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A Trivial Result of Applied Statistical Mechanics in Traffic Flow

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Abstract

We present, in this paper, the model of deterministic traffic flow based on periodic boundary conditions where the vehicles are identical, the variables of the flow are considered to be discrete including velocities and positions. Using the information entropy function as the key parameter, we study the variation of the velocity as function of the density. Given two phases of the flow, we demonstrate that in the congested phase, a critical density corresponds to the maximum of the entropy where all the velocity values of the traffic have the same probability, this density corresponds to half the allowed maximum value of the traffic. In the second part, we conduct a numerical simulations using cellular automaton model based on Nagel Schrekenberg model in the deterministic case, the results of the entropy of the velocity as function of density are presented.

Keywords: *traffic flow, entropy, velocity, critical density, cellular automaton*

1. Introduction

In applied statistical mechanics, the macroscopic properties are derived based on the average values of individual particles (vehicles). Indeed, the different phases of matter can be compared with the different phases of traffic flow, in statistical mechanics; the studied system can generally be in one of the three different phases that are solid, liquid or gaz. We can project these phases on the dynamics of traffic flow if we consider the inter-distance between cars in one dimensional flow. By considering the average length of the car as 7.5 m, at very low densities, if the average distance between the cars is in the order of 1 Km for example, the traffic is in the free flow phase that can assimilated to gas of particles. If the average distance is in the order of 10 m per example, the flow in the beginning of the congested phase, finally if the density equals unity such that the average distance is in order of 1 m, all the cars are stopped and the traffic is in completely jammed state which corresponds to the solid phase.

Therefore, the principles and concepts of statistical mechanics can be successfully applied on traffic engineering. Indeed, several researches demonstrated the results of statistical physics principle on traffic flow where empirical observations were theoretically and numerically explained using methods borrowed from statistical physics. In studies reported in [1,2], the application of computational methods on traffic flow including cellular automaton models and asymmetric simple exclusion process yielded explanations of phase transitions in the traffic induced by open boundary conditions and by on- and off-ramps [1].

Different models of traffic were discussed in [3], where the results of methods such as mean field theory reproduced the fundamental diagrams of flow and density for different velocities. From the perspective of probability distributions functions, principles of equilibrium statistical physics using random matrix theory were applied in [4], to model the headway distribution of headway traffic. For non equilibrium statistical physics of interacting particles, simulations of non deterministic systems [5] yielded exactly similar

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results with theoretical ones in the case of open boundary conditions. Focusing on fundamental concepts of statistical mechanics, we present in this paper the application of the entropy function on deterministic traffic flow with periodic boundary conditions. We treat the problem of vehicle's velocity using information entropy, based on standard model of free flow and congested phases, we derive a critical velocity density where the congested phase contains two secondary phases.

In the second section, we present a brief review of the maximum entropy principle using the expression of Shannon entropy. In the third section, we study the statistics of the velocity of the deterministic model of traffic flow using the information entropy; we derive the expression of the critical velocity density using fundamental diagrams of the traffic. In the fourth section, we present the simulation results obtained using cellular automaton model based on deterministic case, we illustrate the variation of the entropy of velocity as a function of the density and the average velocity.

2. Maximum Entropy Principle

We begin this section by introducing a brief definition of information entropy [6], given a number of microstates Ω of a system with bounded energy, the probability of microstate i is $p(i)$ with normalization condition:

$$\sum_{i=1}^{\Omega} p(i) = 1 \tag{1}$$

The system is defined with entropy function h where the accessible information of the system is minimal when h is maximal; the entropy function is defined by the relation:

$$h = -\sum_{i=1}^{\Omega} p(i)\log(p(i)) \tag{2}$$

The system is in equilibrium when all the microstates have the same probability, to demonstrate this concept; we search for $\max\{h\}$ using Lagrange multiplier method that is defined as:

$$F = h - \lambda\left(\sum_{i=1}^{\Omega} p(i) - 1\right) \tag{3}$$

Where λ is Lagrange multiplier. F is a function of probabilities $p(i)$ therefore, the maximum is reached when the first order derivative equals zero, it is given by:

$$\frac{\partial F}{\partial p(i)} = \frac{\partial}{\partial p(i)}\left(-\sum_{i=1}^{\Omega} p(i)\log(p(i)) - \lambda\left(\sum_{i=1}^{\Omega} p(i) - 1\right)\right) = -\log(p(i)) - 1 - \lambda = 0 \tag{4}$$

The probability is then given by:

$$p(i) = e^{-1-\lambda} \tag{5}$$

Using the normalization condition, we obtain the following result:

$$\sum_{i=1}^{\Omega} p(i) = \sum_{i=1}^{\Omega} e^{-1-\lambda} = \Omega e^{-1-\lambda} = 1 \tag{6}$$

Therefore, the probability of the microstate i is then given as:

$$p(i) = \frac{1}{\Omega} \tag{7}$$

The entropy h is maximal when all the microstates have the same probability and the information about the system is minimal. The maximum value is:

$$Max\{h\} = \log(\Omega) \tag{8}$$

In traffic flow, we apply the concept of maximum entropy to analyze some properties of the system dynamics, using the velocity data. We consider a model of lattice of length L with periodic boundary condition, each site can either be empty or occupied by vehicle and given the density of the system, the phase of the flow is derived using the microscopic variables.

3. Velocity Variation

The system of N cars on L sites are described by the values of position and velocity $\left(x_g, v_g\right)$ for $g=1,...,N$. The global phase space gives the information of the dynamics of the system, based on probabilistic approach. Given a discrete system in both position and velocity, the vehicles cannot run with speeds higher than the allowed maximum velocity $v_{\max}$, the N vehicles can run with integer value between 0 and $v_{\max}$, the number microstates is $\Omega = v_{\max}+1$. To analyze the velocity distribution of cars, we introduce the information entropy of the velocity:

$$h = -\sum_{i=0}^{v_{\max}} p(i)\log(p(i)) \tag{9}$$

Based on the principle of maximum entropy, the maximum value of the entropy is $Max\{h\} = \log(v_{\max}+1)$, where all the velocities have the same probability which is given by:

$$p(i) = \frac{1}{\Omega} = \frac{1}{v_{\max}+1} \tag{10}$$

The first interesting result we derive is that the value of the probabilities that maximize the velocity entropy, equal the critical density [7] of the deterministic flow which is given by:

$$\rho_c = \frac{1}{1+v_{\max}} \tag{11}$$

Next, we study the variation of the entropy with respect the global parameters of the flow. For simplicity, we consider a deterministic model which consists of free flow and congested phases, the flow increases and decreases linearly with critical density $\rho_c = (1+v_{\max})^{-1}$. The equation of the flow as function of the density in the deterministic limit is given by:

$$J = \begin{cases} v_{\max}\rho & if \quad \rho < \rho_c \\ v_{\max}\rho_c \dfrac{(\rho-1)}{(\rho_c-1)} & if \quad \rho \geq \rho_c \end{cases} \tag{12}$$

Given the relation $J = <v>\rho$, the velocity model is $<v> = J/\rho$ where all the vehicles run with their maximum velocity in the free flow phase, while in the congested phase the average velocity decreases non linearly. The relation of the average speed is given by the following equation:

$$<v> = \begin{cases} v_{\max} & if \quad \rho < \rho_c \\ v_{\max}\rho_c \dfrac{(\rho-1)}{\rho(\rho_c-1)} & if \quad \rho \geq \rho_c \end{cases} \tag{13}$$

At critical point, the entropy h is maximal if all the velocity values have the same probability, the corresponding average value in the equilibrium is half of the maximum value:

$$< v >= \frac{1}{N}\sum_{i=1}^{N} v_i = \sum_{j=0}^{v_{\max}} p(j)j = \frac{1}{v_{\max}+1}\sum_{j=0}^{v_{\max}} j = \frac{v_{\max}}{2} \tag{14}$$

Where v_i is the speed of the i[th] vehicle. The variation of the velocity in the system can be characterized by three phases. In the free flow phase I defined by the range $[0, \rho_c]$, the entropy h is zero since the average speed is the maximum speed which means that $p(v_{\max}) = 1$, all the cars are running with the same velocity. In the first congested phase II, other values of the velocity began to appear in the system, as the number of cars N increases, deceleration due to hindrance is mandatory, the entropy increases non linearly until the average velocity of the system is half the maximum value where all the probabilities of the speed have the same value, the entropy reaches its maximum value $Max\{h\} = \log(v_{\max}+1)$ which corresponds to critical density that we call ρ_v. The first congested phase is defined by the range $[\rho_c, \rho_v]$.

In the second congested phase III, the number of cars continues to increase, the jam increases and the gap between the vehicles decreases, which means that larger values of the velocity decreases in the system, the probabilities of the velocities become different, hence the entropy starts to decreases and tends to zero at $\rho = 1$ where we have single value of the speed $< v >= 0$ such that $p(0) = 1$ and $h(\rho = 1) = 0$, the range of this phase is $[\rho_v, 1]$. The critical density of the maximum entropy is obtained by solving the equation $< v >= v_{\max}/2$ which yields to the following expression:

$$\rho_v = \frac{2\rho_c}{(\rho_c+1)} = \frac{2}{(2+v_{\max})} \tag{15}$$

To illustrate the different discussed phases, we present in Figure 1, the configuration using the value $v_{\max} = 4$ where we illustrate the dynamics of the velocity in the congested phase.

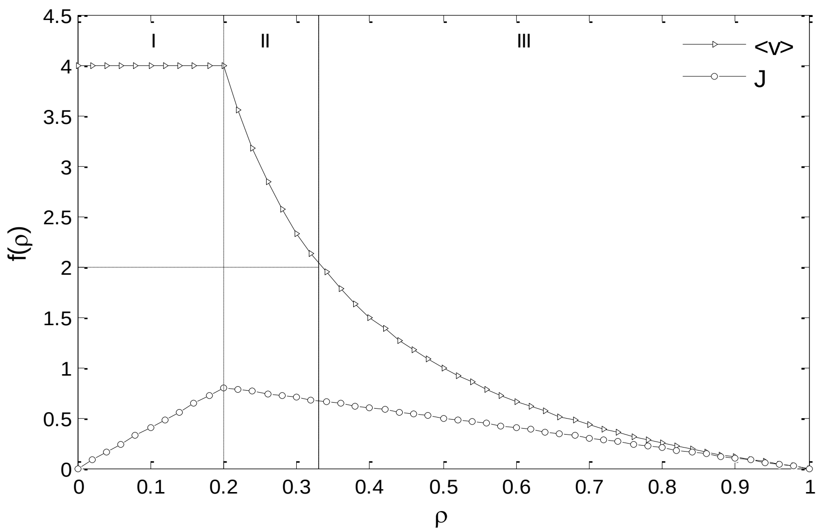

Figure 1. Fundamental Diagrams of Deterministic Traffic Flow with $v_{\max} = 4$, Illustrating Two Parts of the Congested Phase In Terms of Velocity Values

Note that at the critical point ρ_v the probabilities of the velocity values are known, therefore it is possible to compute the velocity statistics. In the next section we present a numerical results of the entropy evaluation using cellular automaton model.

4. Numerical Evaluation

Microscopic approach of traffic flow simulation is based on the parallel update of variables of individual cars including position x and velocity v, The most well known cellular automaton based solution is Nagel Schrekenberg model [8] that is based on four rules of space and time variables. In this model, a road is divided into lattice of L sites where each site can have one of the binary values $\{0,1\}$, each site can be occupied by a car or empty, the acceleration and deceleration of car is based on updating the velocity states where the possible values are $0,1,2,..., v_{\max}$.

Given a number of N cars on road, the system is defined by density $\rho = N / L$, in the case of periodic boundary conditions, the density is constant $\rho(t) = \rho$ and the parameters of the system are N, L, $v_{\max}$ and the probability of braking p. The simulation is based on updating the matrix of position and velocity $X \in N^{L\times T}$ and $V \in N^{L\times T}$ for period T where X represents the space-time diagram of the flow demonstrating the different phases of the flow, and V represents the matrix of velocities of the N cars. Nasch model is based on the following parallel update rules:

- If the velocity of the n^{th} car is less than $v_{\max}$ then $v_n \rightarrow \min\{v_n + 1, v_{\max}\}$.
- Given the distance between two consecutive cars $d_n = x_{n+1} - x_n - 1$, if $v_n > d_n$ then $v_n \rightarrow \min\{v_n, d_n\}$.
- If $v_n > 0$, then the velocity is decreased by unity with probability p, $v_n \rightarrow \max\{v_n - 1, 0\}$.
- The position of the car is updated with new velocity by $x_n \rightarrow x_n + v_n$.

The probabilities of the velocity $p(i)$ can be calculated using the following procedure:

- For $i = 0$ to $v_{\max}$.
- $w = 0$.
- For $x = 1$ to L and for $t = t_0$ to $T + t_0$, if $V(x,t) = i$ then $w \leftarrow w + 1$.
- If $w \neq 0$ then $p(i) = w / (T \times L)$.

We consider a ring with $L = 400$ sites where each vehicle can occupy one site. In the simulation, the measurements are made from the last 500 time steps. For each maximum value $v_{\max} = \{1, 2\}$ and for each value of density, the velocities are initially distributed randomly. We present the results for the deterministic case with braking probability $p = 0$, we conduct a numerical verification of the fundamental diagrams. In Figure 2, we present the flow J and average velocity $< v >$ with respect to the density ρ.

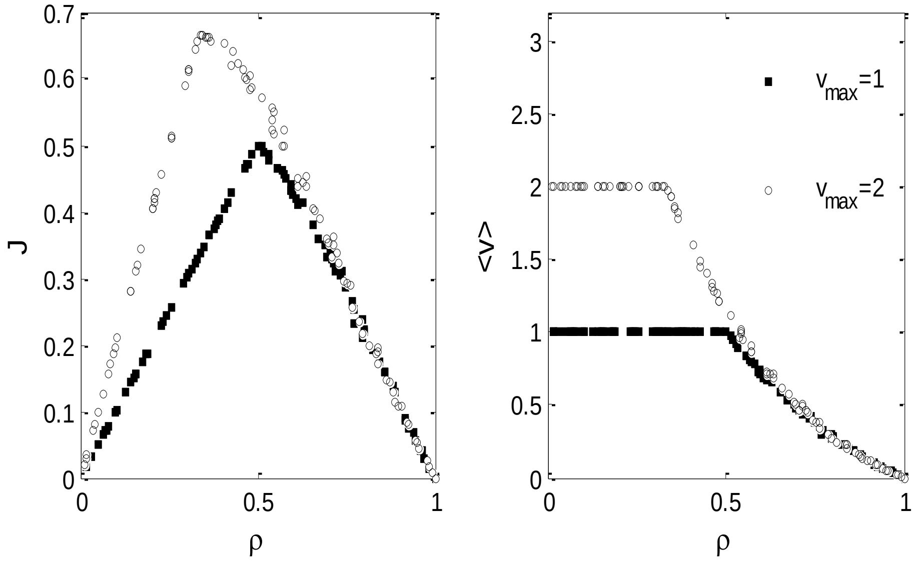

Figure 2. Fundamental Diagrams using Nasch Model with $p = 0$: left, (ρ, J) Diagram. Right, $(\rho, <v>)$ Diagram

In the free flow phase, its obvious that since the average velocity is the maximum velocity $<v> = v_{\max}$, then the unique value of entire flow is $v_{\max}$ where the probability is $p(v_{\max}) = 1$, in other words, as illustrated in Figure 3, the entropy is zero in this phase.

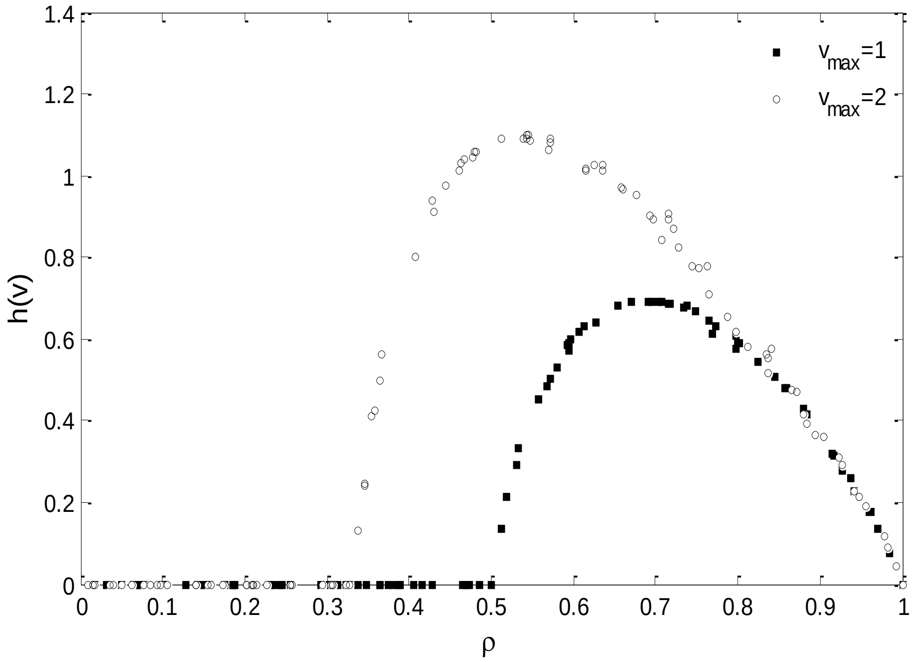

Figure 3. Entropy of Velocity $h(v)$ w.r.t density ρ for Different Values of $v_{\max}$ and $p = 0$

For each case of $v_{\max}$, a sharp transition of the entropy function occurs at critical density ρ_c which separates free flow and congested phases, as the density increases different values of the velocity of the traffic contribute to the increase of the entropy, if the all the velocity values have the same probability, the entropy functions are maximal and correspond to the critical velocity density ρ_v. For the case of $v_{\max} = 1$, the estimated critical density is $\rho_v \approx 0.69$ and for the second case of $v_{\max} = 2$, the estimation is $\rho_v \approx 0.54$. In the second part of the congested phase, the entropy functions decrease as the equilibrium between the different values of the velocity is not maintained. For the jam density $\rho = 1$, we have the probability of the smallest value of speed $p(v = 0) = 1$. As complementary simulation, we present in Figure 4 the variation of the entropy with respect to the average velocity $< v >$. The maximum of the entropy corresponds to half of the maximum velocity $v/2 = [0.5, 1]$.

We consider the result of $v_{\max} = 1$, we can derive the theoretical expression of the entropy as function of the average velocity, the equation is simply the information entropy with binary states, which is in this case two values of the velocity $\{0,1\}$, thus we the entropy function is given as:

$$h = -< v > \log(< v >) - (1 - < v >)\log(1 - < v >) \qquad (16)$$

In Figure 5, we present the variation of the entropy obtained from numerical simulation comparatively to its theoretical function. As perspective of this study, further analysis of the entropy of the velocity can be made using the cellular automaton model based non deterministic case [9] where the braking probability $p \neq 0$. A second possible extension consists of the studying the entropy of velocity using Velocity Dependent-Randomization model [10] where the probability of braking depends on the velocity.

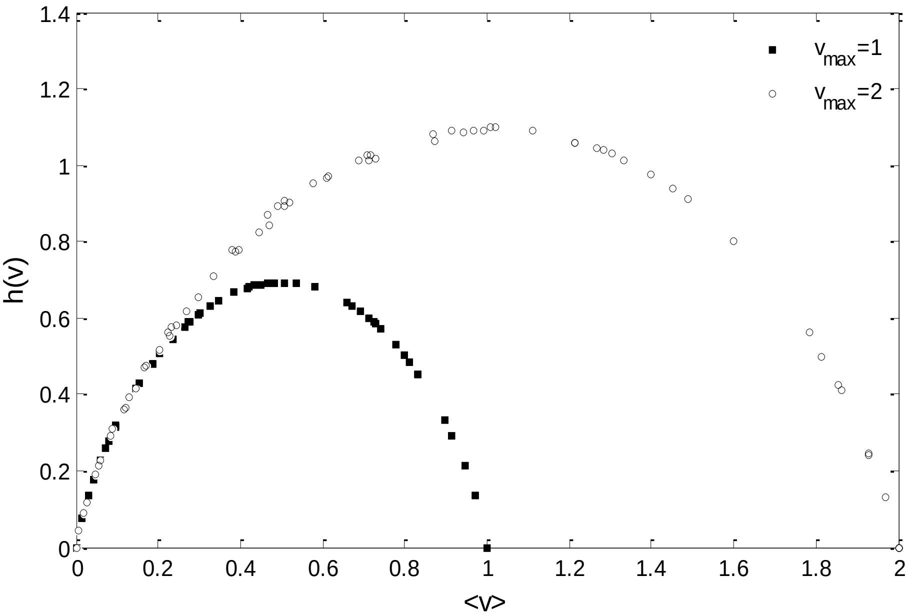

Figure 4. Entropy of Velocity $h(v)$ w.r.t Average Velocity $< v >$ for Different Values of $v_{\max}$ and Braking Probability $p = 0$

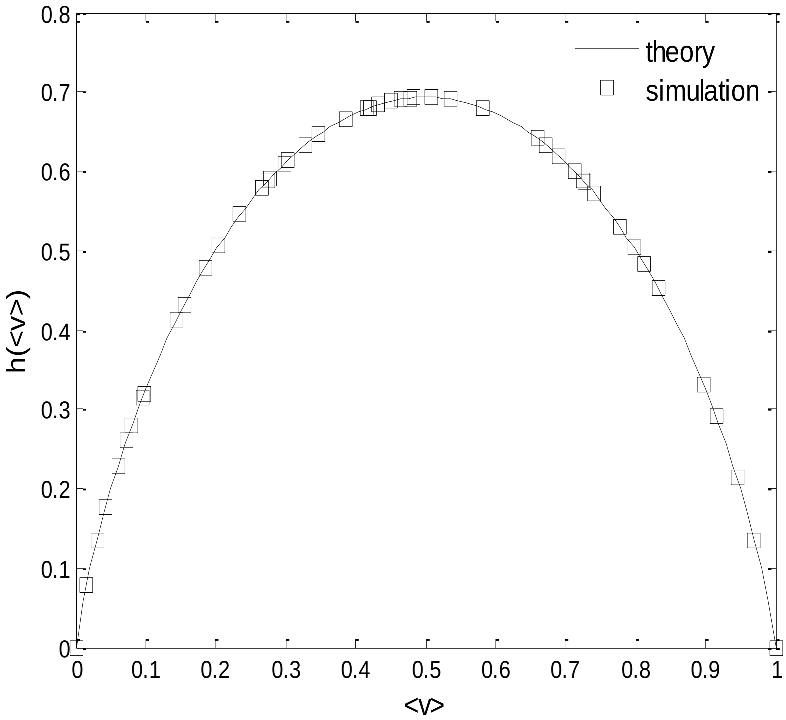

Figure 5. Theoretical and Numerical Variation of the Entropy of Velocity as a Function of Average Velocity, based on Deterministic Nagel Schrekenberg Model for $v_{\max} = 1$

6. Conclusion

In this paper, we have partially reviewed the properties of the deterministic traffic flow based on discrete position and velocity values where the macroscopic variables of the traffic are derived using the averages of the microscopic variables which is the concept of statistical mechanics. Focusing on the information entropy as the principle parameter, we have analyzed the velocity variable, using Shannon entropy, with respect to the fundamental diagram of the average velocity and density. The deterministic model of the traffic is based on periodic boundary conditions where the flow is characterized by free flow and congested phases. The analysis of the entropy of the velocity demonstrated that the maximum value of the entropy function corresponds to a critical velocity density where all the velocity values have the same probability. In the second part, we have conducted a numerical simulation using cellular automaton based on Nagel Schrekenberg model in the deterministic limit, illustrating the variation of the entropy of the velocity as function of the density.

Appendix

The computation of the average velocity <v> given that all the velocity values have the same probability value, $p(j) = (1+v_{\max})^{-1}$ for $j = 0,1,\ldots,v_{\max}$, is described by the equation (14), it is based on the following finite order sum equation:

$$\sum_{i=0}^{N} i = \frac{N(N+1)}{2} \tag{17}$$

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