

The matrix inequalities (22a) and (22b) can be rewritten as:

$$\begin{pmatrix} PG_{ii}+(*) & PZ_{ii} \\ (*) & -P \end{pmatrix} + \left(\begin{bmatrix} 0 \\ P \end{bmatrix} P^{-1} \begin{bmatrix} 0 & P \end{bmatrix}\right) < 0 \quad \text{for} \quad i=1,2,\ldots,r \tag{23a}$$

$$\begin{pmatrix} P\left(\dfrac{G_{ij}+G_{ji}}{2}\right)+(*) & P\left(\dfrac{Z_{ij}+Z_{ji}}{2}\right) \\ (*) & -P \end{pmatrix}+\left(\begin{bmatrix}0\\P\end{bmatrix}P^{-1}\begin{bmatrix}0 & P\end{bmatrix}\right)\le 0 \text{ for } 1\le i<j\le r \tag{23b}$$

While applying to (23a) et (23b) the Schur's Lemma , see *Annexe*, we get the results of Theorem 1.

The obtained stability conditions (18a) and (18b) are not linear. However, by making variable changes and some matrix transformations such as pre- and post-multiplying them by the diagonal matrix $\left(\left[P^{-1}, P^{-1}, P^{-1}\right]\right)$, replacing $G_{ij}=A_i-B_i K_j$, $H_{ij}=B_i N_j$, $Z_{ij}=G_{ij}+H_{ij}$, $S=P^{-1}$, $U_i=K_i S$ and $V_i=N_i S$, (18a) and (18b) can become linear, see Theorem 2.

Theorem 2 – The closed-loop continuous system (17) is quadratically globally stable if there exists a common symmetric positive definite matrix $S=P^{-1}$ and vectors U_i and V_i of appropriate dimensions satisfying the following LMI formulation:

$$\begin{pmatrix} A_i S-B_i U_i+(*) & A_i S-B_i U_i+B_i V_i & 0 \\ (*) & -S & S \\ (*) & (*) & -S \end{pmatrix}<0 \text{ for } i=1,2,\ldots,r \tag{24a}$$

$$\begin{pmatrix} \dfrac{\left(A_i+A_j\right)S-B_i U_j-B_j U_i}{2}+(*) & \dfrac{\Theta(i,\, j)}{2} & 0 \\ (*) & -S & S \\ (*) & (*) & -S \end{pmatrix}\le 0 \text{ for } 1\le i<j\le r \tag{24b}$$

with:

$$\Theta(i,\, j)=\left(A_i+A_j\right)S-B_i U_j-B_j U_i+B_i V_j+B_j V_i$$

The gains K_i and N_i are given by:

$$K_i=U_i P \text{ and } N_i=V_i P \text{ for } i=1,2,...,r \tag{25}$$

4. Stability conditions based on the MQFL function

If the polyquadratic Lyapunov function, which is given by:

$$V\left(e(t)\right)=e^T(t)\sum_{i=1}^{r}\mu_i\left(z(t)\right)P_i\, e(t) \tag{26}$$

where μ_i are defined in (10), then, the continuous-time derivate of V is:

$$\dot{V}\left(e(t)\right) = \dot{e}^T(t)\left(\sum_{i=1}^{r}\mu_i(t)P_i\right)e(t) + e^T(t)\left(\sum_{i=1}^{r}\mu_i(t)P_i\right)\dot{e}(t) + e^T(t)\left(\sum_{i=1}^{r}\frac{\partial\mu_i(t)}{\partial t}P_i\right)\dot{e}(t) < 0 \tag{27}$$

Is used, we can get no information about the monotony of the quantity $\frac{\partial\mu_i(t)}{\partial t}$, and therefore the matrix inequalities (27) cannot be given in terms of LMI. To surpass this problem, we suggest to use the MQFL function which locally finds a symmetric positive definite matrix P_i for each model of the basis. The proposed MQFL function is given by:

If ($z_1(t)$ is F_{1i}) and … and ($z_p(t)$ is F_{pi}) Then $V_i\left(e(t)\right) = e^T(t)\, P_i\, e(t)$ for $i = 1, 2, \ldots, r$ (28)

The obtained stability conditions, given in Theorem 3, are necessary and not sufficient, *ie*, if the stability is verified to every model it is not necessarily true for the global model.

Theorem 3 – The closed-loop continuous system (17) is globally stable if there exist r symmetric positive definite matrices $S_q = P_q^{-1}$, for $q = 1, 2, \ldots, r$, and vectors U_{iq} and V_{iq} of appropriate dimensions satisfying the following LMI formulation:

$$\begin{pmatrix} A_i S_q - B_i U_{iq} + (*) & A_i S_q - B_i U_{iq} + B_i V_{iq} & 0 \\ (*) & -S_q & S_q \\ (*) & (*) & -S_q \end{pmatrix} < 0 \text{ for } i, q = 1, 2, ..., r \tag{29a}$$

$$\begin{pmatrix} \frac{A_{ij} S_q - B_i U_{jq} - B_j U_{iq}}{2} + (*) & \frac{\Theta(i,\, j,\, q)}{2} & 0 \\ (*) & -S_q & S_q \\ (*) & (*) & -S_q \end{pmatrix} \leq 0 \text{ for } 1 \leq i < j \leq r \text{ and } q = 1, 2, \ldots, r \tag{29b}$$

with:

$$\Theta(i,\, j,\, q) = A_{ij} S_q - B_i U_{jq} - B_j U_{iq} + B_i V_{jq} + B_j V_{iq}$$

The gains K_i and N_i are given by:

$$K_i = U_{ii} P_i \text{ and } N_i = V_{ii} P_i \text{ for } i = 1, 2, ..., r \tag{30}$$

Proof: Consider the MQFL function defined in (28), its time derivate $\dot{V}_q\left(e(t)\right) = \dfrac{d\,V_q\left(e(t)\right)}{d\,t}$ is given by:

$$\dot{V}_q\left(e(t)\right) = \dot{e}^T(t)\; P_q\; e(t) + e^T(t)\; P_q\; \dot{e}(t) < 0 \tag{31}$$

Then, substituting (17) in (31) leads to:

$$\dot{V}_q\left(e(t)\right) = \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix}^T \left(\sum_{i=1}^{r} \left(\mu_i(t)\right)^2\; \Lambda_{ii}^q + 2 \sum_{i<j} \mu_i(t)\; \mu_j(t) \left(\frac{\Lambda_{ij}^q + \Lambda_{ji}^q}{2} \right) \right) \begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} < 0 \tag{32}$$

with:

$$\Lambda_{ii}^q \equiv \begin{pmatrix} P_q\, G_{ii} + (*) & P_q\, Z_{ii} \\ (*) & 0 \end{pmatrix} \text{ for } i, q = 1, 2, \ldots, r \tag{33a}$$

$$\frac{\Lambda_{ij}^q + \Lambda_{ji}^q}{2} \equiv \begin{pmatrix} P_q \left(\dfrac{G_{ij} + G_{ji}}{2} \right) + (*) & P_q \left(\dfrac{Z_{ij} + Z_{ji}}{2} \right) \\ (*) & 0 \end{pmatrix} \text{ for } 1 \le i < j \le r \text{ and } q = 1, 2, \ldots, r \tag{33b}$$

As all $0 \le \mu_i \le 1$ and $\begin{bmatrix} e(t) \\ x_d(t) \end{bmatrix} \ne 0_{\mathbb{R}^2}$, (32) is equivalent to:

$$\Lambda_{ii}^q < 0 \text{ for } i, q = 1, 2, \ldots, r \tag{34a}$$

$$\frac{\Lambda_{ij}^q + \Lambda_{ji}^q}{2} \le 0 \text{ for } 1 \le i < j \le r \text{ and } q = 1, 2, \ldots, r \tag{34b}$$

The obtained matrix inequalities (34a) and (34b) can be rewriten as follows:

$$\begin{pmatrix} P_q\, G_{ii} + (*) & P_q\, Z_{ii} \\ (*) & -P_q \end{pmatrix} + \left(\begin{bmatrix} 0 \\ P_q \end{bmatrix} P_q^{-1} \begin{bmatrix} 0 & P_q \end{bmatrix} \right) < 0 \text{ for } i, q = 1, 2, \ldots, r \tag{35a}$$

$$\begin{pmatrix} P_q \left(\dfrac{G_{ij} + G_{ji}}{2} \right) + (*) & P_q \left(\dfrac{Z_{ij} + Z_{ji}}{2} \right) \\ (*) & -P_q \end{pmatrix} + \left(\begin{bmatrix} 0 \\ P_q \end{bmatrix} P_q^{-1} \begin{bmatrix} 0 & P_q \end{bmatrix} \right) \le 0 \text{ for } 1 \le i < j \le r \text{ and } q = 1, 2, \ldots, r \tag{35b}$$

While applying to (35a) and (35b) the Schur's Lemma, we get:

$$\begin{pmatrix} P_q\, G_{ii} + (*) & P_q\, Z_{ii} & 0 \\ (*) & -P_q & P_q \\ (*) & (*) & -P_q \end{pmatrix} < 0 \;\text{ for }\; i, q = 1, 2, \ldots, r \tag{36a}$$

$$\begin{pmatrix} P_q \left(\dfrac{G_{ij} + G_{ji}}{2} \right) + (*) & P_q \left(\dfrac{Z_{ij} + Z_{ji}}{2} \right) & 0 \\ (*) & -P_q & P_q \\ (*) & (*) & -P_q \end{pmatrix} \le 0 \;\text{ for }\; 1 \le i < j \le r \;\text{ and }\; q = 1, 2, \ldots, r \tag{36b}$$

While pre- and post-multiplying the obtained matrix inequalities (36a) and (36b) by a diagonal matrix $\left(\left[P_q^{-1}, P_q^{-1}, P_q^{-1}\right]\right)$, replacing $G_{ij} = A_i - B_i\, K_j$, $H_{ij} = B_i\, N_j$, $Z_{ij} = G_{ij} + H_{ij}$ and making the following variable changes $S_q = P_q^{-1}$, $U_{iq} = K_i\, S_q$ and $V_{iq} = N_i\, S_q$, we get the results of Theorem 3.

5. Numerical example

Consider a nonlinear unstable system as follows [16], [17]:

$$\dot{x}_1(t) = x_1(t) + x_2(t) + \sin\left(x_3(t)\right) - 0.1\, x_4(t) + \left(1 + x_1^2(t)\right) u(t) \tag{37a}$$

$$\dot{x}_2(t) = x_1(t) - 2\, x_2(t) \tag{37b}$$

$$\dot{x}_3(t) = x_1(t) + x_1^2(t)\, x_2(t) - 0.3\, x_3(t) \tag{37c}$$

$$\dot{x}_4(t) = \sin\left(x_3(t)\right) - x_4(t) \tag{37d}$$

Assume that $x_1(t) \in [-a, a]$ and $x_3(t) \in [-b, b]$ where a and b are positive numbers. Using the sector nonlinearity concept, the nonlinear continuous system (37) is represented by the global TS fuzzy model (14) with $z_1(t) = x_1(t)$, $z_2(t) = x_3(t)$ and $r = 4$.

The premise membership functions are given by:

$$x_1^2 = F_{11}\, a^2 + F_{13}\, 0 \quad\text{and}\quad \sin(x_3) = F_{31}\, x_3 + F_{32}\, \frac{\sin(b)}{b}\, x_3$$

The fuzzy sets $F_{12} = F_{11}$, $F_{14} = F_{13}$, $F_{33} = F_{31}$ and $F_{34} = F_{32}$ are in $[0, 1]$ and verify $F_{11} + F_{13} = 1$ and $F_{31} + F_{32} = 1$ with:

$$F_{11} = \frac{x_1^2}{a^2} \quad \text{and} \quad F_{31} = \begin{cases} \dfrac{b\ \sin(x_3) - x_3\ \sin(b)}{x_3\left(b - \sin(b)\right)} & \text{for} \quad x_3 \neq 0 \\ 1 & \text{for} \quad x_3 = 0 \end{cases}$$

The consequent matrices are given by:

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & a^2 & -0.3 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & \left(\sin(b)/b\right) & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & a^2 & -0.3 & 0 \\ 0 & 0 & \left(\sin(b)/b\right) & -1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -0.3 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 1 & 1 & \left(\sin(b)/b\right) & -0.1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -0.3 & 0 \\ 0 & 0 & \left(\sin(b)/b\right) & -1 \end{pmatrix}, \quad B_1 = B_2 = \begin{pmatrix} 1+a^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad B_3 = B_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In the simulation, assume that $a = 1.4$, $b = 0.7$ and choose the initial states $\begin{bmatrix} -0.3 & 0.7 & 0.6 & -0.4 \end{bmatrix}^T$.

Remark 2: Each of the fourth models of the basis contains an unstable pole.

While solving the LMI (24a), (24b), (29a) and (29b), we find the following quadratic Lyapunov matrix P and the MQFL matrices P_i, for $i = 1, 2, \ldots, 4$:

$$P = \begin{pmatrix} 0.1152 & 0.0156 & 0.0471 & 0.0021 \\ & 0.1058 & 0.0070 & 0.0071 \\ & & 0.0644 & -0.0170 \\ (*) & & & 0.0376 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0.6141 & 0.1384 & 0.2832 & -0.0088 \\ & 0.4173 & 0.0766 & 0.0183 \\ & & 0.3028 & -0.0705 \\ (*) & & & 0.1358 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.6277 & 0.1424 & 0.2907 & -0.0096 \\ & 0.4179 & 0.0790 & 0.0180 \\ & & 0.3069 & -0.0708 \\ (*) & & & 0.1357 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 0.6762 & 0.2604 & 0.3343 & -0.0128 \\ & 0.5637 & 0.1673 & 0.0134 \\ & & 0.3655 & -0.0805 \\ (*) & & & 0.1516 \end{pmatrix} \qquad P_4 = \begin{pmatrix} 0.6277 & 0.1424 & 0.2907 & -0.0096 \\ & 0.4179 & 0.0790 & 0.0180 \\ & & 0.3069 & -0.0708 \\ (*) & & & 0.1357 \end{pmatrix}$$

The gains K_i and N_i defined in (25) in the quadratic case and (30) in the MQFL case are given in Table 1 and Table 2, respectively.

Table 1. Gains values K_i and N_i in the quadratic case

$K_1 = [1.3805 \quad 0.9864 \quad 1.0174 \quad -0.1366]$	$N_1 = [1.0427 \quad 0.6486 \quad 0.6795 \quad -0.1028]$
$K_2 = [1.3809 \quad 0.9890 \quad 0.9825 \quad -0.1244]$	$N_2 = [1.0431 \quad 0.6511 \quad 0.6716 \quad -0.0906]$
$K_3 = [3.1822 \quad 2.1119 \quad 2.3276 \quad -0.3124]$	$N_3 = [2.1934 \quad 1.1264 \quad 1.7241 \quad -0.2167]$
$K_4 = [3.1875 \quad 2.1278 \quad 2.2815 \quad -0.2953]$	$N_4 = [2.1182 \quad 1.1278 \quad 1.3611 \quad -0.8201]$

Table 2. Gains values K_i and N_i in the MQFL case

$K_1 = [1.9367 \quad 1.1984 \quad 1.3103 \quad -0.1803]$	$N_1 = [1.5988 \quad 0.8606 \quad 0.9724 \quad -0.1465]$
$K_2 = [1.9713 \quad 1.2092 \quad 1.2927 \quad -0.1697]$	$N_2 = [1.6177 \quad 0.8911 \quad 0.9613 \quad -0.1359]$
$K_3 = [4.8771 \quad 3.7605 \quad 3.4949 \quad -0.4634]$	$N_3 = [3.9941 \quad 2.6733 \quad 2.4798 \quad -0.3774]$
$K_4 = [4.4795 \quad 2.6120 \quad 2.9872 \quad -0.4062]$	$N_4 = [3.9325 \quad 1.7612 \quad 2.3167 \quad -0.3401]$

Figure 1 illustrates the trajectories of the state variables x_1, x_2, x_3 and x_4 of the continuous system (37) described by the closed-loop TS fuzzy model (14) and the chosen constant desired state variable x_d given by a step of amplitude 0.5.

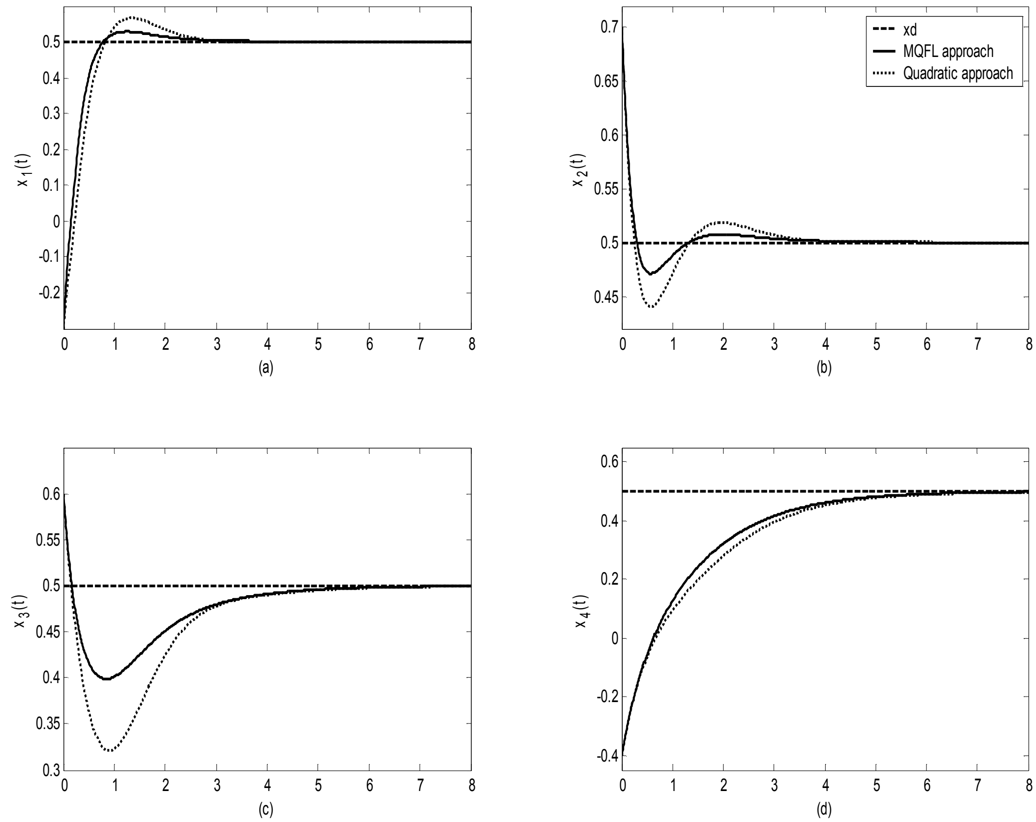

Fig. 1. Trajectories of the state variables x_1, x_2, x_3 and x_4 and the desired state vector x_d

in the cases of the MQFL function (solid line) and the quadratic function (dotted line)

In the Figure 2, are given the trajectories of the global fuzzy control law u in the two cases.

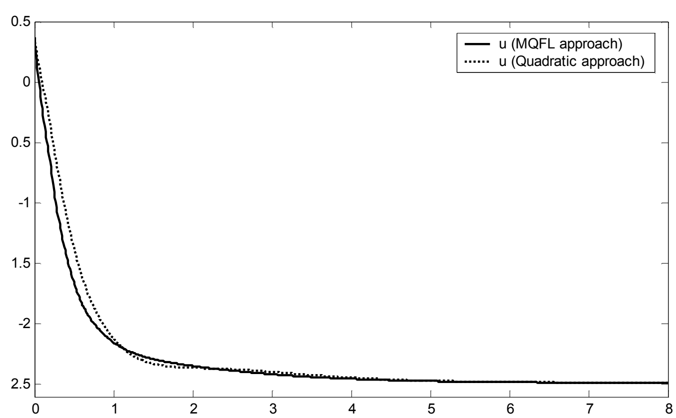

Fig. 2. Trajectories of u

When the MQFL or the quadratic Lyapunov functions are used, the feedback K_i and the tracking N_i gains guarantee the stability of the continuous system (37), initially unstable, described by the fuzzy closed-loop (17). Thus, all the trajectories of the state variables x_i can track the desired state vector x_d very well. The global fuzzy model response is better in the MQFL case than in the quadratic Lyapunov one.

6. Conclusion

In this paper, a contribution in tracking control for the Takagi-Sugeno fuzzy model is considered. Using the MultiQuadratic Fuzzy Lyapunov (MQFL) approach, a new LMI formulation is suggested in order to allow the convergence of the state vector of the system to a desired one. This approach guarantees the local stability of each model of the basis; however, the stability of the global model cannot be concluded. When comparing the MQFL and the quadratic Lyapunov approaches, the results of simulation, in the case of a fourth order continuous nonlinear system, are satisfactory. Our works will focus mainly on seeking sufficient LMI-based stability conditions for the global model using the introduced polyquadratic Lyapunov approach.

Appendix [Lemma of Schur]

Let the matrices M , L and Q of appropriate dimensions where $M = M^T$ and $Q = Q^T > 0$ [12], [13]:

$$M + L^T Q L < 0 \quad \Leftrightarrow \quad \begin{pmatrix} M & L^T \\ L & -Q^{-1} \end{pmatrix} < 0$$

7. Refferences

[1] K. Tanaka and H. O. Wang, “Fuzzy control systems design and analysis. A linear matrix inequality approach”, (John Wiley and Sons, New York, 2001).
[2] T. Takagi and M. Sugeno, “Fuzzy identification of systems and its applications to modeling and control”, IEEE Trans. Syst. Man., Cybern, vol. SMC-15, no. 1, pp. 116-132, Jan. 1985.
[3] M. Vidyasagar, Nonlinear System Analysis, Prentice-Hall, Englewood Cliffs, 1993.
[4] H. O. Wang, K. Tanaka and M. F. Griffin, “An approach to fuzzy control of nonlinear systems: stability and design issues”, IEEE Trans. on Fuzzy Syst., vol. 4, no. 4, pp. 14-23, Feb. 1996.
[5] K. Tanaka and M. Sugeno, “Stability and design of fuzzy control systems”, Fuzzy Set and Systems, vol. 45, no. 2, pp. 135-156, 1992.
[6] C. H. Fang, Y. S. Liu, L. Hong and C. H. Lee, “A new LMI-based approach to relaxed quadratic stabilization of TS fuzzy control systems”, IEEE Trans. on Fuzzy Syst., vol. 14, no. 3, pp. 386-397, June 2006.

[7] K. Tanaka, T. Ikeda and H. O. Wang, “Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs”, IEEE Trans. on Fuzzy Syst., vol. 6, no. 2, pp. 250-265, May 1998.
[8] E. Kim and H. Lee, “New approaches to relaxed quadratic stability condition of fuzzy control systems”, IEEE Trans. on Fuzzy Syst., vol. 8, no. 5, pp. 523-534, Oct. 2000.
[9] J. Joh, Y. H. Chen and R. Langari, “On the stability issues of linear Takagi-Sugeno fuzzy models”, IEEE Trans. on Fuzzy Syst., vol. 6, no. 3, pp. 402-410, Aug. 1998.

[10] Y. Morère, Design and implementation of control law for Takagi-Sugeno fuzzy models, PhD. Thesis, LAMIH, University of Valenciennes and Hainant-Cambrésis, Jan. 2001 (in French).
[11] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishann, Linear matrix inequalities in system and control theory (SIAM, Philadelphia, PA, 1994).
[12] P. Gahinet, A. Nemi rovski, A. J. Laub and M. Chilali, LMI control toolbox, the Math Works (Natick, MA, 1995).
[13] M. Chadli, D. Maquin and J. Ragot, “On the stability analysis of multiple model systems”, ECC, Porto, pp.1894-1899, 2001.
[14] M. Johansson, A. Rantzer and K. Arzen, “Piecewise quadratic stability for affine Sugeno systems, Fuzzy IEEE’98, Anchorage, Alaska, 1998.
[15] C. Ghorbel, A. Abdelkrim and M. Benrejeb, “Observers for continuous nonlinear systems containing unknown parameters and described by Takagi-Sugeno fuzzy model”, Int. Journal of Control and Intelligent Systems, vol. 38, n°. 2, pp. 103-109, April 2010.
[16] C. Ghorbel and A. Abdelkrim, “ Contrôle de poursuite d’un système complexe décrit par l’approche algébrique de Kharitonov : Approche LMI ”, Revue des Sciences et Technologies de l’Automatique (e-sta), vol. 7, n°. 1, pp. 42-47, March 2010.
[17] C. Ghorbel, A. Abdelkrim and M. Benrejeb, “An adaptive fuzzy control of continuous nonlinear systems”, IEEE, 6th International Multi-Conference on Systems, Signals and Devices, SSD’09, March 23-26, 2009, Djerba, Tunisia.

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