

# Quadratic Stabilizability for Polytopic Uncertain Continuous-Time Switched Linear Systems Composed of Two Subsystems

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## Abstract

*In this paper we study quadratic stabilizability via state feedback for continuous-time switched linear systems whose subsystem's matrices are represented as a polytopic linear combination of vertex matrices. Necessary and sufficient conditions for polytopic uncertain continuous-time switched linear systems composed of two subsystems to be quadratically stabilizable via state feedback are proved.*

*Keywords: Switched Linear Systems, Quadratic Stabilizability, Polytopic Uncertain Systems*

## 1 Introduction

The so-called switched system is represented as the family of subsystems with switching rule which concerns with various environmental factors and different controllers. For such switched systems it is important to investigate the stability and stabilizability as fundamental problems. Until now many results on stability and stabilizability problems for various types of switched systems have been studied (e.g., [1],[3]-[6],[10],[12]-[14]). Further, many interesting results for various problems of switched systems have been written in some books (e.g., [7],[8],[11]).

On the other hand, from the practical viewpoint, it is important to investigate switched systems which contain uncertain parameters. Recently, Lin and Antsaklis[9] and Zhai, Lin and Antsaklis[15] investigated the stability and stabilizability problems for both continuous-time and discrete-time uncertain switched linear systems. Especially, the paper[15] investigated the quadratic stabilizability problem via state feedback for polytopic uncertain switched linear system, and sufficient conditions for the switched system which is composed of two subsystems to be quadratically stabilizable were given. However, necessary and sufficient conditions for the switched system to be quadratically stabilizable have not been given even with two subsystems for both continuous-time and discrete-time cases.

In this paper we study quadratic stabilizability via state feedback for continuous-time switched linear systems whose subsystem's matrices are represented as a polytopic linear combination of vertex matrices. In Section 2 necessary and sufficient conditions for polytopic uncertain switched linear systems to be quadratically stabilizable via state-feedback are proved for the case of two subsystems by using the so-called "S-procedure Lemma". The obtained results are extensions of the results of Feron[4] to polytopic uncertain case. Finally, Section 3 gives some concluding remarks.

## 2 Polytopic Uncertain Switched Systems

Consider the following continuous-time switched linear system

$$\Sigma_s : \dot{x}(t) = A_{\sigma(x,t)}x(t), \quad x(0) = x_0$$

where  $x(t) \in \mathbb{R}^n$  is the state vector,  $\sigma(x, t) : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \{1, 2\}$  is a switched rule which depends on the state  $x$  and time  $t$ , and  $\mathbb{R}^+$  is the set of non-negative real numbers.

Then, the above switched system  $\Sigma_s$  is composed of two continuous-time subsystems

$$\Sigma_i : \dot{x}(t) = A_i x(t), \quad i = 1, 2.$$

The architecture of the switched system  $\Sigma_s$  is shown as Figure 1.

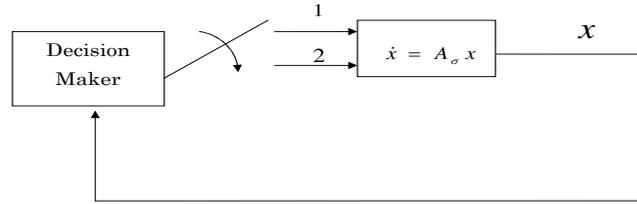


Figure 1. Architecture of switched system  $\Sigma_s$

In this section we consider the case that two subsystems have polytopic uncertainties described as

$$A_i = \sum_{j=1}^{N_i} \mu_{i,j} A_{i,j}, \quad \sum_{j=1}^{N_i} \mu_{i,j} = 1, \quad (i = 1, 2), \quad (1)$$

where  $A_{i,j}$  ( $j = 1, 2, \dots, N_i$ ) are the vertex matrices of the polytopic matrix  $A_i$ ,  $N_i$  is the number of the vertex matrices  $A_{i,j}$  and  $\mu_{i,j}$  ( $j = 1, 2, \dots, N_i$ ) are polytopic uncertain parameters for each  $i = 1, 2$ .

Now, we give the definition of quadratic stabilizability via state feedback for the switched system  $\Sigma_s$ .

**Definition 2.1** The switched linear system  $\Sigma_s$  is said to be quadratically stabilizable via state feedback if there exist a Lyapunov function of the form  $V(x) = x^T P x$ , a positive number  $\epsilon (> 0)$

and a switched rule  $\sigma(x, t)$  such that

$$\frac{d}{dt}V(x) < -\epsilon x^T x$$

for all trajectory  $x$  of the system  $\Sigma_s$ . □

When we investigate the quadratic stabilizability problem of switched systems, the following assumption is given. Because, if there exists a positive definite matrix  $P_i > 0$  ( $i = 1, 2$ ) such that

$$A_{i,j}^T P_i + P_i A_{i,j} < 0 \quad (j = 1, 2, \dots, N_i),$$

then the subsystem  $\Sigma_i$  is quadratically stable. Then, we can always activate the stable subsystem and therefore the problem becomes trivial one.

**Assumption 2.2** Assume that there does not exist positive definite matrices  $P_i > 0$  ( $i = 1, 2$ ) such that

$$A_{i,j}^T P_i + P_i A_{i,j} < 0 \quad (j = 1, 2, \dots, N_i). \quad \square$$

Before giving the main theorem, we need the following lemma which is known as the so-called "S-procedure Lemma".

**Lemma 2.3** [2],[7] Let  $T_0$  and  $T_1 \in \mathfrak{R}^{n \times n}$  be two symmetric matrices. Consider the following two conditions:

$$x^T T_0 x > 0 \text{ whenever } x^T T_1 x \geq 0 \text{ and } x \neq 0 \quad (2)$$

and

$$\exists \tau_1 \geq 0 \text{ such that } T_0 - \tau_1 T_1 > 0. \quad (3)$$

Then, the condition (3) always implies the condition (2). If there is some  $x_0 \neq 0$  such that  $x_0^T T_1 x_0 > 0$ , then condition (2) implies the condition (3). □

The following theorem gives necessary and sufficient conditions for the polytopic uncertain switched linear system to be quadratically stabilizable via state feedback.

**Theorem 2.4** Suppose that Assumption 2.2 holds. The polytopic uncertain switched linear system  $\Sigma_s$  with (1) is quadratically stabilizable via state feedback if and only if there exist  $\lambda_{k,j} \in [0, 1]$  ( $k = 1, \dots, N_1; j = 1, \dots, N_2$ ) such that  $\lambda_{k,j} A_{1k} + (1 - \lambda_{k,j}) A_{2j}$  are simultaneously asymptotically stable, that is, there exists a positive definite matrix  $P > 0$  such that

$$[\lambda_{k,j} A_{1k} + (1 - \lambda_{k,j}) A_{2j}]^T P + P [\lambda_{k,j} A_{1k} + (1 - \lambda_{k,j}) A_{2j}] < 0,$$

equivalently

$$\lambda_{k,j} (A_{1k}^T P + P A_{1k}) + (1 - \lambda_{k,j}) (A_{2j}^T P + P A_{2j}) < 0.$$

**Proof** (Sufficiency) The proof of sufficiency was given by Zhai et al.[15].

(Necessity) Suppose that the polytopic uncertain switched linear system  $\Sigma_s$  with (1) is quadratically stabilizable via state feedback. Then, there exist an  $\epsilon > 0$ ,  $P > 0$  and a switched rule  $\sigma(x, t)$  such that

$$\begin{aligned}
 \frac{d}{dt} V\{x(t)\} &= \frac{d}{dt} \{x^T(t)Px(t)\} \\
 &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
 &= \{A_{\sigma(x,t)}x(t)\}^T Px(t) + x^T(t)P\{A_{\sigma(x,t)}x(t)\} \\
 &= x^T(t)\{A_{\sigma(x,t)}^T P + PA_{\sigma(x,t)}\}x(t) \\
 &< -\epsilon x^T(t)x(t)
 \end{aligned} \tag{4}$$

for all trajectory  $x(t)$  of the system  $\Sigma_s$ . Then, it follows from (4) that we have either

$$x^T(t)(A_1^T P + PA_1)x(t) < -\epsilon x^T(t)x(t) \tag{5}$$

or

$$x^T(t)(A_2^T P + PA_2)x(t) < -\epsilon x^T(t)x(t) \tag{6}$$

for arbitrary state  $x(t)$ .

Since  $A_1$  and  $A_2$  are asymptotically unstable from Assumption 2.2 and (5) or (6), there exists an  $x(t)$  satisfying

$$\begin{cases} x^T(t)(A_1^T P + PA_1)x(t) < -\epsilon x^T(t)x(t) \\ \text{whenever } x^T(t)(A_2^T P + PA_2)x(t) \geq -\epsilon x^T(t)x(t) \end{cases} \tag{7}$$

or

$$\begin{cases} x^T(t)(A_2^T P + PA_2)x(t) < -\epsilon x^T(t)x(t) \\ \text{whenever } x^T(t)(A_1^T P + PA_1)x(t) \geq -\epsilon x^T(t)x(t). \end{cases} \tag{8}$$

Suppose that there exists an  $x(t)$  satisfying (7). Then,

$$\begin{cases} -x^T(t)(A_1^T P + PA_1 + \epsilon I)x(t) > 0 \\ \text{whenever } x^T(t)(A_2^T P + PA_2 + \epsilon I)x(t) \geq 0. \end{cases} \tag{9}$$

If we consider (1) in (9), we have

$$\left\{ \begin{array}{l} -x^T(t) \left\{ \left( \sum_{j=1}^{N_1} \mu_{1,j} A_{1,j} \right)^T P + P \left( \sum_{j=1}^{N_1} \mu_{1,j} A_{1,j} \right) + \epsilon I \right\} x(t) > 0 \\ \text{whenever} \\ x^T(t) \left\{ \left( \sum_{j=1}^{N_2} \mu_{2,j} A_{2,j} \right)^T P + P \left( \sum_{j=1}^{N_2} \mu_{2,j} A_{2,j} \right) + \epsilon I \right\} x(t) \geq 0, \end{array} \right. \quad (10)$$

where  $\sum_{j=1}^{N_i} \mu_{i,j} = 1$  ( $i = 1, 2$ ).

Now, let choose elements  $\mu_{1,k}$  and  $\mu_{2,j}$  ( $k = 1, \dots, N_1$ ;  $j = 1, \dots, N_2$ ) in (10) as

$$\begin{aligned} \mu_{1,k} &= 1 \quad (\text{i.e. } \mu_{1,\ell} = 0, \ell \neq k), \\ \mu_{2,j} &= 1 \quad (\text{i.e. } \mu_{2,p} = 0, p \neq j) \end{aligned}$$

for  $k = 1, \dots, N_1$  and  $j = 1, \dots, N_2$ . Then, we have

$$\left\{ \begin{array}{l} -x^T(t)(A_{1,k}^T P + P A_{1,k} + \epsilon I)x(t) > 0 \\ \text{whenever } x^T(t)(A_{2,j}^T P + P A_{2,j} + \epsilon I)x(t) \geq 0. \end{array} \right. \quad (11)$$

Define the following two symmetric matrices in (11) as

$$\begin{aligned} T_0 &:= -(A_{1,k}^T P + P A_{1,k} + \epsilon I), \\ T_1 &:= A_{2,j}^T P + P A_{2,j} + \epsilon I. \end{aligned}$$

Now, there exists an  $x_0$  such that  $x_0^T T_1 x_0 > 0$ . In fact, if  $x_0^T T_1 x_0 \leq 0$ , that is,

$$x_0^T (A_{2,j}^T P + P A_{2,j}) x_0 \leq -\epsilon x_0^T x_0$$

for all elements  $x_0$  which contradicts Assumption 2.2.

Hence, it follows from Lemma 2.3 that there exists a  $\tau_{k,j} \geq 0$  such that  $T_0 - \tau_{k,j} T_1 > 0$ , that is

$$\begin{aligned} &(A_{1,k}^T P + P A_{1,k} + \epsilon I) + \tau_{k,j} (A_{2,j}^T P + P A_{2,j} + \epsilon I) < 0 \\ \Leftrightarrow &(A_{1,k}^T P + P A_{1,k}) + \tau_{k,j} (A_{2,j}^T P + P A_{2,j}) < -(1 + \tau_{k,j}) \epsilon I \\ \Leftrightarrow &\frac{1}{1 + \tau_{k,j}} (A_{1,k}^T P + P A_{1,k}) + \frac{\tau_{k,j}}{1 + \tau_{k,j}} (A_{2,j}^T P + P A_{2,j}) < -\epsilon I. \end{aligned}$$

Define  $\lambda_{k,j} := \frac{1}{1 + \tau_{k,j}} \in [0, 1]$ . Then, we get

$$[\lambda_{k,j}A_{1,k} + (1 - \lambda_{k,j})A_{2,j}]^T P + P[\lambda_{k,j}A_{2,j} + (1 - \lambda_{k,j})A_{2,j}] < -\epsilon I, \quad (12)$$

for  $k = 1, \dots, N_1$  and  $j = 1, \dots, N_2$ . Similarly, if there exists an  $x(t)$  satisfying (8), we can get the inequality (12) in the same manner. This completes the proof of necessity.  $\square$

If we consider no uncertainties in the switched system  $\Sigma_s$ , Theorem 2.4 is reduced to the following corollary which was given by Feron[4].

**Corollary 2.5** [4] Suppose that  $N = 2$  and Assumption 2.2 is satisfied. Further, suppose that subsystems  $A_i$  ( $i = 1, 2$ ) do not contain uncertain parameters. Then, the switched linear system  $\Sigma_s$  is quadratically stabilizable via state feedback if and only if there exists a  $\lambda \in [0, 1]$  such that  $\lambda A_1 + (1 - \lambda)A_2$  is asymptotically stable, that is, there exist a  $\lambda \in [0, 1]$  and a positive definite matrix  $P > 0$  such that

$$[\lambda A_1 + (1 - \lambda)A_2]^T P + P[\lambda A_1 + (1 - \lambda)A_2] < 0,$$

equivalently

$$\lambda(A_1^T P + P A_1) + (1 - \lambda)(A_2^T P + P A_2) < 0. \quad \square$$

### 3 Concluding Remarks

In this paper, we investigated quadratic stabilizability problem via state feedback for polytopic uncertain continuous-time switched linear system in the sense that the subsystem's matrices are represented as a polytope of vertex matrices. Necessary and sufficient conditions for the polytopic uncertain switched linear systems composed of two subsystems to be quadratically stabilizable via state feedback were proved by using the results of Zhai et al.[15] and S-procedure Lemma (Lemma 2.3). The obtained results are extensions of the results of Feron[4] to a polytopic uncertain case.

As future studies, it remains to investigate necessary and sufficient conditions for the polytopic uncertain continuous-time switched linear system which has more than two subsystems to be quadratically stabilizable via state feedback.

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