A Novel Parametric Controller Design Method for Hopf Bifurcation System

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Abstract

In this paper, using nonlinear state feedback, a novel general formula is proposed for the problem of Hopf bifurcation system control. This method does not increase dimensions of the system and keeps equilibria of the original system unchanged. The Rössler system is used as an example to illustrative the application of the method. Using Hurwitz criterion, the constraints of the parametric controller are derived. The idea of cylindrical algebraic decomposition (CAD) is employed to compute the constraints to find the parameter ranges of the designed controller, and the controller can be designed to stabilize the system using any feasible control parameters in the ranges. The designed controller not only can stabilize two non-symmetric equilibria but also can keep the Hopf bifurcations at the equilibria of the Rössler system. Computer simulation results are presented to demonstrate the theoretical results.

Keywords: *Parametric controller, Hopf bifurcation control, Cylindrical Algebraic Decomposition (CAD), Rössler system*

1. Introduction

In the Hopf bifurcation control, the basic task is to change the system dynamics in order to achieve the desired properties, such as delaying the onset of an inherent bifurcation, changing the parameter value of an existing bifurcation point, stabilizing a bifurcated branch, modifying the shape or type of a bifurcation chain, optimizing the system performance near a bifurcation point, or a combination of some of these objectives [1-2].

The technique of Hopf bifurcation control has been widely applied to solve physical and engineering problems (including electronic circuits [3-5], power systems [6-7], chemical [8], *etc.*). The control of Hopf bifurcation system has become a prime subject in nonlinear dynamics and has been widely investigated in recent years [9-12]. Many techniques of Hopf bifurcation control have been found. For example, in [10], a small-world network model with the delay feedback is considered and with the controller, one can change the critical value of bifurcation and thus enlarge the stable region. Paper [11] presents and discusses a methodology to include in the power system security assessment and control functions a module to deal with Hopf Bifurcations. All these studies are effective, but the Hopf bifurcation problems and the designed controllers are special. In some research, the design methods of Hopf bifurcation control have been discussed [13-16]. The designed controller better than before, but they are still limited to general applications. This fact motivates our work for the paper.

More notable is the works in [1, 20], using parameterization technology and nonlinear state feedback, a general explicit formula is derived for controlling Hopf bifurcations. The formula, which can be applied to many kind of dynamic systems, and not change the equilibria of the original system. In [1], the method is applied to the Hopf bifurcation control of the Lorenz equation and Rössler system, and the computer simulation results

are presented to confirm the analytical predictions. However, the controller designed by the method has its disadvantages: (i) the form is complex; (ii) the order is high; (iii) the controller parameter space has not been solved completely; (iv) the equilibria are included in the controller. In [20], the method in [1] has been improved, and the Rössler system is used as an illustrative example to present the new method. The form of designed controller is simplified, and the order is lower. The approach applies the graphical technique to find the stability boundaries of equilibrium points in parameter space, and then the parameters have been solved completely.

Although the controller form in [20] is simpler than which in [1], it is still complex and the equilibria are also included in the controller. In [21], the further application has been done, but the paper does not improve the controller. Therefore, in this paper a novel parametric controller design method is proposed for Hopf bifurcation system. The designed controller is composed of the system functions and the polynomial of system states, and keeps the equilibria of origin system unchanged. Not included the equilibrium point of the system, the formula is general and it is simpler than the one in [20]. The stability constraints of the controller parameters at equilibria are obtained using the Hurwitz criterion, and the complete parameters space of the controller can be found using the graphical technique (Cylindrical Algebraic Decomposition, CAD). The Rössler system is used as an example to illustrative the application of the approach.

In this work, the method of controller design for Hopf bifurcation systems and its proof is present in Section II. In Section III, using the Rössler system as an example, the process of the controller design is illustrated and the controller parameter space is solved. In Section IV, the computer simulation results are presented. Finally, the attained conclusions are presented in Section V.

2. A General Function of Hopf Bifurcation System and Parametric Controller Design

Before discussing the design of Hopf bifurcation control system, a general form for the system representation is presented. The proposed system description is not restricted to Hopf bifurcation, and it can be applied to study the control of other singularities or bifurcations such as zero bifurcations. To be more specific, consider the following general nonlinear system:

$$\dot{X} = F(X, \mu) \tag{1}$$

Where the dot denotes differentiation with respect to time t, $X = \left(x_1, x_2, \cdots, x_n\right)^T$ is a n-dimensional state vector, while $\mu \in R$ is a scalar parameter, called bifurcation parameter (Note, generally, one may assume that μ is an m-dimensional vector for $m \geq 1$.). $F(X, \mu) = \left[f_1(X, \mu), f_2(X, \mu), \cdots, f_n(X, \mu)\right]^T$ is a matrix of nonlinear polynomial system that has Hopf bifurcations. Let X^* be an equilibrium of the system, $i.e.$ $F(X^*, \mu) \equiv 0$, for any value of μ.

In this paper, we design a Hopf bifurcation controller for the 3 dimensional system, then consider the following theorem.

Theorem 1: A 3 dimension system can be considered as:

$$\begin{cases} \dot{x}_1 = f_1(X, \mu) \\ \dot{x}_2 = f_2(X, \mu) \\ \dot{x}_3 = f_3(X, \mu) \end{cases} \tag{2}$$

where, $X = [x_1, x_2, x_3]^T$. Let $\det(J) \neq 0$, J is the Jacobian at equilibrium X^* of system (2). The parametric controller can be designed as:

$$u(X,a) = \left[u_1(X,a), u_2(X,a), u_3(X,a) \right]^T ,$$

Generally, only one of $u_1(X,a)$, $u_2(X,a)$ and $u_3(X,a)$ need to be designed, the others will be choose to zero. Here, the control can be written as:

$$u_j(X,a) = g(X,a)f_j(X,a) + \sum_{k=1}^{n} c_{0k} f_k(X,a) \tag{3}$$

where $j = 1,2,3$, $k = 1,2,3$, $k \neq j$, $u_j(X,a)$ is composed of the system vector basis and

$$g(X,a) = c_1 + c_2 x_1 + c_3 x_2 + \cdots + c_{n+1} x_n + c_{n+2} x_1^2 + c_{n+3} x_2^2 + \cdots + c_{2n+1} x_n^2 \tag{4}$$

$\{c_{0k}, c_1, c_2, ..., c_{2n+1} \in R\}$, $n = 3$ are the parameters of the designed controller. If the controller parameters can satisfy the constraints of the Hurwitz criterion, the system (2) can be controlled by selecting the controller parameter values.

Proof:

To be more simplifying, generally, the parametric controller can be designed as:

$$u_2(X,\mu) = g(X) \cdot f_2(X,\mu) + c_{01} f_1(X,\mu) \tag{5}$$

here, $g(X) = (c_1 + c_2 x_1 + c_3 x_2^2 + c_4 x_3^2)$, $\{c_{01}, c_1, c_2, c_3, c_4 \in R\}$, and c_{01}, c_1, c_2, c_3, c_4 are controller parameters.

Add the controller (5), the system can be rewritten as:

$$\begin{cases} \dot{x}_1 = f_1(X,\mu) \\ \dot{x}_2 = f_2(X,\mu) + u_2(X,\mu) \\ \dot{x}_3 = f_3(X,\mu) \end{cases}$$

for this closed loop system, the Jacobian is given as:

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_1}{\partial x_3} \\ j_{21} & j_{22} & j_{23} \\ \dfrac{\partial f_3}{\partial x_1} & \dfrac{\partial f_3}{\partial x_2} & \dfrac{\partial f_3}{\partial x_3} \end{bmatrix}$$

where,

$$j_{21} = \frac{\partial f_2}{\partial x_1} + \frac{\partial f_2}{\partial x_1} g(X) + c_2 f_2 + c_{01} \frac{\partial f_1}{\partial x_1}$$

$$j_{22} = \frac{\partial f_2}{\partial x_2} + \frac{\partial f_2}{\partial x_2} g(X,\mu) + 2c_3 f_2 x_2 + c_{01} \frac{\partial f_1}{\partial x_2}$$

$$j_{23} = \frac{\partial f_2}{\partial x_3} + \frac{\partial f_2}{\partial x_3} g(X,\mu) + 2c_4 f_2 x_3 + c_{01} \frac{\partial f_1}{\partial x_3}$$

Let the system equilibrium is $X_0 = (x_{10}, x_{20}, x_{30})$, then the system characteristic equation at the equilibrium can be given as:

$$G(\lambda) = \det(\lambda I - J) = a_0 \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

where,

$$a_0 = 1$$

$$a_1 = \left[-c_{01} \frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_2}{\partial x_2} g(X,\mu) - \frac{\partial f_3}{\partial x_3} - 2c_3 f_2 x_2 \right]_{X=X_0}$$

$$a_2 = \Bigg[(\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3})g(X,\mu) - c_2\frac{\partial f_1}{\partial x_2}f_2 - 2c_4 x_3\frac{\partial f_3}{\partial x_2}f_2$$

$$+ 2c_3 f_2(\frac{\partial f_3}{\partial x_3} + \frac{\partial f_1}{\partial x_1})x_2 + c_{01}(\frac{\partial f_3}{\partial x_3}\frac{\partial f_1}{\partial x_2} - \frac{\partial f_1}{\partial x_3}\frac{\partial f_3}{\partial x_2}) + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} + \frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}$$

$$- \frac{\partial f_1}{\partial x_3}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} \Bigg]_{X=X_0}$$

$$a_3 = \Bigg[(-\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_3} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_2}$$

$$+ \frac{\partial f_1}{\partial x_3}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_1})g(X,\mu) + (-2c_3\frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_3}x_2 + 2c_4\frac{\partial f_1}{\partial x_1}\frac{\partial f_3}{\partial x_2}x_3 - 2c_4\frac{\partial f_1}{\partial x_2}\frac{\partial f_3}{\partial x_1}x_3 + c_2\frac{\partial f_1}{\partial x_2}\frac{\partial f_3}{\partial x_3}$$

$$+ 2c_3\frac{\partial f_1}{\partial x_3}\frac{\partial f_3}{\partial x_1}x_2 - c_2\frac{\partial f_1}{\partial x_3}\frac{\partial f_3}{\partial x_2})f_2 - \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} + \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_3}$$

$$- \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_3}\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\frac{\partial f_2}{\partial x_1}\frac{\partial f_3}{\partial x_2} + \frac{\partial f_1}{\partial x_3}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_1} \Bigg]_{X=X_0}$$

here, $f_i = f_i(x_1,x_2,x_3)$, $i=1,2,3$.

It can be seen, when $X = X_0$, the values of $\dfrac{\partial f_i}{\partial x_j}$ and f_i are fixed, here $i=1,2,3$, $j=1,2,3$. Therefore, values and symbols of a_1, a_2 and a_3 can be changed by c_{01}, c_1, c_2, c_3, c_4. According to the Hurwitz criterion, the constraints of the controller parameters can be written as the following:

$$\Sigma : \begin{cases} a_1 > 0 \\ a_2 > 0 \\ a_3 > 0 \\ g \geq 0 \end{cases}$$

(6)

here, $g = a_1 a_2 - a_3$.

So, if the constraints Σ can be satisfied, the system (2) can be controlled by selecting the values of controller parameter c_{01}, c_1, c_2, c_3 and c_4.

Remarks: By no means the control formula given in Eq. (3) and Eq. (4) are a unique control law. There are many other feasible controllers.

As shown in the next section, as an example, the Rössler system will be applied to illustrate the effectiveness of the controller designed in Theorem 1.

3. An Example

The Rössler system can exhibit complex dynamics, including equilibria, limit cycles, and chaos. In this section, we will apply it as an example to demonstrate the calculation method of designed controller parameters and to prove the effectiveness of the controller.

3.1. Rössler System Description

The Rössler system can be described as

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases} \tag{7}$$

where $X = [x, y, z]^T$, a, b, c are adjustable parameters, and. Here, a is selected as bifurcation parameter. We suppose that (x_0, y_0, z_0) is an equilibrium of the system (7), then $y_0 = -z_0$, $x_0 = az_0$.

3.2. Controller Design

According to equation (3), the controller of the Rössler system is designed as:

$$u_2(X, a) = g(X) \cdot f_2(X, a) + c_{01} f_1(X, a) ,$$

where

$$g(X) = (c_1 + c_2 x + c_3 y^2 + c_4 z^2)$$

and c_{01}, c_1, c_2, c_3, c_4 are the parameters of the controller. $c_{01} \in R$, $c_1 \in R$, $c_2 \in R$, $c_3 \in R$, $c_4 \in R$. To be more simplifying, here $c_{01} = c_0$, $c_3 = c_4 = 0$ are considered, then $u_2(X, a)$ can be rewritten as:

$$u_2(X, a) = (c_1 + c_2 x)(x + ay) + c_0(-y - z) \tag{8}$$

The order of the designed controller (8) is 2. It is easy to see that the order in this paper is lower than the one in [20], where it is 3.

Adding the controller $u_2(X, a)$ to original system (7), the new system with nonlinear state feedback controls is:

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay + u_2(X, a) \\ \dot{z} = b + z(x - c) \end{cases} \tag{9}$$

The controller (8) does not change the equilibria of the original system (7). In order to calculate the parameters of $u_2(X, a)$, evaluating the Jacobian of system (9) at the equilibrium yields:

$$J_E = \begin{bmatrix} 0 & -1 & -1 \\ 1 + c_1 + c_2 a z_0 & a + a c_1 + c_2 a^2 z_0 - c_0 & -c_0 \\ z_0 & 0 & a z_0 - c \end{bmatrix}$$

which gives the characteristic polynomial :

$$G(\lambda) = \det(\lambda I - J_E) = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

where

$$a_1 = c - c_2 a^2 z_0 - a - a c_1 - a z_0 + c_0 \tag{10}$$

$$a_2 = c_1 + c_2 a z_0 - c_2 a^2 b - c a c_1 + a^2 z_0 + c c_0 + 1 + a^2 z_0 c_1 - a z_0 c_0 + z_0 - ac \tag{11}$$

$$a_3 = 2c_2 ab - 2a z_0 + c + c c_1 - c c_2 a z_0 - 2 z_0 a c_1 \tag{12}$$

Using the controller parameter constraints (6), the stability conditions of the equilibria are

$$a_j > 0 , \quad j = 1, 2, 3 \tag{13}$$

and

$$g \geq 0 \tag{14}$$

The equation (13) must be satisfied, which imply that the static bifurcation does not exist in controlled Rössler system. In equation (14), $g > 0$ is a system stability condition, at $g = 0$ Hopf bifurcation occurs.

3.3. System Constraints Calculation

The Rössler system trajectories change significantly at various system parameters, considering the system parameters as $a = 2/5$, $b = 2$, $c = 4$. While without the controller in the system, the trajectory is in chaotic state at the equilibrium. In the following, the numerical results of the constraints based on the inequalities (13) and (14) are calculated as:

$$a_1 = c_0 + \frac{18}{5} - \frac{2}{5}c_1 - \frac{4}{25}c_2 z_0 - \frac{2}{5}z_0$$

$$a_2 = 4c_0 - \frac{2}{5}c_0 z_0 - \frac{3}{5}c_1 + \frac{4}{25}c_1 z_0 - \frac{8}{25}c_2 + \frac{2}{5}c_2 z_0 + \frac{29}{25}z_0 - \frac{3}{5}$$

$$a_1 = 4c_1 - \frac{4}{5}c_1 z_0 + \frac{8}{5}c_2 - \frac{8}{5}c_2 z_0 - \frac{4}{5}z_0 + 4$$

From system (7), the equilibria are

$$E_e^- = (2 - \frac{4}{5}\sqrt{5}, -(5 - 2\sqrt{5}), 5 - 2\sqrt{5})$$

$$\approx (0.211145618, -0.527864045, 0.527864045)$$

$$E_e^+ = (2 + \frac{4}{5}\sqrt{5}, -(5 + 2\sqrt{5}), 5 + 2\sqrt{5})$$

$$\approx (3.788854382, -9.472135955, 9.472135955)$$

Firstly, to satisfy the constraints in (13) and (14), the Hurwitz criterion parameters at equilibrium E_e^- are

$$a_1^- = c_0 - \frac{2}{5}c_1 - \frac{4}{25}(5 - 2\sqrt{5})c_2 + \frac{8}{5} + \frac{4}{5}\sqrt{5}$$

$$\approx c_0 - 0.4c_1 - 0.08445824720c_2 + 3.38854382$$

$$a_2^- = (2 + \frac{4}{5}\sqrt{5})c_0 + (\frac{1}{5} - \frac{8}{25}\sqrt{5})c_1 + (\frac{42}{25} - \frac{4}{5}\sqrt{5})c_2 + \frac{26}{5} - \frac{58}{25}\sqrt{5}$$

$$\approx 3.788854382c_0 - 0.5155417528c_1 - 0.1088543820c_2 + 0.0123222922$$

$$a_3^- = \frac{8}{5}\sqrt{5}c_1 + (-\frac{32}{5} + \frac{16}{5}\sqrt{5})c_2 + \frac{8}{5}\sqrt{5}$$

$$\approx 3.577708764c_1 + 0.7554175280c_2 + 3.577708764$$

$$g^- = a_1^- a_2^- - a_3^-$$

$$\approx 1.515541753c_0^2 + 0.08248668045c_1^2 + 0.0036774601c_2^2 - 0.8124334022c_0 c_1$$

$$- 0.1715417528c_0 c_2 + 0.03483340224c_1 c_2 + 5.140879227c_0 - 2.131893444c_1$$

$$- 0.45013995857c_2 - 1.414380124$$

Secondly, the Hurwitz criterion parameters at equilibrium E_e^+ are

$$a_1^+ = c_0 - \frac{2}{5}c_1 - \frac{4}{25}(5 + 2\sqrt{5})c_2 + \frac{8}{5} - \frac{4}{5}\sqrt{5}$$

$$\approx c_0 - 0.4c_1 - 1.515541753c_2 - 0.188854382$$

$$a_2^+ = (2 - \frac{4}{5}\sqrt{5})c_0 + (\frac{1}{5} + \frac{8}{25}\sqrt{5})c_1 + (\frac{42}{25} + \frac{4}{5}\sqrt{5})c_2 + \frac{26}{5} + \frac{58}{25}\sqrt{5}$$

$$\approx 0.211145618c_0 + 0.9155417530c_1 + 3.468854382c_2 + 10.38767771$$

$$a_3^+ = -\frac{8}{5}\sqrt{5}c_1 + (-\frac{32}{5} - \frac{16}{5}\sqrt{5})c_2 - \frac{8}{5}\sqrt{5}$$

$$\approx -3.577708764c_1 - 13.55541753c_2 - 3.577708764$$

$$g^+ = a_1^+ a_2^+ - a_3^+$$

$$\approx 0.084458247c_0^2 - 0.1464866804c_1^2 - 2.10287746c_2^2 + 0.332433402c_0c_1$$
$$+ 1.259541753c_0c_2 - 1.110033402c_1c_2 + 4.139120773c_0 - 0.300106556c_1$$
$$- 1.137060041c_2 + 0.646380124$$

Then the constraints of the system from (13) can be given as:

$$\begin{cases} a_j^- > 0 \\ a_j^+ > 0 \end{cases}, \quad j = 1, 2, 3; \tag{15}$$

It can be noted that if $c_2 = 0$; $a_3^- > 0$ and $a_3^+ > 0$ will never occur at same time. Therefore, the condition of $c_2 \neq 0$ must be satisfied for the given design. From $a_3^- > 0$ and $a_3^+ > 0$, following inequalities can be obtained:

$$3.577708764c_1 + 0.7554175280c_2 + 3.577708764 > 0 \quad \text{and}$$
$$-3.577708764c_1 - 13.55541753c_2 - 3.577708764 > 0.$$

These can be rewritten as:

$$\begin{cases} c_1 > -1.0 - 0.2111456180c_2 \\ c_1 < -1.0 - 3.788854383c_2 \end{cases} \tag{16}$$

here, if $c_2 >= 0$, the two inequalities cannot be satisfied at same time. Therefore, $c_2 < 0$ and $c_1 > -1$.

It is noted that, from the Cylindrical Algebraic Decomposition (CAD) theory, the problem of solving the controller parameters c_1, c_2 and c_3 can be transformed into three-dimensional space segmentation problem, and their characteristic polynomials are the dividing planes of the space. The divided sub-space is continuous and the characteristics of all points are consistent; therefore, a feature of any point in the subspace can explain the subspace features. The basic range of c_1 and c_2 have been identified in (16). In order to simplify the process, $c_2 = -2$ is chosen in this case. So the three-dimensional segmentation problem is converted into a two-dimensional space segmentation problem.

Then conditions can be simplified to

$$a_1^- = c_0 - 0.4c_1 + 3.557770876 \tag{17}$$

$$a_2^- = 3.788854382c_0 - 0.5155417528c_1 + 0.2300310562 \tag{18}$$

$$a_3^- = 3.577708764c_1 + 2.066873708 \tag{19}$$

$$g^- = 1.515541753c_0^2 + 0.08248668045c_1^2 - 0.8124334022c_0c_1$$
$$+ 5.483962733c_0 - 2.201560248c_1 - 0.4993903666 \tag{20}$$

$$a_1^+ = c_0 - 0.4c_1 + 2.842229124 \tag{21}$$

$$a_2^+ = 0.211145618c_0 + 0.9155417530c_1 + 3.449968946 \tag{22}$$

$$a_3^+ = -3.577708764c_1 + 23.53312630 \tag{23}$$

$$g^+ = 0.084458247c_0^2 - 0.1464866804c_1^2 + 0.3324334020c_0c_1$$
$$+ 1.620037267c_0 + 1.919960248c_1 - 5.491009634 \tag{24}$$

The stability boundaries of the system for the designed controller can be obtained from (17) ~ (24).

3.4. Calculation of Parameters Using CAD

In general, it is difficult to calculate the parameters fully from the conditions provided in equations (17) to (24). CAD is suitable for parameter range solving problems, and has been applied to the system design [17]. In this paper, CAD is used to solve the range of parameters c_0 and c_1. As $c_2 = -2$ is selected, in the designed controller there are two parameters (c_0 and c_1) that need to be calculated. A set of curves for the conditions (17) to (24) has been drawn on the $c_0 - c_1$ plane in Figure 1. Thereafter, the regions where the Rössler system can be controlled are extracted. The parameters c_0 and c_1 in each region are examined to identify the set of values that satisfy the criteria. Finally, region Q is found, which is surrounded by $a_3^- > 0$, $g^- > 0$, $a_3^+ > 0$, $g^+ > 0$. L_1 is part of the curves of g^- (from point A to B) and L_2 is part of the curves of g^+ (from point B to C).
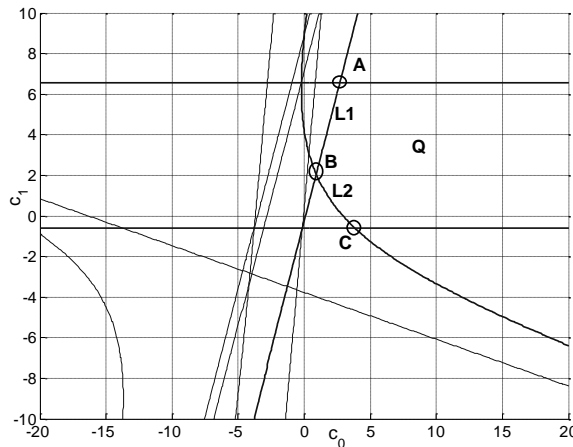

Figure 1. System Constraints Curves in $c_0 - c_1$ Plane

Point A, B, and C are three specific points on $c_0 - c_1$ plane, where A is the crossover point of conditions (20) and (23). From (20) and (23), $c_1 = 6.577708766$, and $c_0 = 2.698244265$ are obtained. B is the crossover point of conditions (20) and (24). According to the two equations, the coordinate of B can be obtained, and these are found as $c_1 = 2.061683145$ and $c_0 = 0.9048934015$. C is the crossover point of condition (19) and (24), and the values of parameters corresponding to this point are $c_1 = -0.5777087640$, $c_0 = 3.801520372$.

4. The Simulations

In this section, the parameters of the controller are calculated based on different cases of the system and the simulations will be done. The states of the selected Rössler system are in chaos state at the equilibria without the application of controller. After the addition of controller, the system states can be controlled within a stable range. There are four cases of dynamic situations near the two equilibria E_e^- and E_e^+ : (i) both the equilibria become stable without Hopf bifurcations; (ii) E_e^+ is stable while E_e^- has a supercritical Hopf bifurcation; (iii) E_e^- is stable while E_e^+ has a supercritical Hopf bifurcation; and (iv) both equilibria have Hopf bifurcations.

In order to ensure that there is no bifurcation in all of the four cases, the conditions $a_i^- > 0$ and $a_i^+ > 0$ must be satisfied. The four cases are decided by g^- and g^+. In fact, if $g^- < 0$ the system cannot be stabilized at E_e^-, and if $g^+ < 0$ the system cannot be stabilized at E_e^+. When $g^- = 0$, the system trajectory at E_e^- is Hopf bifurcation, and limit cycle is stable. Similarly at E_e^+ the trajectory is Hopf bifurcation when $g^+ = 0$.

In the following, the values of c_0 and c_1 will defined for each of the four cases.

Case (i): Both equilibria become stable without Hopf bifurcations. Both equilibria E_e^- and E_e^+ of the controlled system (9) are stable and no bifurcations occur. Then the following inequalities must be satisfied.

$$\begin{cases} g^- > 0 \\ g^+ > 0 \end{cases}$$

For this case c_0 and c_1 should be selected in Q only, avoiding any boundary. Here these are selected as $c_0 = 1.5$, $c_1 = 2$ for simulation.

Case (ii): E_e^+ is stable while E_e^- has a Hopf bifurcation. For this case, following constraints must be satisfied:

$$\begin{cases} g^- = 0 \\ g^+ > 0 \end{cases}$$

Here c_0 and c_1 should be selected on the line L_1 but it should not include A or B. Hence, $c_0 = 1.5$ is selected, and $c_1 = 3.562109985$ is obtained from (20) and (24).

Case (iii): E_e^- is stable while E_e^+ has a Hopf bifurcation. Similar to the Case (ii), the following conditions should be satisfied.

$$\begin{cases} g^- > 0 \\ g^+ = 0 \end{cases}$$

Where c_0 and c_1 should be selected on the line L_2 but it must not include B or C. Hence, $c_0 = 1.5$ is selected, and $c_1 = 1.287395373$.

Case (iv): Both equilibria have Hopf bifurcations. Following condition must be satisfied.

$$\begin{cases} g^- = 0 \\ g^+ = 0 \end{cases}$$

This case only be satisfied at the crossover point of the curves $g^- = 0$ and $g^+ = 0$, where it is the point B in Figure 1, resulting in $c_0 = 0.9048934015$ and $c_1 = 2.061683145$. At this case, the controller can keep the Hopf bifurcations from the equilibria of the Rössler system.

Then the controller parameters have been calculated for all the four cases. Simulations have been performed in order to verify the effectiveness of the controller, and numerical results for the four cases are shown in Figures 2~5, where $a = 2/5$, $b = 2$, $c = 4$.

Figures 2(a) and 2(b) show that both E_e^- and E_e^+ are stable when the conditions given in (41) are satisfied [Case (i)]. Where the selected controller parameter values are $c_0 = 1.5$, $c_1 = 2$, $c_2 = -2$; with $(x_0, y_0, z_0) = (0.3, 0.5, 0.7)$ for (a), and $(x_0, y_0, z_0) = (3.5, -9.5, 11.0)$ for (b).
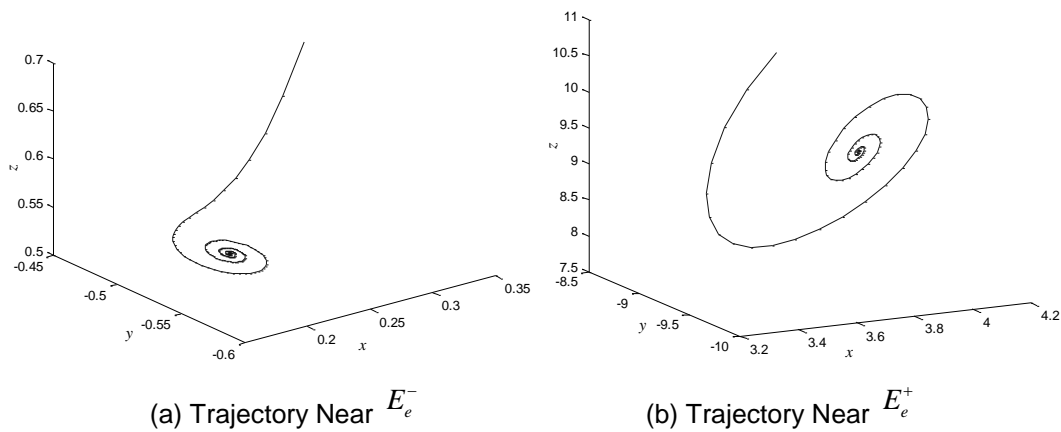

(a) Trajectory Near E_e^- (b) Trajectory Near E_e^+

Figure 2. Trajectories of The Rössler System (12) when both E_e^- and E_e^+ are Stable

(a) Trajectory Near E_e^- (b) Trajectory Near E_e^+
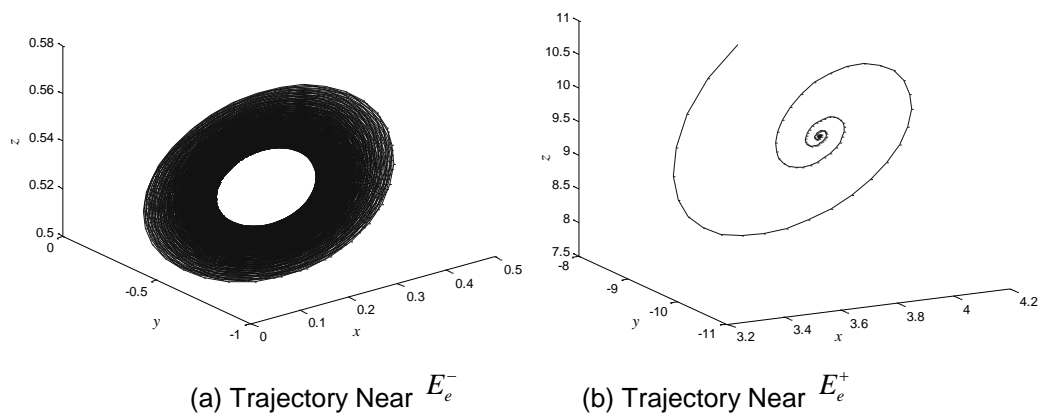
Figure 3. Trajectories of The Rössler System (12) when E_e^- is Hopf Bifurcation and E_e^+ is Stable

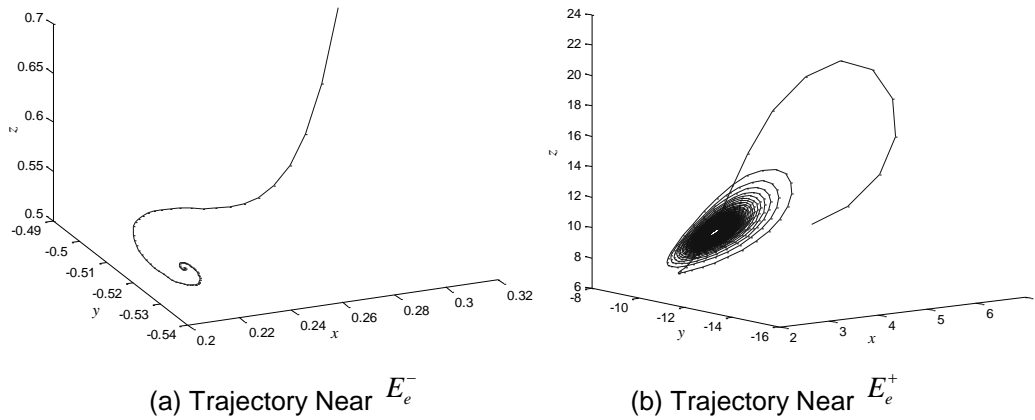(a) Trajectory Near E_e^-

(b) Trajectory Near E_e^+

Figure 4. Trajectories of The Rössler System (12) when E_e^+ is Hopf Bifurcation and E_e^- is Stable

Figure 3(a) shows a stable limit cycle bifurcating from E_e^-, while 3(b) shows that E_e^+ is still stable [Case (ii)]. Where the selected controller parameter values are $c_0 = 1.5$, $c_1 = 3.562109985$, $c_2 = -2$; with $(x_0, y_0, z_0) = (0.4, 0.5, 0.5)$ for (a), and $(x_0, y_0, z_0) = (3.5, -9.5, 11.0)$ for (b).

Figure 4(a) shows E_e^- is stable, and 4(b) shows a stable limit cycle bifurcating from E_e^+ [Case (iii)]. Where the selected controller parameter values are $c_0 = 1.5$, $c_1 = 1.287395373$, $c_2 = -2$; with $(x_0, y_0, z_0) = (0.3, 0.5, 0.7)$ for (a), and $(x_0, y_0, z_0) = (6.0, -9.0, 9.0)$ for (b).
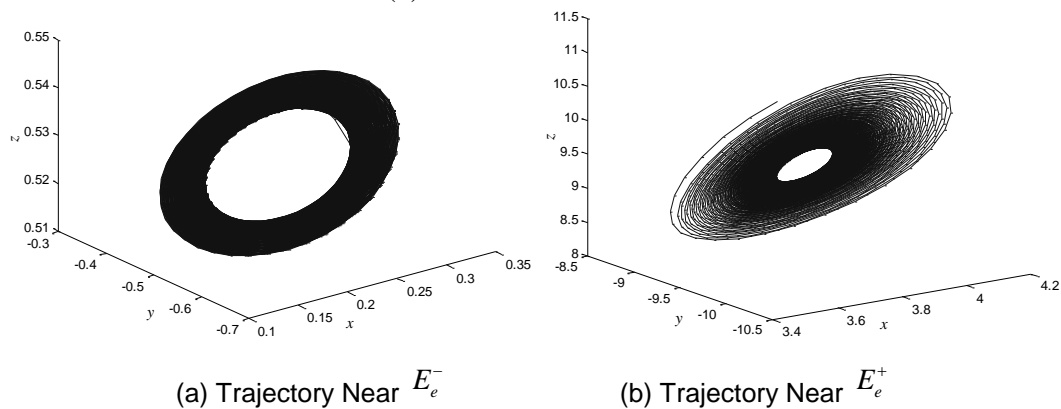

(a) Trajectory Near E_e^-

(b) Trajectory Near E_e^+

Figure 5. Trajectories of The Rössler System (12) when both E_e^- and E_e^+ are Hopf Bifurcations

Figures 5(a) and 5(b) show that both E_e^- and E_e^+ are unstable [Case (iv)], and they are surrounded by stable limit cycles. Where the selected controller parameter values are $c_0 = 0.9048934015$, $c_1 = 2.061683145$, $c_2 = -2$; with $(x_0, y_0, z_0) = (0.3, -0.52, 0.52)$ for (a), and $(x_0, y_0, z_0) = (3.7, -9.5, 10.5)$ for (b).

5. Conclusion

In this paper a Hopf bifurcation controller design method is proposed for nonlinear systems. The designed controller does not affect the equilibrium of the original system. As an example, the Rössler system is used to demonstrate the controller design and parameter solving process of the proposed method. The range of the controller parameters is established using CAD. Furthermore, four set of parameter values are given for different cases of Hopf bifurcation at two equilibria of the system. The analytical predictions have been verified by numerical simulation. Results show that the proposed controller design method be effectively applied to stabilize the system states. When the system parameters are same, compared to the existing works, the form of the designed controller in this paper is simpler and the order lower.

Acknowledgments

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