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An Application of Eigenvalues Statistics on Sensor Array Processing

Youssef Khmou

Sultan Moulay Slimane University, Morocco
khmou.y@gmail.com

Abstract

The properties of the eigenvalues in array processing are important to study where they depend on several functions of physical parameters including the media of propagation, the geometry and the number of sensors in base station. For source number detection and angular interferometry of radiating sources, where the objective is to detect the number of radiating sources during the period of observation and to implement an angular beam scan in order to spatially study the coherence effect of wave fields, eigenvalues analysis is an important step that enables to separate the sources of interest and to isolate the interfering sources that can be other radiating sources or simply the noise field that is generated in the circuits of the antennas, or as result of diffuse reflections in the medium of propagation. In this paper, we study the statistics of the eigenvalues of the spectral matrix obtained from signal vectors, their computations are based on the trace. We discuss the eigenvalues statistics of Hermitian matrices, next, we apply first and second order statistics to estimate the threshold between signal and noise eigenvalues, some numerical results are presented using large array of sensors and closely sources.

Keywords*: Eigenvalues statistics, array processing, threshold, trace, Hermitian matrix*

1. Introduction

In the field of electronic engineering, array processing is an ongoing field of research divided into many parts including, but not limited to, channel estimation [1], modulation [2] and demodulation of transmitted signals, narrow and broadband [3] transmission schemes and spatial interferences study, this last case is focused on spatial and temporal coherence effects of propagating waves from the sources towards the intercepting array of sensors.

Special processing of induced voltages in receiving array of antennas is performed to estimate the coherence effects by detecting the number of radiating sources which is known as model order estimation [4], these steps are based on eigenvalues of spectral matrix computed from the array of signals on the base station. In the other hand, the bearing estimation and waveforms retrieval is performed using high resolution spectral and algebraic methods [5,6], the term high resolution refers to the case where it is possible to correctly estimation the angles of arrival of radiating sources where the angular differences between different angles of incidence are less than the Rayleigh angular resolution limit of the array, this metric is also known as Half Power Beam Width.

Bearing estimation relies on the spectral matrix and angular beam scan which permit the estimation of the properties of sources and those of propagation medium [7]. Among the high resolution techniques that rely on eigenvalues properties is the ESPRIT (Estimation of Signal Parameters via Rotational Invariance) algebraic method [8,9], another method based on angular spectrum and peak detection procedure is the EG [10]

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(Ermolaev and Gershman) where the constructed operator is based on chosen threshold value of the eigenvalues of spectral matrix. In fact, the signal vector at base station can be decomposed into two complementary parts, signal and noise subspaces [5,6,7] that are orthogonal to each other, each part corresponds to a set of eigenvalues of spectral matrix, the EG operator [10] is computed based on threshold value between signal and noise eigenvalues. Theoretically, the signal eigenvalues have higher magnitudes and noise eigenvalue is degenerate q times where q in the number of sensors of the array minus the number of present sources, the threshold is strictly superior to the noise eigenvalue and strictly inferior to the smallest signal eigenvalue.

An alternative approach to estimate the threshold of the two sets of eigenvalues was proposed using QR or LU decomposition of spectral matrix [11]. Another methodology of noise subspace approximation was proposed using the threshold obtained from first and second order statistics of eigenvalues [12], the threshold used consists of selecting the value that attenuates the function used to approximate the noise subspace, the resulted operator has almost binary eigevalues such as one and null eigenvalues correspond to noise and signal subspaces respectively.

In this paper, we focus on large array of sensors, we study a possible alternative solution of the threshold selection using first and second order statistics of eigenvalues [13], without performing the eigendecomposition of spectral matrix, this proposition is supported by some numerical simulations. In the second part, we describe the statistics of eigenvalues of Hermitian matrices using the trace, next we present the signal model of array processing and the theorem we are based upon to estimate the threshold of eigenvalues. In the third part, we perform a comparative analysis between the proposed and existing models using large array and few radiating sources.

2. Eigenvalues Statistics of Hermitian Matrices

In this section, we present a brief discussion of the spectra of Hermitian matrices and their statistics using the trace. Let us consider a complex matrix $M \in C^{N\times N}$, M is Hermitian if it equals its transpose conjugate pair, the definition is based on the relation:

$$M = M^{+} \tag{1}$$

Where $M^{+} = M^{*T}$, the part we are focused on is the eigenvalues, the decomposition of such matrices exists and is given by the following relationship:

$$\begin{aligned} M &= \sum_{i}^{N} \lambda_i u_i u_i^{+} = U \Lambda U^{+} \\ M u_i &= \lambda_i u_i \end{aligned} \tag{2}$$

$U \in C^{N\times N}$ is complex vectorial space of M , λ_i is the i^{th} eigenvalue that corresponds to the i^{th} eigenvector, Λ is diagonal matrix whose elements are λ_i. For Hermitian matrices, the eigenvalues are always real $\lambda_i \in R$. We limit our study by examining fourth order statistics, the mean eigenvalue denoted by letter m can be computed using the trace as the following:

$$m = \frac{1}{N} Tr(M) = \frac{1}{N} \sum_{i=1}^{N} M_{ii} \tag{3}$$

Based on this value, the second order statistic or the standard deviation s is given by the following relation:

$$s = \sqrt{<\lambda^2> - <\lambda>^2} = \sqrt{\frac{Tr(M^2)}{N} - m^2} \tag{4}$$

Next, the third order statistic called skewness that we denote by letter μ is given by:

$$\mu = \frac{1}{s^3}\left(\frac{Tr(M^3)}{N} - \frac{3}{N^2}(Tr(M)Tr(M^2)) + \frac{2}{N^3}Tr(M)^3\right) \tag{5}$$

Finally, the fourth order statistic called Kurtosis k can be computed similarly to the above metrics, by the equation:

$$k = \frac{1}{s^4}\left(\frac{Tr(M^4)}{N} - \frac{1}{N^2}(4Tr(M^3)Tr(M)) + \frac{1}{N^3}(6Tr(M^2)Tr(M)^2) - \frac{3}{N^4}(3Tr(M)^4)\right) \tag{6}$$

Similarly, n^{th} statistic, if it exists, can be computed using the binomial theorem as the following:

$$m^n = <(\lambda - <\lambda>)^n> = <\sum_{i=0}^{n} C_n^i \lambda^i (-<\lambda>)^{n-i}> = \sum_{i=0}^{n} C_n^i <\lambda^i> (-<\lambda>)^{n-i} \tag{7}$$

$$= \sum_{i=0}^{n} C_n^i \frac{Tr(M^i)}{N}\left(-\frac{Tr(M)}{N}\right)^{n-i}$$

Let us illustrate an example of random and real symmetric matrix defined by $M = 0.5\times(H + H^T)$ where H_{ij} are independent and identically distributed random variables $H_{ij} \sim \mathrm{N}(0,1)$. Staring from dimension $N = 2$ to $N = 100$, we compute the eigenvalues statistics, as illustrated by Figure 1, each statistic of given value of N is averaged over 100 trials.

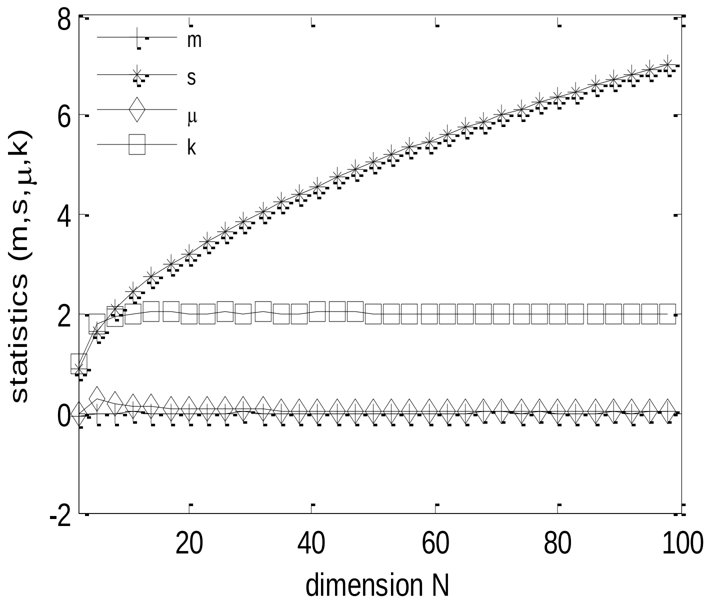

Figure 1. Mean, Standard Deviation, Kurtosis and Skewness of Symmetric Random Matrix W.R.T Dimension

Given the case where only the statistics of matrix spectra is needed, the above relations can be applied without using the eigendecomposition given in equation (1). To illustrate one possible application of trace based statistics, we present in the next section, an application in field of bearing estimation with large array of sensors.

3. Eigenvalue Threshold of Array Signal Model

In array signal processing, bearing estimation [5,6,7] is based on observations (samples) collected from an array of antennas, the superimposed waves pass through the array system and induce a voltage at sensors, the signal model is based on angle interferometry which is a function of geometrical arrangement of sensors, in order to simplify the problem, let us consider a case of uniform linear array with N omni-directional and identical sensors with same consecutive distance d which equals half of the wavelength of radiating sources $d = \lambda / 2$, based on K samples and superposition of P punctual sources in the far field region, the signal model relatively to the normal of the array is given by the relation:

$$x(t) = \sum_{i=1}^{P} s_i(t) a(\theta_i) + n(t) \quad (8)$$

For $t=1,...,K$ and angles of incidence $\theta_1,...,\theta_P$ of plane waves [5], it is assumed that the sensors are vertically placed in (x,y) plane and the waves are linearly polarized $E=(0,0,E_z)$, $a(\theta_i)$ is the steering vector which is written in the following form using the first element as phase reference:

$$a(\theta_i) = \left(1, e^{-j2\pi\lambda^{-1} d \sin(\theta_i)}, ..., e^{-j2\pi\lambda^{-1} d (N-1)\sin(\theta_i)}\right)^T \quad (9)$$

The steering matrix of the uniform linear array $A \in C^{N\times P}$ is written by the relation $A = [a(\theta_1),...,a(\theta_P)]$. Bearing estimation algorithms [5,6,7] rely generally on second order statistics of data $x(t)$, its theoretical expression is given by :

$$\Gamma = < xx^+ > = A < ss^+ > A^+ + \sigma^2 I_N \quad (10)$$

$< ss^+ >$ is the correlation matrix of P waveforms, σ^2 is the noise power and I_N is the identity matrix. Bearing estimation requires a threshold of the spectrum of Γ described in descending order $\lambda_1 \geq \lambda_2 \geq ... \lambda_P > \lambda_{P+1} \square \lambda_N = \sigma^2$, the minimum eigenvalue is degenerate $N-P$ times.

The threshold value is in the range $\lambda_c \in]\lambda_{P+1}, \lambda_P[$, several solutions were proposed to estimate λ_c, among the alternative solutions is the QR and LU decompositions [11] of spectral matrix that are given by:

$$\Gamma = \begin{pmatrix} L_{11} & 0 \\ L_{21} & I_{N-P} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{pmatrix} \quad (11)$$

The dimensions of matrices U_{22} and R_{22} are $N-P \times N-P$, according the relations presented in [11] $\lambda_{P+1} < \| U_{22} \|_2 < \lambda_P$ and $\lambda_{P+1} < \| R_{22} \|_2 < \lambda_P$, the threshold can be chosen as $\lambda_c = \| U_{22} \|_2$ or $\lambda_c = \| R_{22} \|_2$.

Let us consider the application of eigenvalues statistics on the present model of signals. For the first application, given that the different waveforms $s(t)$ are uncorrelated between each other and white noise model of $n(t)$ [6], the first order statistic of eigenvalues given in equation (3) is an estimate of sum of signal and noise powers [14], it is given by:

$$< \lambda >= \frac{Tr(\Gamma)}{N} = \sum_{i=1}^{P} \sigma_i^2 + \sigma^2 \tag{12}$$

As a second application, the above relation and second order statistic $\Delta\lambda$, given in equation (4), can be employed to estimate the threshold λ_c using a theorem of eigenvalues bounds [15], given Hermitian matrix Γ, the lower and upper bounds of smallest and largest eigenvalues λ_N and λ_1 respectively are bounded by [15]:

$$\begin{aligned} &< \lambda > -\Delta\lambda\sqrt{N-1} \le \lambda_N \le \; < \lambda > -\frac{\Delta\lambda}{\sqrt{N-1}} \\ &< \lambda > +\frac{\Delta\lambda}{\sqrt{N-1}} \le \lambda_1 \le \; < \lambda > +\Delta\lambda\sqrt{N-1} \end{aligned} \tag{13}$$

If the condition of large number of sensors N relatively to the number of radiating sources P is verified, the upper bound of smallest eigenvalue can be an alternative estimate the threshold λ_c as the following:

$$\lambda_{cs} =< \lambda > -\frac{\Delta\lambda}{\sqrt{N-1}} \tag{14}$$

The threshold is in general, applied to estimate the number of radiating sources and for angular beam scan to estimate the angles of incidence of superimposed wave fields using as example the EG [9] and Lorentzian [12] operators. In order to quantify the performance of the three presented functions above for approximating the parameter λ_c, we present in the next section, a numerical simulation using large array and closely sources.

4. Numerical Comparison

In this section, we run a computer simulation based on $L=100$ trials for each value of Signal to Noise Ratio SNR in the range $[-5\text{ dB}, 20\text{ dB}]$, the configuration of antennas-sources is chosen as wave field consisting of $P=4$ radiating sources that are linearly polarized, uncorrelated and have complex random envelopes $s(t)$ with zero mean and same power $\sigma_s^2 = 1\text{W}$, the angles of incidence are $\theta = [5^\circ, 10^\circ, 15^\circ, 18^\circ]$.

The distance between sensors is half the wavelength $d = \lambda / 2$ and the number of samples is $K=200$. For each value of SNR, we compute the average, over L trials, of smallest signal eigenvalue λ_P and largest noise eigenvalue λ_{P+1} in order to construct the threshold range $[\lambda_{P+1}, \lambda_P]$, next we compare the estimate of threshold using the functions $\| U_{22} \|_2$, $\| R_{22} \|_2$ and λ_{cs} defined by equation (14). For precise quantification of the performance of the three functions, we test the estimated values according to the number of sensors, in the first part, we consider that the number of antennas is the double of the number of sources $N = 8$, Figure 2 represents the comparison.

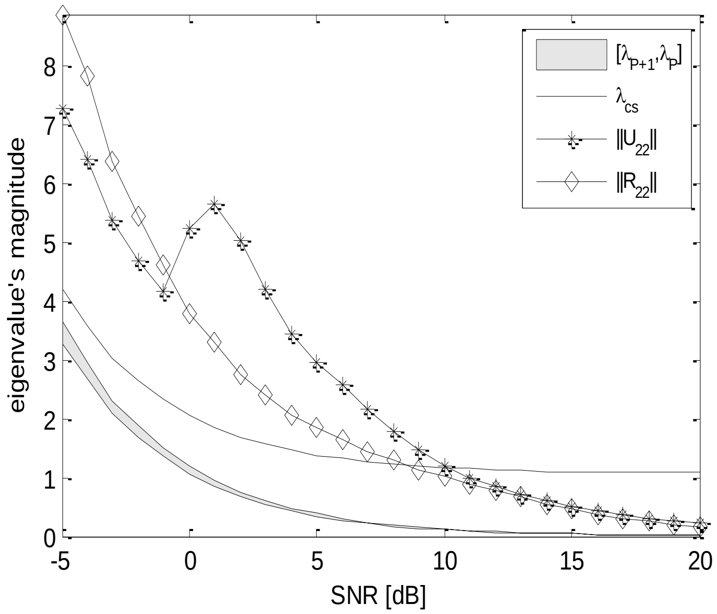

Figure 2. Comparison of Different Thresholds λ_c of Eigenvalues, $N = 8$

The colored area represents the range $[\lambda_{P+1}, \lambda_P]$, the lines denoted by the notations '-*-' and '-◊-' represent the norms $\| U_{22}\|_2$ and $\| R_{22}\|_2$ respectively, and the continuous line represents the threshold defined in equation (14), we remark that the three functions cannot estimate the threshold λ_c in this case of $N = 8$, however the norms $\| U_{22}\|_2$ and $\| R_{22}\|_2$ converge to an accurate estimate staring from SNR=20 dB. To compare this remark with that of large array, we repeat the simulation using N=20 sensors, the result of comparison are illustrated in Figure 3.

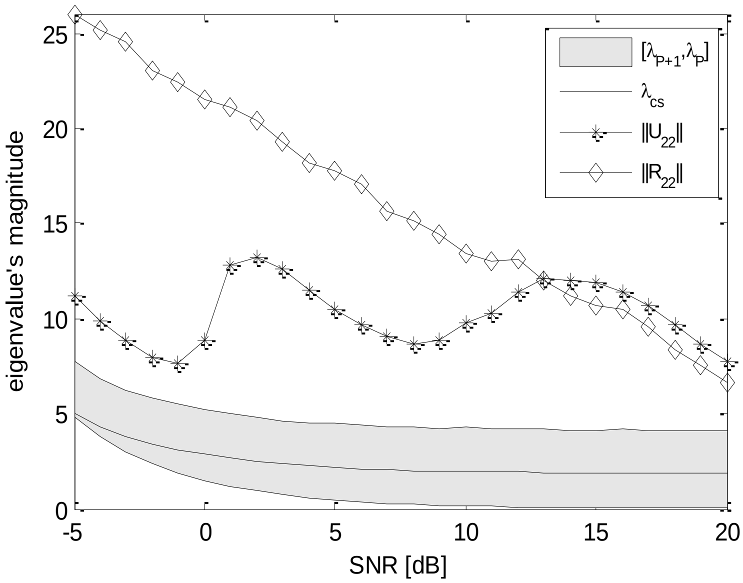

Figure 3. Comparison of Different Thresholds λ_c of Eigenvalues, $N = 20$

In this case, the estimated threshold λ_{sc} defined by equation (14) can correctly estimate the threshold as we can remark that $\lambda_{cs} \in [\lambda_{P+1}, \lambda_P]$, however the two norms $\| U_{22} \|_2$ and $\| R_{22} \|_2$ are out of the valid range, an intuitive explanation for this result is that for large number of sensors, the dimensions of the blocks U_{22} and R_{22} become larger with dimensions $N-P \times N-P$ and consequently the norms become higher.

For complementary study of this simulation results, we analyze the problem by peak detection of localization functions, we choose the optimal value of $SNR = 10$ dB, the EG operator is computed for the three threshold criteria using parameter $m = 10$ as the following:

$$P_{EG} = \left(\left(\frac{\Gamma}{\lambda_c} \right)^m + I_N \right)^{-1} \tag{15}$$

Using the same conditions of array number of sensors, we present in Figure 4 the results of three spectra $f(\theta)$ averaged using L=100 Monte Carlo trials, for different criteria of threshold.

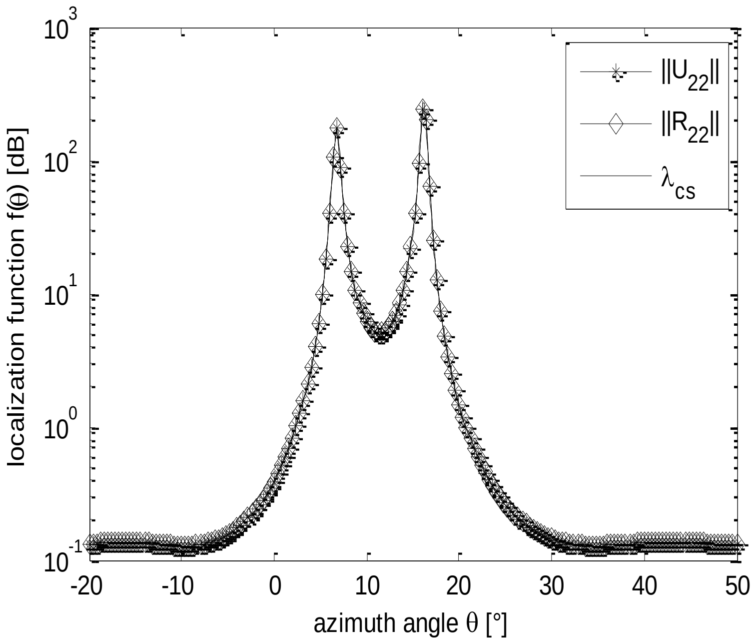

Figure 4. Comparison of Different Spectra $f(\theta)$, $SNR = 10$ dB, $N = 8$

In this case, the number of peaks is less that the exact number of present sources and the three localization functions have the same response of angular beam scan. Next, we test the spectra for $N = 20$ as illustrated in Figure 5.

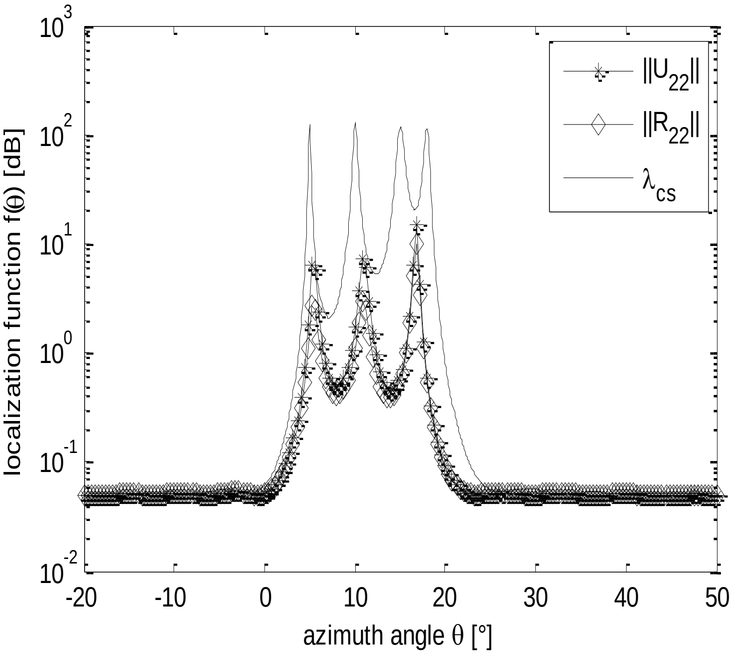

Figure 5. Comparison of Different Spectra $f(\theta)$, $SNR=10$ dB, $N=20$

The peaks of the localization function based on statistical threshold λ_{cs} indicate the exact incident angles contrarily to the two other spectra. As concluding remark, for large array of sensors, eigenvalues first and second order statistics can be employed to estimate the threshold between signal and noise eigenvalues for the purpose of angular interferometry.

5. Conclusion

This paper is devoted to the subject of array signal processing, especially the properties of eigenvalues of spatial covariance matrix computed from observations generated by an array of sensors. In the first part, we have explained the calculations of eigenvalues statistics of Hermitian matrices using the trace which is the sum of diagonal elements. Next, we have demonstrated an application of first and second order statistics in the context of bearing estimation where for some techniques, a threshold between signal and noise eigenvalues is mandatory to accurately estimate the projector into the noise subspace. To support this proposition, we have performed some numerical simulations comparatively to the existing approaches of threshold estimation, the results obtained from sufficient number of trials, for each value of signal powers, demonstrated that the statistical threshold is valid in the case of large array relatively to the number of radiating sources.

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References

[1] A. Alkhateeb, O. El Ayach, G. Leus and R. W. Heath, “Channel Estimation and Hybrid Precoding for Millimeter Wave Cellular Systems”, in IEEE Journal of Selected Topics in Signal Processing, vol. 8, no. 5, **(2014)**, pp. 831-846.

[2] M. P. Daly and J. T. Bernhard, "Directional Modulation Technique for Phased Arrays", in IEEE Transactions on Antennas and Propagation, vol. 57, no. 9, **(2009)**, pp. 2633-2640.
[3] J. J. Lee, "Photonic wideband array antennas", in IEEE Transactions on Antennas and Propagation, vol. 43, no. 9, **(1995)**, pp. 966-982.
[4] J. Rissanen, "Modeling by the Shortest Data Description", Automatica, vol. 14, **(1978)**, pp. 465–471.
[5] H. Krim and M. Viberg, "Two decades of array signal processing research: the parametric approach", Signal Processing Magazine, IEEE, vol.13, no.4, **(1996)**, pp.67, 94.
[6] Y. Khmou, S. Safi and M. Frikel, "Comparative Study between Several Direction of Arrival Estimation Methods", Journal of Telecommunications and Information Technology, **(2014)**, pp.41-p.48.
[7] F. B. Gross, "Smart Antennas for Wireless Communications with Matlab", McGraw-Hill Professional, **(2005)**.
[8] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques", Acoustics, Speech and Signal Processing, IEEE Transactions on, vol.37, no.7, **(1989)**, pp.984, 995.
[9] V.T. Ermolaev and A.B. Gershman, "Fast algorithm for minimum-norm direction-of-arrival estimation", Signal Processing, IEEE Transactions on, vol.42, no.9, **(1994)**, pp.2389, 2394.
[10] T. B. Lavate, V. K. Kokate and A. M. Sapkal, "Performance Analysis of MUSIC and ESPRIT DOA Estimation Algorithms for Adaptive Array Smart Antenna in Mobile Communication", 2010 Second International Conference on Computer and Network Technology, Bangkok, **(2010)**, pp. 308-311.
[11] J. Marot, C. Fossati and S. Bourennane, "Fast subspace-based source localization methods", 2008 5th IEEE Sensor Array and Multichannel Signal Processing Workshop, Darmstadt, **(2008)**, pp. 203-206.
[12] Y. Khmou, S. Safi and M. Frikel, "Lorentzian Operator for Angular Source Localization with Large Array", Journal of Telecommunications and Information Technology, vol. 4, **(2015)**, pp.98-105.
[13] T. Taniguchi, S. Sha and Y. Karasawa, "Statistical Distribution of Eigenvalues of Correlation Matrices in i.i.d MIMO Channels under Rayleigh Fading", 2005 IEEE 16th International Symposium on Personal, Indoor and Mobile Radio Communications, Berlin, **(2005)**, pp. 537-541.
[14] Y. Khmou and S. Safi, "New radiolocation technique for intelligent antennas", 2015 Intelligent Systems and Computer Vision (ISCV), Fez, **(2015)**, pp. 1-4.
[15] H. Wolkowicz and G.P.H. Styan, "Bounds for eigenvalues using traces", Linear Algebra and its Applications, ISSN 0024-3795, vol. 29, **(1980)**, pp. 471-506.

