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Phase Plane Analysis of a Host and Commensal Ecological Model –A Special Case Study

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Abstract

This paper focuses mainly on the phase plane analysis for two species commensal-host ecological model. In this model, the commensal has considered with mortality rate and the host is being immigrated at constant rate. Further the model is restricted with the limited resources. The behavior of this ecological model is observed around the equilibrium points through phase -plane analysis method by taking a set of selected values of the parameters in the basic model equations.

***Keywords**: Commenalisim, Host species, Commensal species, Non-linear system, Equilibrium points, and phase plane diagrams*

2010 Mathematics Subject Classification: 92D25, 92D40.

1. Introduction

Any principle is a professed rule or natural phenomena which helps us to decide various factors in the investigating system. The fundamental principle of phase plane analysis is to identify initially the points where the solution trajectories must be exactly horizontal or exactly vertical lines. These lines will divide the main portion of the phase plane into a sub portions. The general direction of trajectories in each sub portion can be identified around the equilibrium points. It is often possible to see the general behavior of the trajectories, even without explicitly solving them. This technique helps us to know the long term behavior of the solutions rather than their precise values. Several authors Colinvaux [6], Cushing [7], Simmons [21], Freedman [8-9], Meyer [13], Lotka [11], Volterra [22], Kapur [10], Lakshmi Narayan [12], Phanikumar and Pattabhi Ramacharyulu [14-16], Acharyulu and Pattabhi Ramacharyulu [1-5], Seshagiri Rao [17-20] have been contributed their research work in different ecological interactions between the species namely Mutualism, Neutralism, Ammensalism, Commensalism, Prey-Predators and Competition etc. Some of them discussed the sustainability of the ecological models by taking harvesting rates in the basic model equations.

The main aim of this paper is to present phase plane diagrams and the behavior of this ecological interaction between the mortal commensal species and host species within the limited natural resources.The host species is being immigrated at a constant rate. This model is constituted by a couple of first order non-linear ordinary differential equations. The nature of this model (stability and/or instability) is observed through directed filed lines and the trajectories around the possible two equilibrium points in the first quadrant of the phase plane at selected parameter values in the basic model equations.

2. Basic Model Equations

The growth equations for this ecological model comprising the mortal commensal and the host species in which the host species is immigrated at constant .

(i). Equation for the growth rate of the Mortal Commensal species (S_1) is

$$\frac{dN_1}{dt} = a_{11} N_1 [- e_1 - N_1 + cN_2] \tag{1}$$

(ii). Growth rate equation for the Host species (S_2) is

$$\frac{dN_2}{dt} = a_{22} [k_2 N_2 - N_2^2 + I_2] \tag{2}$$

with the initial conditions $N_i(0) = N_{i0} \geq 0, \ (i = 1,2)$ (3)

where

$N_1(t)$, $N_2(t)$: The population rates of both the commensal (S_1) and the host (S_2) at time t.

d_1 : The mortal rate of the commensal (S_1).

a_2 : The rate of natural growth of the host (S_2).

$a_{ii} \ (i = 1,2)$: The rate of decrease of the commensal and the host due to the limitations of its natural resources.

a_{12} : The rate of increase of the commensal (S_1) due to the support given by the host (S_2).

$k_2 (= a_2 / a_{22})$: The carrying capacity of S_2 .

$c (= a_{12} / a_{11})$: The coefficient of the commensal.

$e_1 (= d_1 / a_{11})$: The mortality coefficient of S_1 .

$i_2 (= a_{22} I_2)$: The coefficient of immigration of the host.

I_2 : The immigration of S_2 per unit time.

The state variables $N_1(t)$ and $N_2(t)$ as well as all the model parameters d_1 , a_2 , a_{11} , a_{22} , a_{12} , e_1 , k_2 ,c, i_2 , I_2 are assumed to be non-negative constants.

3. Equilibrium Points

The system under investigation has only two equilibrium points (E_1 - E_2) given by $\frac{dN_1}{dt} = 0, \ \frac{dN_2}{dt} = 0$.

(A). Commensal washed out equilibrium state (E_1):

$$\overline{N_1} = 0 \quad ; \quad \overline{N_2} = \frac{k_2 + \sqrt{k_2^2 + 4I_2}}{2} \tag{4}$$

(B). The Co-existent state (E_2):

$$\overline{N_1} = c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4I_2}}{2}\right) - e_1 \;\; ; \;\; \overline{N_2} = \frac{k_2 + \sqrt{k_2^{\,2} + 4I_2}}{2} \qquad (5)$$

This exists only when $e_1 < c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4I_2}}{2}\right)$. When $e_1 = c\left(\frac{k_2 + \sqrt{k_2^{\,2} + 4I_2}}{2}\right)$ then E_2 merges with the equilibrium state E_1.

4. Phase - Plane Diagrams

The equilibrium points given by $\frac{dN_1}{dt} = 0$, $\frac{dN_2}{dt} = 0$ which are the turning points in the variation of N_1 and N_2 with respect to time t. The lines (straight lines and/or curves) are given by $\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$ in the $N_1 - N_2$ plane may be referred as the threshold lines or Null clines. These lines or curves divide the first quadrant of the $N_1 - N_2$ plane (since $N_1 \geq 0$, $N_1 \geq 0$) into regions which are called the threshold regions. The diagram shows the threshold lines and regions are known as the threshold/phase-plane diagram. This diagram provides the direction of variations of the species around the stable/unstable equilibrium points. The nature of this ecological model around the equilibrium points is discussed by phase plane diagrams.

4.1 The threshold Diagram for equilibrium point E_1

The threshold lines/ null clines supplied by $\frac{dN_1}{dt} = 0$, $\frac{dN_2}{dt} = 0$ .The main portion of this phase plane splits into three regions I , II and III in the first quadrant (i.e., $N_1 \geq 0$, $N_2 \geq 0$) are shown in Figure 1.

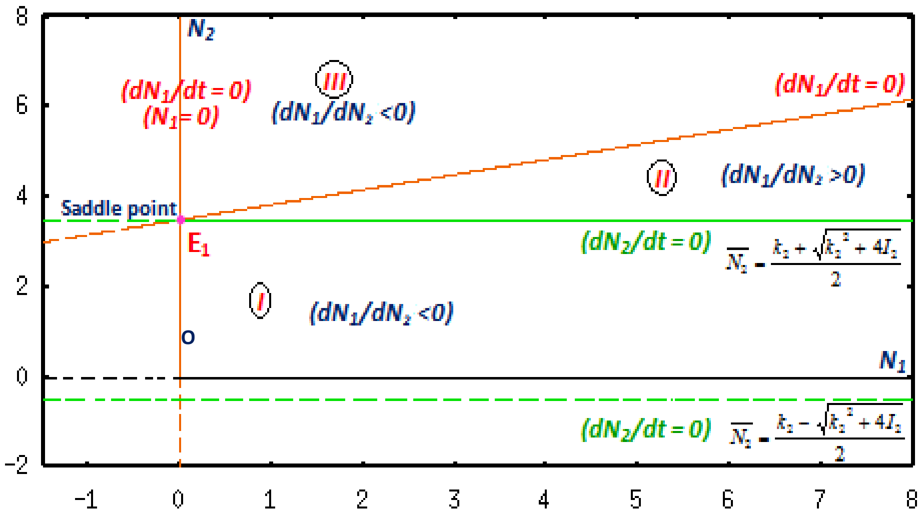

Figure 1. Threshold Regions

4.1.1. Threshold Regions

Region I: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} > 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$
& the trajectories move up and left.

Region II: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} > 0$ then $N_1(t)$ is an increasing function of $N_2(t)$ & the trajectories move down and left.

Region III: Here $\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$ & the trajectories move down and right.

Figure 2 shows the direction of the field lines around the equilibrium point E_1 in the threshold regions for $a_{11} = 0.1$, $e_1 = 10.5$, $c = 3$, $a_{22} = 0.1$, $k_2 = 3$, $I_2 = 1.75$.

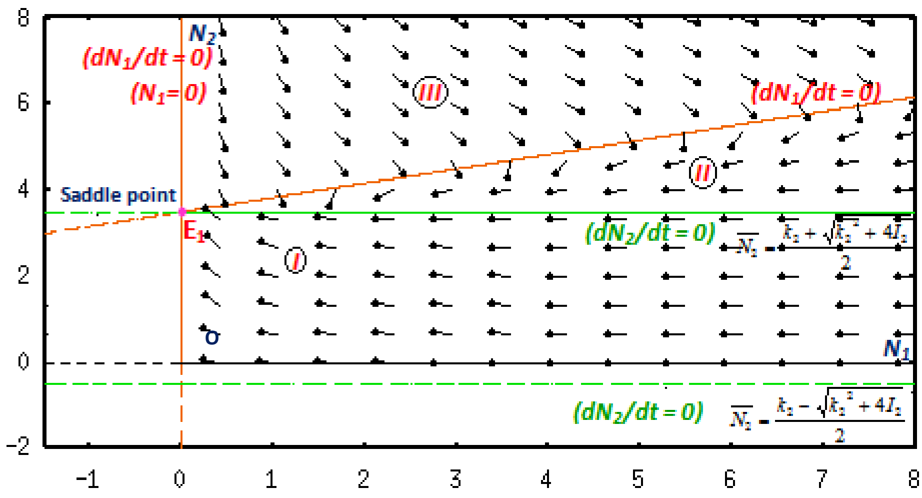

Figure 2. Threshold Diagram for E_1

4.2. The Threshold Diagram for Equilibrium Point E_2

The null clines are given by $\frac{dN_1}{dt} = 0, \frac{dN_2}{dt} = 0$,now divide the phase plane into four regions I , II , III and IV in the first quadrant (*i.e.*, $N_1 \geq 0$, $N_2 \geq 0$) which is shown in Figure 3.

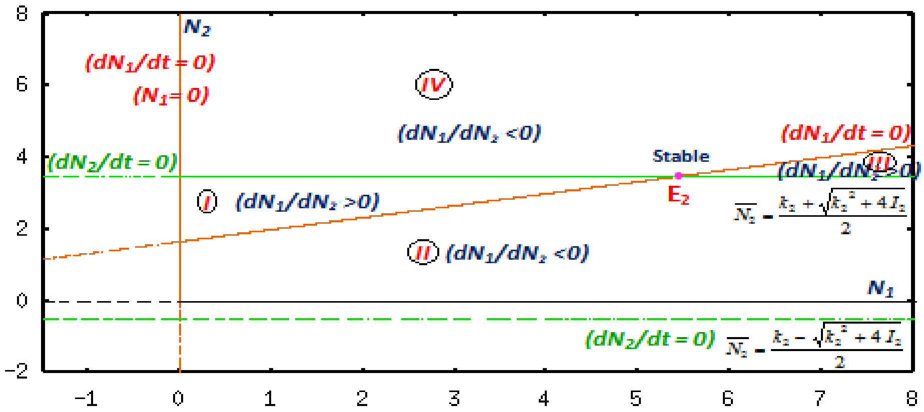

Figure 3. Threshold Regions

4.2.1. Threshold Regions

Region I: In this region $\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} > 0 \Rightarrow \frac{dN_1}{dN_2} > 0$ then $N_1(t)$ is an increasing function of $N_2(t)$ and the trajectories move up and right.

Region II: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} > 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$ and the trajectories move up and left.

Region III: Here $\frac{dN_1}{dt} < 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} > 0$ then $N_1(t)$ is an increasing function of $N_2(t)$ and the trajectories move down and left.

Region IV: Here $\frac{dN_1}{dt} > 0$ and $\frac{dN_2}{dt} < 0 \Rightarrow \frac{dN_1}{dN_2} < 0$ then $N_1(t)$ is a decreasing function of $N_2(t)$ and the trajectories move down and right.

Figure 4 shows the direction of the field lines and the trajectories around the equilibrium point E_2 in the threshold regions for $a_{11} = 0.1$, $e_1 = 5$, $c = 3$, $a_{22} = 0.1$, $k_2 = 3$, $I_2 = 1.75$.

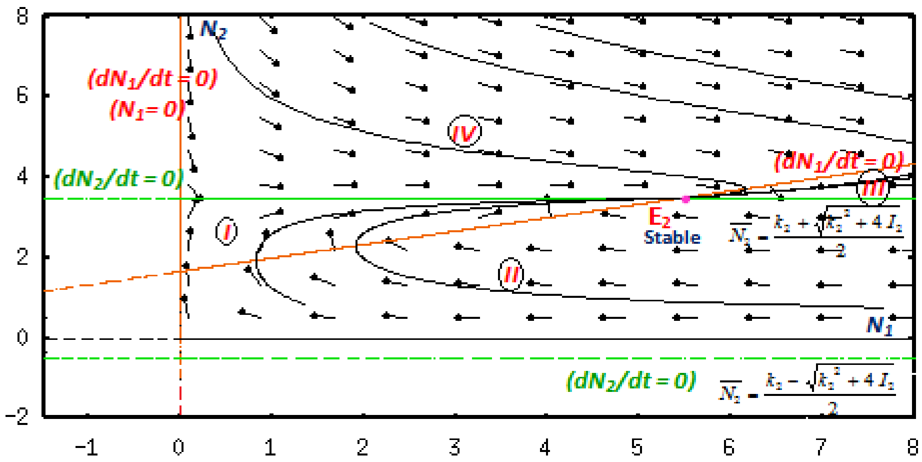

Figure 4. Threshold Diagram for E_2

5. Conclusions

(i). The ecological system is always stable for the lower values of the mortal coefficient (e_1) of the commensal species.
(ii). If the values of the mortal coefficient of the commensal species are more than the immigration coefficients of the host then one can see that the system is not sustainable for longer time.
(iii).If the supported commensal coefficient by the host is more than the mortal coefficient of the communal species than the ecological system is always stable.

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