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H∞ Tracking Control for Nonlinear Fractional-Order Systems Via Output Feedback Design

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Abstract

This paper studies the H_∞ tracking control problem for nonlinear fractional-order systems based on linear matrix inequalities (LMIs) method. Firstly, the stability analysis problem of the closed-loop systems is solved in terms of fractional-order Lyapunov theory. Then, our aim is to design an observer-based output feedback controller, such that though the unavoidable phenomenon of external disturbances and nonlinear items is fully considered, the resulting closed-loop system is asymptotically stable with a prescribed H_∞ performance level. Algorithms based on properly formulated LMIs are established for the existence of an admissible controller and the observer-based output feedback controller parameters. Finally, a numerical example is provided to illustrate the effectiveness of the proposed method.

Keywords: *Fractional-order systems, tracking control, output feedback control, H_∞ performance*

1. Introduction

Fractional calculus has a history almost as long as classical calculus, but it was once thought of as a pure mathematical problem and therefore was not applied to engineering practice widely. Fractional-order (FO) calculus is now widely accepted taking benefit of the fractional operator capacity for modeling various physical phenomena, such as thermal systems [2], batteries [19], neurons [1], with less parameters then integer order systems. That is why efficient stability analysis and controller design methods have been developed to study their properties. Concerning stability, in terms of linear matrix inequality, the stability conditions have been given for continuous-time FO systems of order $0 < \alpha < 1$ in [20] and of order $1 \le \alpha < 2$ in [21]. For FO-LTI systems with interval parameters, the stability and the controllability problems have been addressed for the first time in [16] and [4], respectively. Performances were also considered in [10], where a method to evaluate the H_2 norm of a FO system. Furthermore, concerning the extension of H_∞ theory [24] to FO systems, analysis results on the computation of H_∞ norm for FO system have recently been published in [6]. The H_∞ state feedback and output feedback controllers for FO systems were proposed in [7] and [13], respectively. With the extension of Youla-Kucera parameterization, the H_∞ control problem for LTI FO system was addressed in [14]. Further researches for L_∞ norm was considered in some of them as [6] and [8] as instance.

On the other hand, design of robust tracking controller for uncertain nonlinear systems has aroused a growing interest in the past years. In general, tracking control design is more general and more difficult than the stabilization control design. Various systems including strict-feedback systems [23], networked control systems [15], neural networks [12], stochastic Lagrangian systems [5], *etc.*, the tracking controller design methods have

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been proposed. For tracking control problem with H_∞ performance, reference [22] has discussed the fuzzy control design method for nonlinear systems with a guaranteed H_∞ model reference tracking performance, and a novel neural-network-based robust H_∞ control strategy is proposed for the trajectory following problem of robot manipulators in [25] and so on. But for FO systems so far, very few works exist for the problem of H_∞ tracking controller design.

In this paper, we consider the design of tracking controllers for nonlinear fractional-order systems. The system to be considered is described by a state-space model with nonlinear function and external disturbance. The aim is to obtain a model reference based output feedback tracking control law. This one includes a observer-based output feedback controller and external disturbance attenuation based on an H_∞ criterion. First the stability problem of the closed-loop system with the output feedback tracking controller is investigated. Then a tracking controller design method that yields closed-loop systems with H_∞ performance specification is investigated. Algorithms based on properly formulated LMIs are developed for the above different cases. When these LMIs are feasible, an explicit of a desired observer-based output feedback tracking controller is also given.

Notations: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). The notation M^T represents the transpose of the matrix M. $I_{n\times n}$ denotes the $n \times n$ identity matrix. In symmetric block matrices, "*" is used as an ellipsis for terms induced by symmetry. Matrices, if not explicitly stated, are assumed to have appropriate dimensions.

2. Preliminaries and Problem Formulation

In this section, we first give the definition of fractional-order differentiation. There are several forms of definitions for fractional derivative, such as Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grünald-Letnikov fractional derivative, and so on. In this paper, we adopt the following definitions for fractional derivative, the R-L derivative, on one hand, defined as [17]

$$ {}^{RL}D^\alpha f(t) = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{a+1-n}} d\tau \quad , \tag{1} $$

or the Caputo derivative on the other, defined as [18]

$$ {}^{C}D^\alpha f(t) = \frac{d^a f(t)}{dt^a} = \frac{1}{\Gamma(n-a)} \int_0^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{a+1-n}} \quad , \tag{2} $$

where n is an integer satisfying $n-1 < a < n$, $\Gamma(\bullet)$ is the Gamma function and is defined by the integral

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \; . $$

Property 1: Between the two definitions, R-L and Caputo fractional derivatives, there are the following relations:

$$ {}^{RL}D^\alpha f(t) = {}^{C}D^\alpha f(t) \sum_{k=0}^{n} \frac{t^{k-\alpha}}{\Gamma(k-a+1)} f^{(k)}(0), n-1 < a < n \quad , \tag{3} $$

$$ {}^{C}D^\alpha f(t) = {}^{RL}D^\alpha [f(t) - \sum_{k=0}^{n} f^{(k)}(0) \frac{t^k}{k!}], n-1 < a < n \quad . \tag{4} $$

Let us consider the R-L fractional derivative of order α, then we have

$$^{RL}D^{\alpha}(a) = \frac{at^{-\alpha}}{\Gamma(1-\alpha)}, \tag{5}$$

where α is a positive constant.

Considering the following fractional-order systems with external disturbances and the nonlinear functions:

$$D^{\alpha}x(t) = Ax(t) + Bu(t) + f(x(t)) + D_{\omega}\omega(t), 0 < \alpha < 1 \tag{6}$$

$$y(t) = Cx(t), \tag{7}$$

where α is the time fractional derivative order. $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the control input, $y(t) \in R^s$ is the measured output, $\omega(t) \in R^s$ is a bounded external disturbance and $f(x(t))$ is a nonlinear function. The system matrices A, B, C and D_{ω} are known real constant matrices with appropriate dimensions.

Assumption 1 The nonlinear function $f(x(t))$ is assumed to satisfy the Lipschitz condition:

$$\| f(x(t)) - f(\hat{x}(t)) \| \le \gamma \| x(t) - \hat{x}(t) \|,$$

where γ is the Lipschitz constant.

To derive an output control law, an additional observer is added. This one is based on the nominal model with nonlinear functions:

$$D^{\alpha}\hat{x}(t) = A\hat{x}(t) + Bu(t) + f(\hat{x}(t)) + L[y(t) - \hat{y}(t)], 0 < \alpha < 1 \tag{8}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{9}$$

where $\hat{x}(t) \in R^n$ is the estimated state and L is the observer gain.

To specify the desired trajectory, consider the following reference model:

$$D^{\alpha}x_r(t) = A_r x_r(t) + r(t), \tag{10}$$

where $x_r(t)$ is the reference state, A_r is a specified asymptotically stable matrix, and $r(t)$ is a bounded reference input.

The attenuation of external disturbances is guaranteed considering the H_{∞} performance related to the tracking error $x_r(t) - x(t)$ as follows.

Definition 1 The H_{∞} norm is given by

$$\eta = \sup_{\|\phi(t) \neq 0\|} \frac{\| x_r(t) - x(t) \|}{\| \phi(t) \|}, \tag{11}$$

where $\eta > 0$ is a positive number and $\phi(t) = \begin{bmatrix} \omega(t) \\ r(t) \end{bmatrix}.$

Now, we consider the following output feedback controller for the FO system (6)-(7)

$$u(t) = -K[x_r(t) - \hat{x}(t)], \tag{12}$$

where K is a constant matrix to be determined.

Let us introduce a new state variable

$$\tilde{x}(t) = \left[[x(t) - \hat{x}(t)]^T \;\; [x(t) - x_r(t)]^T \;\; x_r^T(t) \right]^T$$

then, combining the control law (12), the system (6)-(7) and the observer (8)-(9), one obtains, after some easy manipulations, the following closed-loop system:

$$D^{\alpha}\tilde{x}(t)=\tilde{A}\tilde{x}(t)+\tilde{D}_{\omega}\phi(t)+\tilde{I}\xi(x(t),\tilde{x}(t)), \tag{13}$$

where

$$\tilde{A}=\begin{bmatrix} A-LC & 0 & 0 \\ -BK & A+BK & A-A_r \\ 0 & 0 & A_r \end{bmatrix}, \tilde{D}_{\omega}=\begin{bmatrix} D_{\omega} & 0 \\ D_{\omega} & -I \\ 0 & I \end{bmatrix},$$

$$\tilde{I}=\begin{bmatrix} I & 0 & 0 \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}, \xi(x(t),\tilde{x}(t))=\begin{bmatrix} f(x(t)-f(\hat{x}(t)) \\ f(x(t))-f(x_r(t)) \\ f(x_r(t)) \end{bmatrix}.$$

Note that, with the state vector $\tilde{x}(t)$, (11) can be rewritten with $H=\begin{bmatrix}0 & I & 0\end{bmatrix}$:

$$\eta=\sup_{\|\phi(t)\neq 0\|}\frac{\|H\tilde{x}(t)\|}{\|\phi(t)\|}. \tag{14}$$

The objective now is to compute the gains K and L to ensure the asymptotic stability of the closed-loop system (13) guaranteeing the H_{∞} tracking performance (14) for all $\phi(t)$.

3. Main Results

In this section, we give solutions to the H_{∞} stability analysis and the tracking control problems formulated in the previous part. We first give the following results which will be used in the proof of our main results.

Lemma 1 [9] Let $x=0$ be an equilibrium point for the nonautonomous fractional-order system

$$^{RL}D^{\alpha}x(t)={}^{C}D^{\alpha}x(t)=f(t,x), \quad 0<\alpha<1. \tag{15}$$

Assume that there exists a Lyapunov function $V(t,x(t))$ and class- k functions $\beta_i(i=1,2,3)$ satisfying

$$\beta_1(\|x\|)\le V(t,x(t))\le\beta_2(\|x\|), \tag{16}$$

and

$$^{RL}D^{\alpha}(t,x(t))\le-\beta_3(\|x\|), \ \ ^{C}D^{\alpha}(t,x(t))\le-\beta_3(\|x\|). \tag{17}$$

Then the nonlinear fractional-order system (15) is asymptotically stable.

Lemma 2 (Schur Complement) [3] The following linear matrix inequality

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}>0,$$

where $Q=Q^T$, $R=R^T$, is equivalent to

$$\text{(i) } Q>0, R-S^TQ^{-1}S>0,$$

or

$$\text{(ii) } R>0, Q-SR^{-1}S^T>0.$$

Lemma 3 [11] Let a matrix $\Phi<0$, a matrix X with appropriate dimension such that $X^T\Phi X\le 0$, and a scalar ε, the following inequality holds:

$$X^T \Phi X \le -\varepsilon (X^T + X) - \varepsilon^2 \Phi^{-1}.$$

Now, we are in a position to present a solution to the H_∞ tracking control problem for FO system (13).

First, we will present a solution to the H_∞ stability analysis for FO systems (13) with the order $0 < \alpha < 1$.

Theorem 1 The FO system (13) with $0 < \alpha < 1$ is asymptotically stable and its H_∞ norm is bounded by η, if there exists a symmetric positive definite matrix $\tilde{P}$ such that

$$\begin{bmatrix} \Omega & \tilde{P}\tilde{D}_\omega & \tilde{P}\tilde{I} \\ * & -\eta^2 & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{18}$$

where

$$\Omega = \tilde{P}\tilde{A} + \tilde{A}^T \tilde{P} + 2\mu \tilde{P} + \tilde{\gamma}^T \tilde{\gamma} + H^T H \ , \gamma = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & \gamma \\ 0 & 0 & \gamma \end{bmatrix}.$$

Then the stability of the closed-loop system (13) is ensured and the H_∞ tracking control performance (14) is guaranteed with an attenuation level η.

Proof. Consider the following candidate Lyapunov function:

$$V(t) = 2\tilde{x}(t)^T \tilde{P} \tilde{x}(t), \tag{19}$$

where $\tilde{P} = \tilde{P}^T > 0$

Using Property 1, the fractional-order Caputo derivative of (19) is given by

$${}^{C}D^{\alpha} V(t) = {}^{RL}D^{\alpha} \{ 2\tilde{x}(t)^T \tilde{P}\tilde{x}(t) - [\sum_{k=0}^{n} (2\tilde{x}(t)^T \tilde{P}\tilde{x}(t))^{(k)}(0) \frac{t^k}{k!}] k = 0 \},$$

or equivalent to

$$\begin{aligned} {}^{C}D^{\alpha} V(t) = {} & [{}^{RL}D^{\alpha} \tilde{x}(t)^T] \tilde{P}\tilde{x}(t) + \tilde{x}(t)^T \tilde{P} [{}^{RL}D^{\alpha} \tilde{x}(t)] \\ & + 2\sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)} [{}^{RL}D^{k} \tilde{x}(t)] \tilde{P} [{}^{RL}D^{\alpha-k} \tilde{x}(t)] \\ & - 2^{RL} D^{\alpha} [\tilde{x}(0)^T \tilde{P} \tilde{x}(0)]. \end{aligned} \tag{20}$$

Using equation (5), (20) can be modified as follows:

$$\begin{aligned} {}^{C}D^{\alpha} V(t) = {} & [{}^{RL}D^{\alpha} \tilde{x}(t)^T] \tilde{P}\tilde{x}(t) + \tilde{x}(t)^T \tilde{P} [{}^{RL}D^{\alpha} \tilde{x}(t)] \\ & + 2\sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)} [{}^{RL}D^{k} \tilde{x}(t)] \tilde{P} [{}^{RL}D^{\alpha-k} \tilde{x}(t)] \\ & - 2\frac{t^{-\alpha}}{\Gamma(1-\alpha)} [\tilde{x}(0)^T \tilde{P} \tilde{x}(0)]. \end{aligned} \tag{21}$$

For notational convenience of the results formulation, we replace R-L fractional derivative (21) by Caputo fractional derivative, then equation (21) can be rewritten as

$$\begin{aligned} {}^{C}D^{\alpha} V(t) = {} & [{}^{C}D^{\alpha} \tilde{x}(t)^T] \tilde{P}\tilde{x}(t) + \tilde{x}(t)^T \tilde{P} [{}^{C}D^{\alpha} \tilde{x}(t)] \\ & + 2\Upsilon_x(t) - 2\frac{t^{-\alpha}}{\Gamma(1-\alpha)} [\tilde{x}(0)^T \tilde{P} \tilde{x}(0)] \end{aligned} \tag{22}$$

where

$$\Upsilon_x(t)=\sum_{k=1}^{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1+\alpha-k)}[{}^{RL}D^k\tilde{x}(t)\ ]\tilde{P}[{}^{RL}D^{\alpha-k}\tilde{x}(t)]$$

we can consider the following boundedness conditions:

$$\Upsilon_x(t)\le\mu\tilde{x}(t)^T\tilde{P}\tilde{x}(t).$$

Since $2\frac{t^{-\alpha}}{\Gamma(1-\alpha)}[\tilde{x}(0)^T\tilde{P}\tilde{x}(0)]\ge 0$ and substituting (13) into (22), one can easily obtain that

$$\begin{aligned}{}^{C}D^{\alpha}V(t)&\le[\tilde{A}\tilde{x}(t)+\tilde{D}_{\omega}\phi(t)+\tilde{I}\xi(x(t),\tilde{x}(t))]^T\tilde{P}\tilde{x}(t)\\&\quad+\tilde{x}(t)^T\tilde{P}[\tilde{A}\tilde{x}(t)+\tilde{D}_{\omega}\phi(t)+\tilde{I}\xi(x(t),\hat{x}(t))]\\&\quad+2\mu\tilde{x}(t)^T\tilde{P}\tilde{x}(t)\\&=\tilde{x}(t)^T(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\mu\tilde{P})\tilde{x}(t)+\tilde{x}(t)^T\tilde{P}\tilde{D}_{\omega}\phi(t)\\&\quad+\tilde{x}(t)^T\tilde{P}\tilde{I}\xi(x(t),\hat{x}(t))+\phi(t)^T(\tilde{P}\tilde{D}_{\omega})^T\tilde{x}(t)\\&\quad+\xi(x(t),\hat{x}(t))^T(\tilde{P}\tilde{I})^T\tilde{x}(t)\\&\le\tilde{x}(t)^T(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\mu\tilde{P})\tilde{x}(t)+\tilde{x}(t)^T\tilde{P}\tilde{D}_{\omega}\phi(t)\\&\quad+\phi(t)^T(\tilde{P}\tilde{D}_{\omega})^T\tilde{x}(t)+\tilde{x}(t)^T\tilde{P}\tilde{I}(\tilde{P}\tilde{I})^T\tilde{x}(t)\\&\quad+\xi(x(t),\hat{x}(t))^T\xi(x(t),\hat{x}(t))\\&\le\tilde{x}(t)^T(\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\mu\tilde{P}+\tilde{P}\tilde{I}(\tilde{P}\tilde{I})^T+\tilde{\gamma}^T\tilde{\gamma})\tilde{x}(t)\\&\quad+\tilde{x}(t)^T\tilde{P}\tilde{D}_{\omega}\phi(t)+\phi(t)^T(\tilde{P}\tilde{D}_{\omega})^T\tilde{x}(t).\end{aligned}\tag{23}$$

Considering the H_∞ condition in equation (14), we have

$${}^{C}D^{\alpha}V(t)+\tilde{x}(t)^TH^THx(t)-\eta^2\phi(t)^T\phi(t)<0\,. \tag{24}$$

Using inequalities (23), (24) and the fractional direct Lyapunov method in Lemma 1, the sufficient condition can be written as

$$\begin{bmatrix}\tilde{x}(t)\\ \phi(t)\end{bmatrix}^T\begin{bmatrix}\Omega_1 & \tilde{P}\tilde{D}_{\omega}\\ * & -\eta^2\end{bmatrix}\begin{bmatrix}\tilde{x}(t)\\ \phi(t)\end{bmatrix}<0,$$

where

$$\Omega=\tilde{P}\tilde{A}+\tilde{A}^T\tilde{P}+2\mu\tilde{P}+\tilde{P}\tilde{I}(\tilde{P}\tilde{I})^T+\tilde{\gamma}\tilde{\gamma}^T+H^TH$$

By Schur complement Lemma, it is easy to have (18) in Theorem 1. This completes the proof.

Then, the FO output feedback control problem for FO systems (13) with order $0<\alpha<1$ is presented in the following Theorem.

Theorem 2 Consider the FO system (13) with order $0<\alpha<1$ and let $\eta>0$ be a prescribed constant scalar. The H_∞ problem is solvable if there exist nonsingular matrices $N>0$, $P_1>0$, $P_3>0$, Y, Z and scalars μ, γ such that the following conditions are satisfied

$$\begin{bmatrix} -2\beta NI & * & * & * \\ \Lambda_1 & \Lambda_2 & * & * \\ 0 & \Lambda_3 & -\Lambda_4 & * \\ \hat{N} & 0 & 0 & -(\gamma^T\gamma+1)^{-1}I \end{bmatrix} < 0 \tag{25}$$

where

$$\Lambda_1 = \begin{bmatrix} \beta I & 0 & 0 \\ -BY & D_\omega N & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} \Omega_{11} & * & * \\ 0 & \Omega_{22} & * \\ 0 & (A-A_r)^T + \gamma^T\gamma N & \Omega_{33} \end{bmatrix},$$

$$\Lambda_3 = \begin{bmatrix} 0 & -I & P_3 \\ 0 & I & 0 \\ 0 & I & P_3 \end{bmatrix}, \quad \Lambda_4 = \begin{bmatrix} \eta^2 I & * & * \\ 0 & I & * \\ 0 & 0 & I \end{bmatrix}, \quad \hat{N} = \begin{bmatrix} N & 0 & 0 \end{bmatrix}.$$

and

$$\Omega_{11} = \begin{bmatrix} \Psi_{11} & * & * \\ D_\omega^T P_1 & -\eta^2 I & * \\ P_1 & 0 & -I \end{bmatrix},$$

$$\Omega_{22} = NA^T + AN + Y^T B^T + BY + 2\mu N,$$

$$\Omega_{33} = A_r^T P_3 + P_3 A_r + 2\mu P_3 + 2\gamma^T\gamma I,$$

$$\Psi_{11} = P_1(A-LC) + (A-LC)^T P_1 + 2\mu P_1 + \gamma^T\gamma I.$$

Then the asymptotic stability of the closed-loop FO system (13) with order $0<\alpha<1$ is ensured and the H_∞ tracking control performance (14) is guaranteed with an attenuation level η. Furthermore, if a solution exists, the gains K and L are obtained using:

$$K = YN^{-1}, \; L = P_1^{-1}Z. \tag{26}$$

Proof. For a convenient design, let us assume that $\tilde{P} = diag\{P_1, P_2, P_3\}.$ Equation (18) can be rewritten as

$$\begin{bmatrix} \Sigma_{11} & * & * & * \\ \Sigma_{21} & \Sigma_{22} & * & * \\ \Sigma_{31} & \Sigma_{32} & -R & * \\ \Sigma_{41} & \Sigma_{42} & 0 & -I \end{bmatrix} < 0 \tag{27}$$

where

$$\Sigma_{11} = \begin{bmatrix} \Psi_{11} & * \\ -P_2 BK & \Psi_{22} \end{bmatrix}, \quad \Sigma_{21} = \begin{bmatrix} 0 & (A-A_r)^T P_2 + \gamma^T\gamma I \\ D_\omega^T P_1 & D_\omega^T P_2 \end{bmatrix},$$

$$\Sigma_{22} = \begin{bmatrix} A_r^T P_3 + P_3 A_r + 2\mu P_3 + 2\gamma^T\gamma I & * \\ 0 & -\eta^2 I \end{bmatrix}, \quad R = \begin{bmatrix} \eta^2 I & 0 \\ 0 & I \end{bmatrix},$$

$$\Sigma_{31} = \begin{bmatrix} 0 & -P_2 \\ P_1 & 0 \end{bmatrix}, \quad \Sigma_{32} = \begin{bmatrix} P_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_{41} = \begin{bmatrix} 0 & P_2 \\ 0 & P_2 \end{bmatrix}, \quad \Sigma_{42} = \begin{bmatrix} 0 & 0 \\ P_3 & 0 \end{bmatrix},$$

and

$$\Psi_{22} = (A+BK)^T P_2 + P_2(A+BK) + 2\mu P_2 + \gamma^T \gamma I + I.$$

To rearrange the matrices involved in (27), a congruence with the full-rank matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is made. Thus (27) is equivalent to

$$\begin{bmatrix} \tilde{\Sigma}_{11} & * & * & * \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} & * & * \\ 0 & \tilde{\Sigma}_{31} & \tilde{\Sigma}_{32} & * \\ 0 & \Sigma_{41} & \Sigma_{42} & -I \end{bmatrix} < 0 \tag{28}$$

where

$$\tilde{\Sigma}_{11} = \begin{bmatrix} \Psi_{11} & * \\ D_\omega^T P_1 & -\eta^2 I \end{bmatrix}, \quad \tilde{\Sigma}_{21} = \begin{bmatrix} P_1 & 0 \\ -P_2 BK & P_2 D_\omega \end{bmatrix},$$

$$\tilde{\Sigma}_{22} = \begin{bmatrix} -I & * \\ 0 & \Psi_{22} \end{bmatrix}, \quad \tilde{\Sigma}_{31} = \begin{bmatrix} 0 & (A-A_r)^T P_2 + \gamma^T \gamma I \\ 0 & -P_2 \end{bmatrix},$$

$$\tilde{\Sigma}_{32} = \begin{bmatrix} A_r^T P_3 + P_3 A_r + 2\mu P_3 + 2\gamma^T \gamma I & * \\ P_3 & -\eta^2 I \end{bmatrix}.$$

Then, pre- and post- multiplying the LMI (28) by $\mathrm{diag}\{N \; N \; N \; N \; I \; I \; I \; I\}$ and its transpose, respectively, with $N = P_2^{-1}, Y = KN, Z = P_1 L,$ one obtains:

$$\begin{bmatrix} \multicolumn{3}{c}{\tilde{N}\Omega_{11}\tilde{N}} & * & * & * & * & * \\ -BY & D_\omega N & 0 & \tilde{\Omega}_{22} & * & * & * & * \\ 0 & 0 & 0 & (A-A_r)^T + \gamma^T \gamma N & \Omega_{33} & * & * & * \\ 0 & 0 & 0 & -I & P_3 & -\eta^2 I & * & * \\ 0 & 0 & 0 & I & 0 & 0 & -I & * \\ 0 & 0 & 0 & I & P_3 & 0 & 0 & -I \end{bmatrix} < 0 \tag{29}$$

where

$$\tilde{N} = \begin{bmatrix} N & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & N \end{bmatrix},$$

$$\tilde{\Omega}_{22} = NA^T + AN + Y^T B^T + BY + 2\mu N + N(\gamma^T \gamma + I)N.$$

Now, applying Lemma 3 to $\tilde{N}\Omega_{11}\tilde{N}$, it yields

$$\tilde{N}\Omega_{11}\tilde{N} \le -2\beta\tilde{N} - \beta^2\Omega_{11}^{-1}. \tag{30}$$

Then, applying the Schur complement, (30) becomes

$$\tilde{N}\Omega_{11}\tilde{N} \le \begin{bmatrix} -2\beta\tilde{N} & \beta I \\ * & \Psi_{22} \end{bmatrix} < 0. \tag{31}$$

Substituting (31) into (29), we obtain the following inequality

$$\begin{bmatrix} -2\beta NI & * & * \\ \Lambda_1 & \tilde{\Lambda}_2 & * \\ 0 & \Lambda_3 & -\Lambda_4 \end{bmatrix} < 0 \tag{32}$$

where

$$\tilde{\Lambda}_2 = \begin{bmatrix} \Omega_{11} & * & * \\ 0 & \tilde{\Omega}_{22} & * \\ 0 & (A-A_r)^T + \gamma^T\gamma N & \Omega_{33} \end{bmatrix}.$$

Applying Schur complement to (32), the conditions of Theorem 2 hold. Finally, by Theorem 1, the FO systems (13) with order $0 < \alpha < 1$ is asymptotically stable and satisfying the H_∞ performance under the tracking control scheme (8), (9) and (12), with the controller gain in (26). This completes the proof.

4. Numerical Example

In this section, we provide a numerical example to demonstrate the validity of the design method.

Consider the tracking controller design for the fractional-order systems described in (6)-(7) with the following parameters:

$$A = \begin{bmatrix} -5 & 1 \\ 0 & -3 \end{bmatrix}, B = \begin{bmatrix} -2 \\ -3 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D_\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

After trials, the presented simulations are performed with the following tuning:

(1) The reference model was arbitrary chosen with $A_r = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$ Hurwitz to set a desired dynamics to follow.

(2) The values $\gamma = 1, \eta = 10, \mu = 0.1, \lambda = 100$ were arbitrary chosen.

(3) The solutions P_1, P_3, N, Z are computed (if feasible) by solving the LMI condition (25) given in Theorem 2 with classical Matlab LMI Toolbox.

(4) Finally, the gain K and L are obtained from the directive change of variable $K = YN^{-1}, L = P_1^{-1}Z.$

Therefore, for the proposed example, the solutions of Theorem 2 are obtained using the Matlab LMI toolbox and are given by the gains

$$K = \begin{bmatrix} 0.1937 & 0.4621 \end{bmatrix}, L = \begin{bmatrix} 77.6617 \\ 116.4462 \end{bmatrix}$$

and the matrices

$$P_1 = \begin{bmatrix} 52.3438 & 45.8596 \\ 45.8596 & 50.2918 \end{bmatrix}, P_3 = \begin{bmatrix} 5.7204 & -0.7685 \\ -0.7685 & 4.0639 \end{bmatrix},$$

$$N = \begin{bmatrix} 3.4064 & -0.6698 \\ -0.6698 & 2.0166 \end{bmatrix}, Y = \begin{bmatrix} 0.3504 & 0.8201 \end{bmatrix},$$

$$Z = 10^3 \times \begin{bmatrix} 9.4053 \\ 9.4178 \end{bmatrix},$$

and the minimum of H_∞ performance $\eta = 0.9$.

The initial system state $x(0) = \begin{bmatrix} 0.01 & 0 \end{bmatrix}^T$ and observed state $\hat{x}(0) = \begin{bmatrix} 0 & 0.01 \end{bmatrix}^T$ for $r(t) = \begin{bmatrix} 0.01\sin\omega(t) & 0.01\sin\omega(t) \end{bmatrix}^T$. Note that the system is subject external disturbances $\omega(t) = \begin{bmatrix} 0.01\sin\omega(t) & 0.01\sin\omega(t) \end{bmatrix}^T$ and the nonlinear function $f(x(t)) = \sin(\begin{bmatrix} 0.3 & 0.1 \end{bmatrix})x(t).$ The simulation results are shown in Figure 1-Figure 4. Figure 1 depicts the state response of $x_1(t),$ the reference state response of $x_{r1}(t)$ with $\alpha = 0.8$; the state response of $x_2(t)$, the reference state response of $x_{r2}(t)$ are shown in Figure 2 with $\alpha = 0.5$. In Figure 3, the error of $x_1(t) - x_{r1}(t)$ with $\alpha = 0.5, 0.8, 0.9$ are shown; and the error of $x_2(t)$ and $x_{r2}(t)$ are presented in Figure 4 with $\alpha = 0.5, 0.8, 0.9.$

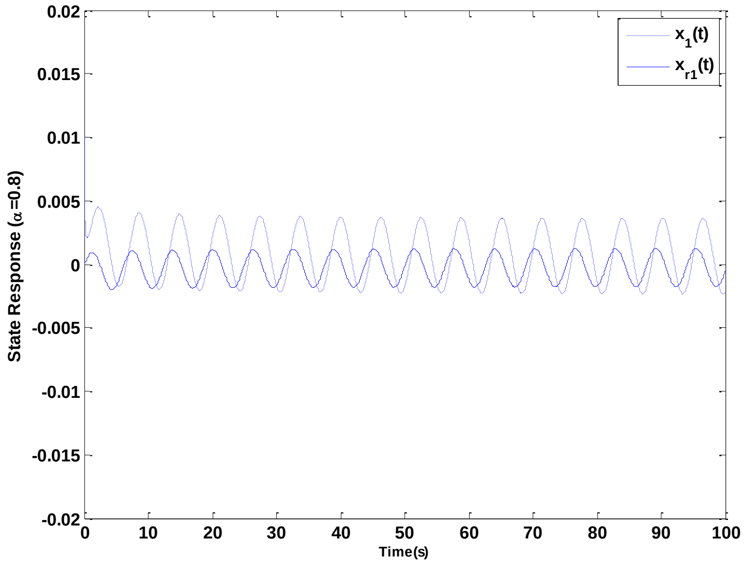

Figure 1. State Response $x_1(t)$ and $x_{r1}(t)$

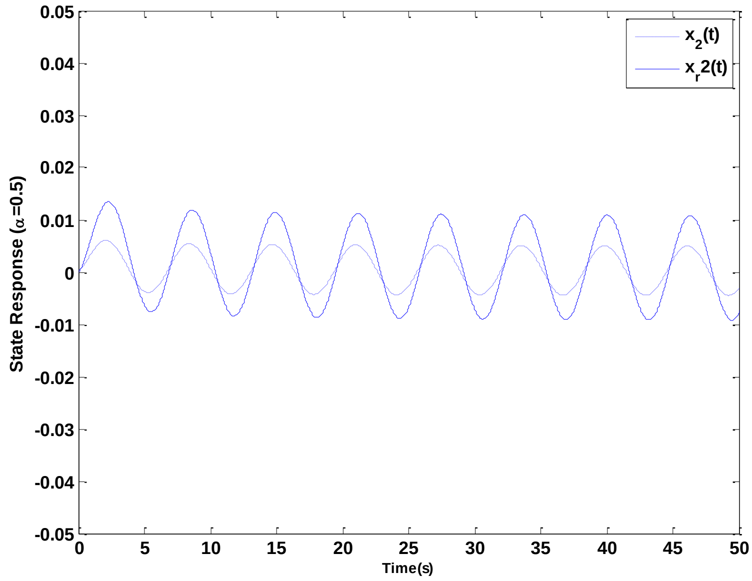

Figure 2. State Response $x_2(t)$ and $x_{r2}(t)$

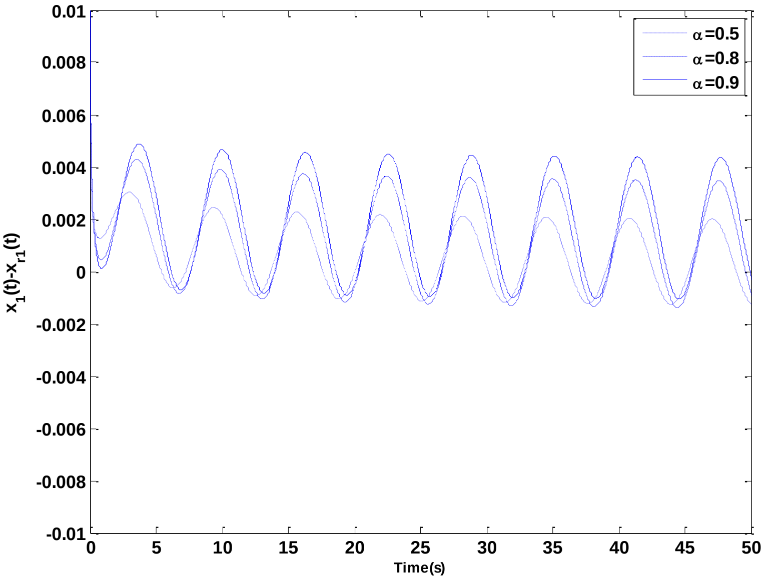

Figure 3. Error of $x_1(t)$ and $x_{r1}(t)$

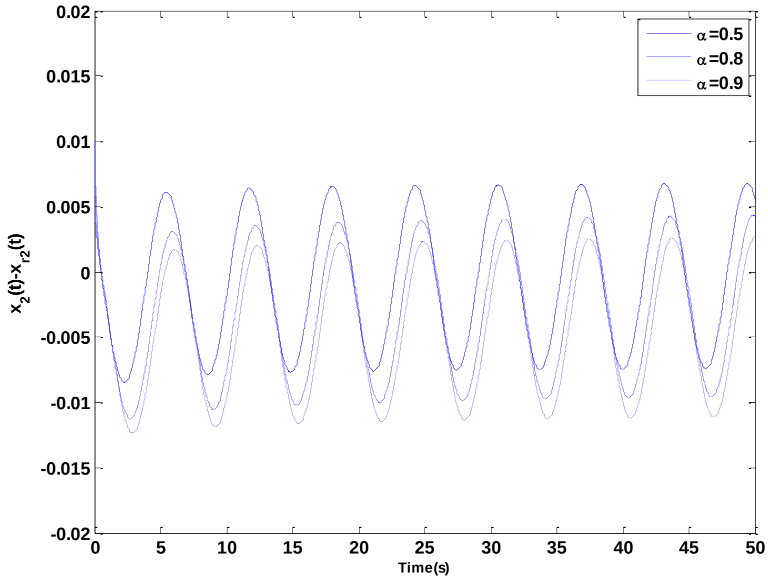

Figure 4. Error of $x_2(t)$ and $x_{r2}(t)$

5. Conclusions

This paper investigates the H_∞ tracking controller design problem for nonlinear fractional-order systems, in which the nonlinear function is assumed to satisfy the Lipschitz condition. By using a Lyapunov functional, some conditions have been established to ensure that the resulting closed-loop system is asymptotically stable with a prescribed H_∞ performance level. An observer-based output feedback controller design method has been proposed for the existence of an admissible controller and corresponding controller parameters. A numerical example has been employed to show the effectiveness of our proposed controller design method.

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